FINITE TYPE SUBMANIFOLDS IN PSEUDO-EUCLIDEAN SPACES AND APPLICATIONS

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§ 1. Introduction.

Let $E_s{}^m$ be the m-dimensional pseudo-Euclidean space with (flat) pseudo-Riemannian metric of signature $(s, m-s)$. And let M be a compact space-like submanifold of $E_s{}^m$. By using the induced Riemannian structure on M, we can define two well-defined numbers p and q associated with the submanifold M in $E_s{}^m$. Here p is a positive integer and q is either $+\infty$ or an integer $\geq p$. The pair $[p, q]$ is called the *order of the submanifold* M (cf. [1]). The submanifold M is said to be of *finite type* if q is finite. Otherwise, M is said to be of *infinite type*. The submanifold M is of finite type if and only if there is a non-trivial polynomial P such $P(\Delta)H=0$; where Δ is the Laplacian Δ on M and H the mean curvature vector of M in $E_s{}^m$.

In this paper, we will give some general results for finite type submanifolds in the pseudo-Euclidean space $E_s{}^m$. By applying these results, we will prove the following. (1) There exist no compact space-like hypersurfaces with constant mean curvature and constant scalar curvature in the anti-de Sitter space-time; (2) Every compact hypersurface with constant mean curvature and constant scalar curvature in a hyperbolic space is a small hypersphere; and (3) If M is a compact space-like hypersurface of the de Sitter space-time, then M has non-zero constant mean curvature and constant scalar curvature when and only when M is mass-symmetric and of 2-type in the Lorentz-Minkowski world.

For the general knowledge on Finite-Type Submanifolds in Euclidean spaces, see [1, 2]. And for the general knowledge on Relativity, see for instance [3, 4].

§ 2. Preliminaries.

Let $E_s{}^m$ be the m-dimensional pseudo-Euclidean space with metric tensor given by

$$(2.1) \qquad g_0 = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2 ,$$

where $(x_1, \cdots, x_m)$ is a rectangular coordinate system of $E_s{}^m$. $(E_s{}^m, g_0)$ is a flat pseudo-Riemannian manifold of signature $(s, m-s)$.

Let c be a point in E_s^m and $r>0$. We put

$$(2.2) \qquad S_s^m(c, r)=\{x\in E_s^{m+1} \mid \langle x-c, x-c\rangle=r^2\},$$

$$(2.3) \qquad H_s^m(c, r)=\{x\in E_{s+1}^{m+1} \mid \langle x-c, x-c\rangle=-r^2\},$$

where $\langle , \rangle$ denotes the indefinite inner product on the pseudo-Euclidean space. It is known that $S_s^m(c, r)$ and $H_s^m(c, r)$ are complete pseudo-Riemannian manifolds of signature $(s, m-s)$ and respective constant sectional curvatures r^{-2} and $-r^{-2}$. $S_s^m(c, r)$ is simply-connected for $s<m-1$; $S_{m-1}^m(c, r)$ is connected and has infinite cyclic fundamental group; and $S_m^m(c, r)$ has two simply-connected components. $S_s^m(c, r)$ and $H_s^m(c, r)$ are called the *pseudo-Riemannian sphere* and the *pseudo-hyperbolic space*, respectively (cf. [6, p. 67].) The point c is called the *center* of $S_s^m(c, r)$ and of $H_s^m(c, r)$. In the following, $S_s^m(0, 1)$ and $H_s^m(0, 1)$ are simply denoted by S_s^m and H_s^m, respectively. S_1^m is called the *de Sitter space-time* $(=de\ Sitter\ world)$ and H_1^m the *anti-de Sitter space-time* $(=anti-de$ *Sitter world*). Both S_1^m and H_1^m are pseudo-Riemannian manifold of signature $(1, m-1)$. The hyperbolic space H^m is defined by

$$(2.4) \qquad H^m=\{x\in E_1^{m+1} \mid \langle x, x\rangle=-1 \text{ and } t>0\},$$

where $t=x_1$ is the first coordinate in E_1^m. H^m is a complete, simply-connected Riemannian manifold of constant sectional curvature -1. E_1^m is called the *Lorentz-Minkowski space-time* $(=Lorentz-Minkowski\ world.)$

Let $\tilde{M}$ be a pseudo-Riemannian manifold with pseudo-Riemannian metric $\tilde{g}$. Denote by $\langle , \rangle$ the associated non-degenerate inner product and by $\tilde{\nabla}$ the metric connection on $\tilde{M}$. A tangent vector X to $\tilde{M}$ is said to be *space-like* (respectively, *time-like* or *light-like*) if $\langle X, X\rangle>0$ or $X=0$ (respectively, if $\langle X, X\rangle<0$ or $\langle X, X\rangle=0$ and $X\neq0$).

