M. FUJIKI
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ON THE GAUSS MAP OF MINIMAL SURFACES IMMERSED IN R^n

BY MASAHIKO FUJIKI

1. Introduction.

The Gauss map of a minimal surface M in R^n can be considered as a holomorphic mapping from M to the complex quadric Q_{n-2} in the complex projective space CP^{n-1} with the Fubini-Study metric of constant curvature 2. This paper is devoted to the question, "If a minimal surface M in R^n has a constant curvature $\hat{K}$ in its Gaussian image, what values of $\hat{K}$ can be possible?".

This question comes from Ricci's classical theorem;

There exists a minimal surface in R^3 which is isometric with M iff (M, ds^2) satisfies Ricci condition:

(i) Gaussian curvature K of M is negative,

(ii) the new metric $d\tilde{s}^2=\sqrt{-K}ds^2$ is flat on M.

The condition (ii) is known to be equivalent to the condition "$\hat{K}\equiv 1$". (see Lawson [2])

Concerning the question, the following are well-known;

(a) If $\hat{K}\equiv 1$, then M must lie fully in R^3 or R^6. And all the minimal surfaces isometric to M make a two parameter family. (Lawson [2])

(b) Minimal surfaces in R^4 which have constant curvature $\hat{K}$ in their Gaussian images are classified as follows;

i. $\hat{K}\equiv 1$, and M lies in some affine R^3,

ii. $\hat{K}\equiv 2$, and M is a holomorphic curve in C^2.

Here C^2 means R^4 with some orthogonal complex structure. (Osserman-Hoffman [5])

(c) And in R^5,

i. $\hat{K}\equiv 1$ or 2, and M lies in R^4 (these are the cases (b).)

ii. $\hat{K}\equiv 1/2$, and the Gaussian image of M can be represented locally as;

$$1/2(1-w^4,\ i+iw^4,\ 2w+2w^3,\ 2iw-2iw^3,\ 2\sqrt{3}iw^2)$$

(Masal'tsev [4])

To get these results, Calabi's theorem [1] plays the main role. Using the

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method in [2], [4] and [5], following results are obtained;

THEOREM A. *For every positive integer* m, *there exists a minimal surface with* $\hat{K}\equiv 1/m$ *in* $2m+1$ *dimensional Euclidean space.*

THEOREM B. *For every integer* $m\geqq 5$, *there exists a minimal surface with* $\hat{K}\equiv 2/(2m-1)$ *in* $2m$ *dimensional Euclidean space.*

THEOREM C. *Let* $k=3, 5$, *or* 7. *Then there exists a minimal surface* M *with* $\hat{K}\equiv 2/k$ *in* $k+3$ *dimensional Euclidean space.*

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2. Preliminaries.

Let M be a surface immersed in R^n. It means that there exists a conformal immersion

$$X: S \longrightarrow R^n, \qquad X=(X_1, X_2, \cdots, X_n)$$

where S is a Riemann surface. Here we define the Gauss map g as follows;

$$g: S \longrightarrow Q_{n-2}=\{z\in CP^{n-1} \mid \sum_i z_i^2=0\}$$

$$g(w)=\frac{\partial X}{\partial w}=\left(\frac{\partial X_1}{\partial w}, \frac{\partial X_2}{\partial w}, \cdots, \frac{\partial X_n}{\partial w}\right)$$

where $w=u_1+iu_2$ is a local coordinate of S.

By definition a surface M is minimal if

$$\Delta X_i=0 \qquad \text{for} \quad i=1, 2, \cdots, n$$

where $\Delta=\dfrac{\partial^2}{\partial u_1^2}+\dfrac{\partial^2}{\partial u_2^2}$.

It is known that $g(w)$ is holomorphic iff M is a minimal surface. (see [5]) In this paper we exclude the case where M is a plane.

Let CP^{n-1} have the Fubini-Study metric with constant holomorphic curvature 2;

$$ds^2=\frac{2\sum_{j<k}|z_j dz_k-z_k dz_j|^2}{\left[\sum_{j=1}^{n}|z_j|^2\right]^2}$$

Let $\hat{K}(p)$ denote the Gaussian curvature of $g(S)\subset Q_{n-2}\subset CP^{n-1}$ at a point $p\in S$. It follows immediately that

$$\hat{K}(p)\leqq 2.$$

3. Results.

Let M be a minimal surface in R^n with $\hat{K}\equiv c$ (constant). Then Calabi's results tell us that c must be the form $2/k$, $k\in N$, and it must satisfy

$$k\leqq n-1. \tag{1}$$

And furthermore, $g(S)$ must be represented locally

$$^t g(w)=Uy_k$$

where U denotes an n×n unitary matrix, and

$$y_k={}^t\left(1, \sqrt{\binom{k}{1}}w, \sqrt{\binom{k}{2}}w^2, \cdots, \sqrt{\binom{k}{k}}w^k, 0, \cdots, 0\right).$$

From the fact that $g(S)\subset Q_{n-2}$, $g(w)$ must satisfy

$$g(w)\cdot {}^t g(w)=0$$

It is equivalent to

$$^t y_k {}^t U U y_k=0 \tag{2}$$

Now we set

$$^t UU=A=(a_{ij}) \qquad i, j=1, \cdots, n.$$

Here A is a symmetric unitary matrix. So, $a_{ij}=a_{ji}$.

THEOREM A. *For every positive integer m, there exists a minimal surface with $\hat{K}\equiv 1/m$ in $2m+1$ dimensional Euclidean space.*

Proof. From the fact

$$\binom{k}{0}-\binom{k}{1}+\binom{k}{2}-\cdots+(-1)^j\binom{k}{k}+\cdots+(-1)^k\binom{k}{k}=0,$$

the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & & -1 & 0 & 0 \\ & & \cdots & & & \\ 0 & 0 & -1 & & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

satisfies the properties of A. Because it is a real orthogonal matrix, it is diagonalizable in the sense of real matrices, and its eigenvalues are 1 or -1. So, the unitary matrix U is easily calculated. q. e. d.

THEOREM B. *For every integer* $m\geqq 5$, *there exists a minimal surface with* $\hat{K}\equiv 2/(2m-1)$ *in* $2m$ *dimensional Euclidean space.*

Proof. Let

$$P(j)=\sum_{i=0}^{j}\binom{2m-1}{i}, \qquad Q(j)=\sum_{i=j+1}^{m-2}\binom{2m-1}{i}.$$

There exists j_0 *s. t.* $P(j_0)\leqq Q(j_0)$, and $P(j_0+1)\geqq Q(j_0+1)$. Now set

$$P=P(j_0), \quad Q=Q(j_0), \quad R=\binom{2m-1}{m-1}.$$

These P, Q and R satisfy the triangle inequality. So, there exist two real numbers θ, φ *s. t.*

$$P+Qe^{i\theta}+Re^{i\varphi}=0 \tag{3}$$

Let us define the symmetric unitary matrix A as follows;

$$a_{s,2m+1-s}=\begin{cases} 1 & \text{for } 1\leqq s\leqq j_0+1 \quad \text{or } 1\leqq 2m+1-s\leqq j_0+1 \\ e^{i\theta} & \text{for } j_0+2\leqq s\leqq m-1 \text{ or } j_0+2\leqq 2m+1-s\leqq m-1 \\ e^{i\varphi} & \text{for } s=m,\ m+1 \end{cases}$$

$$a_{s,t}=0 \qquad \text{for } \ t\neq 2m+1-s$$

It is easy to see that the matrix A is decomposed as

$$A={}^tUU$$

where U is a unitary matrix. Then, from (3), the equation (2) is satisfied.

q. e. d.

THEOREM C. *Let* $k=3, 5,$ *or* 7. *Then there exists a minimal surface* M *with* $\hat{K}\equiv 2/k$ *in* $k+3$ *dimensional Euclidean space.*

Proof. In this case the matrix A is given as follows;

for $i+j=k+2$, $i\neq(k+1)/2, (k+3)/2$, $a_{ij}=1$,

for $(i, j)=((k+1)/2, (k+3)/2), ((k+3)/2, (k+1)/2)$,

$$a_{ij}=\alpha=\left\{\sum_{r=0}^{m-2}\binom{k}{r}\right\}\Big/\binom{k}{m-1}<1 \qquad \text{where } \ m=(k+1)/2,$$

for $(i, j)=((k+1)/2, k+2), ((k+3)/2, k+3), (k+2, (k+1)/2),$

$(k+3, (k+3)/2)$, $\quad a_{ij}=\sqrt{1-\alpha^2}$, for $(i, j)=(k+2, k+3), (k+3, k+2)$, $a_{ij}=-\alpha$,

otherwise, $a_{ij}=0$. q. e. d.

Now, we know Calabi's inequality (1) is best possible when $k\neq 1, 3, 5, 7$. And when $k=1, 3$, the minimum n is 4, 6 respectively. But when $k=5, 7$, the minimum n are unknown. In other words, it is unknown whether minimal surfaces with $K=2/5$ $(2/7)$ exist in R^6, R^7 (in R^8, R^9 respectively), or not.

Remark 1. In theorem A., if $n=3$, then $\hat{K}\equiv 1$ and matrix A must be the form;

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

And,

$$U=\begin{pmatrix} 1/2 & 0 & -1/2 \\ i/2 & 0 & i/2 \\ 0 & 1 & 0 \end{pmatrix}$$

(mod orthogonal transformations in R^3).

From this, we can obtain classical Weierstrass-Enneper's expression formula for classical minimal surfaces.

Remark 2. Also in theorem A., if $n=5$, then $\hat{K}\equiv 1/2$ and

$$U=\begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} \\ i/\sqrt{2} & 0 & 0 & 0 & i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & i/\sqrt{2} & 0 & -i/\sqrt{2} & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}$$

(mod orthogonal transformations in R^5)

Combining the fact that no minimal surfaces with $\hat{K}\equiv 2/3$ exist in R^5, Masal'-tsev's theorem is obtained. (see [4])

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