Let M be a submanifold of $\tilde{M}$. If the pseudo-Riemannian metric tensor $\tilde{g}$ of $\tilde{M}$ induces a pseudo-Riemannian metric (respectively, Riemannian metric) on M, then M is called a *psuedo-Riemannian* (respectively, *space-like*) *submanifold* of $\tilde{M}$.

If M is a pseudo-Riemannian (or space-like) submanifold of $\tilde{M}$, each tangent space is, $T_x(M)$ by definition, a nondegenerate subspace of $T_x(\tilde{M})$. Hence, we have the direct sum decomposition:

$$(2.5) \qquad T_x(\tilde{M})=T_x(M)\oplus T_x^\perp(M),$$

where the normal space $T_x^\perp(M)$ is also nondegenerate.

Let ∇ denote the induced metric connection on M. Then, for any vector fields X, Y tangent to M, we have the following Gauss formula:

$$(2.6) \qquad \tilde{\nabla}_X Y=\nabla_X Y+h(X, Y),$$

where h is the second fundamental form of M in $\tilde{M}$.

Denote by R and $\tilde{R}$ the curvature tensors of M and $\tilde{M}$, respectively. The

Gauss equation is given by

(2.7) $\langle R(X, Y)Z, W \rangle = \langle \tilde{R}(X, Y)Z, W \rangle$

                         $+ \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle$.

Denote by D the linear connection induced on the normal bundle $T^{\perp}(M)$. For each vector field ξ normal to M, the Weingarten formula is given by

(2.8) $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$,

where A_ξ is the Weingarten map with respect to ξ. A_ξ is a self-adjoint endomorphism of the tangent bundle $T(M)$ which can be diagonalized when M is space-like. It is well-known that h and A are related by

(2.9) $\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$.

For the second fundamental form h, we define the covariant differentiation of h by

(2.10) $(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

The Codazzi equation is given by

(2.11) $(\tilde{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z)$.

A normal vector field ξ is said to be *parallel* is $D_X \xi = 0$ for any vector X tangent to M.

Let M be a pseudo-Riemannian submanifold with signature $(t, n-t)$ in $\tilde{M}$. Let $E_1, \cdots, E_n$ be an orthonormal local basis on M such that $E_1, \cdots, E_t$ are time-like and $E_{t+1}, \cdots, E_n$ are space-like. If F is an endomorphism of TM such that $F = (F_{ij})$ with respect to the basis $E_1, \cdots, E_n$, then the *trace* of F is defined by

$$\operatorname{tr} F = \sum_{i=1}^{n} \varepsilon_i F_{ii} ,$$

where $\varepsilon_1 = \cdots = \varepsilon_t = -1$, $\varepsilon_{t+1} = \cdots = \varepsilon_n = 1$ and $F_{ij} = \langle F(E_i), E_j \rangle$. From these, we find

(2.12) $\operatorname{tr} h = \sum_{i=1}^{n} \varepsilon_i h(E_i, E_i)$.

The *mean curvature vector* H of M in $\tilde{M}$ is defined by $H = 1/n \operatorname{tr} h$, where $n = \dim M$. A pseudo-Riemannian submanifold M of $\tilde{M}$ is said to be *minimal* if the mean curvature vector H vanishes identically. A pseudo-Riemannian submanifold M in $\tilde{M}$ is said to be *pseudo-umbilical* if $\langle H, H \rangle \neq 0$ and $A_H = \mu I$ for some function μ on M, where I is the identity transformation.

We need the following.

LEMMA 1. *Let M be a pseudo-Riemannian (in particular, space-like) submanifold of the pseudo-Riemannian sphere $S_s^m(c, r)$ (respectively, the pseudo-hyperbolic*

space $H_{s-1}{}^m(c, r))$ in $E_s{}^{m+1}$. Then the mean curvature vectors H and H' of M in $E_s{}^{m+1}$ and in $S_s{}^m(c, r)$ (respectively, in $H_{s-1}{}^m(c, r)$) satisfy

$$(2.13) \qquad H=H'-\frac{1}{r^2}(x-c) \quad \left(respectively, \; H=H'+\frac{1}{r^2}(x-c)\right).$$

Proof. Let $\varepsilon=1$ or -1 according to M being a submanifold of $S_s{}^m(c, r)$ or of $H_{s-1}{}^m(c, r)$. Let x denote the position vector of M in $E_s{}^{m+1}$. Then we have

$$(2.14) \qquad\qquad\qquad \langle x-c, \; x-c \rangle = \varepsilon r^2.$$

For each vector X tangent to $S_s{}^m(c, r)$ or to $H_{s-1}{}^m(c, r)$ we have $\tilde{\nabla}_X x = X$. Thus, $x-c$ is normal to $S_s{}^m(c, r)$ and to $H_{s-1}{}^m(c, r)$. Moreover, from $\tilde{\nabla}_X x = X$, we have

$$(2.15) \qquad\qquad\qquad A_{x-c} = \tilde{A}_{x-c} = -I,$$

where A and $\tilde{A}$ denote the Weingarten maps of M and $S_s{}^m(c, r)$ (or $H_{s-1}{}^m(c, r)$) in $E_s{}^{m+1}$. Let h' and h be the second fundamental forms of M in $S_s{}^m(c, r)$ (or in $H_{s-1}{}^m(c, r)$) and of $S_s{}^m(c, r)$ (or $H_{s-1}{}^m(c, r)$) in $E_s{}^{m+1}$, respectively. Then we have

$$(2.16) \qquad\qquad h(X, Y) = h'(X, Y) + \tilde{h}(X, Y).$$

Since $h(X, Y) = \varepsilon \langle A_{x-c}X, Y \rangle (x-c)/r^2$, (2.12), (2.15) and (2.16) give (2.13).

$$(\text{Q. E. D.})$$

§3. Some General Results.

Throughout the remaining part of this paper, we assume that M is a connected, n-dimensional, pseudo-Riemannian submanifold of $E_s{}^{m+1}$.

First, we give the following.

LEMMA 2. *A submanifold M of the pseudo-Euclidean space $E_s{}^{m+1}$ is a pseudo-umbilical submanifold with parallel mean curvature vector if and only if M is either a minimal submanifold of a pseudo-Riemannian sphere $S_s{}^m(c, r)$ or a minimal submanifold of a psedo-hyperbolic space $H_{s-1}{}^m(c, r)$ for some $c \in E_s{}^{m+1}$ and $r>0$.*

Proof. Assume that M is a pseudo-umbilical submanifold of $E_s{}^{m+1}$ with parallel mean curvature vector. Then, $\langle H, H \rangle \neq 0$ and, by definition, for any X tangent to M, we have $X\langle H, H \rangle = 2\langle \tilde{\nabla}_X H, H \rangle = 0$. Thus, $\langle H, H \rangle$ is a non-zero constant. We put

$$(3.1) \qquad\qquad \langle H, H \rangle = \frac{\varepsilon}{r^2}, \qquad \varepsilon=1 \;\; \text{or} \;\; -1.$$

Let $A_H = \mu I$. Then from (2.9) and (2.12), we find

$$(3.2) \qquad\qquad\qquad \varepsilon = \mu r^2.$$

We put

(3.3) $y = x + \varepsilon r^2 H$,

where x is the position vector of M in E_s^{m+1}. Then, for any vector X tangent to M, we have

(3.4) $\tilde{\nabla}_X y = \tilde{\nabla}_X x + \varepsilon r^2 \tilde{\nabla}_X H$

                                        $= X - \varepsilon r^2 A_H X = 0$.

Thus y is a constant vector, say c, in E_s^{m+1}. Hence, we find $\langle x-c, x-c \rangle = r^4 \langle H, H \rangle = \varepsilon r^2$. This shows that M lies either in the pseudo-Riemannian sphere $S_s^m(c, r)$ or in the pseudo-hyperbolic space $H_{s-1}^m(c, r)$. Since (3.3) gives

(3.5) $H = -\dfrac{\varepsilon}{r^2}(x - c)$,

Lemma 1 implies that $H' = 0$, i.e., M is a minimal submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$.

Conversely, if M is a minimal submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$, then Lemma 1 shows that $H = -\varepsilon(x-c)/r^2$. This implies that

$$\tilde{\nabla}_X H = -\frac{\varepsilon}{r^2}\tilde{\nabla}_X x = -\frac{\varepsilon}{r^2}X.$$

Thus, $A_H = \varepsilon I/r^2$ and $DH = 0$. Moreover, by applying (2.13) and (3.5) we also find $\langle H, H \rangle = \varepsilon/r^2 \neq 0$. Consequently, M is a pseudo-umbilical submanifold with parallel mean curvature vector.   (Q. E. D.)

Remark 1. If E_s^{m+1} is the Euclidean space E^{m+1}, Lemma 2 is due to Yano and Chen [7]. For pseudo-Euclidean case the "only if" part of Lemma 2 was given in [5].

LEMMA 3. *If M is a submanifold of S_s^m (or H_{s-1}^m) in E_s^{m+1}, then the mean curvature vectors H' and H in S_s^m (or in H_{s-1}^m) and in E_s^{m+1} satisfy*

(3.6) $DH = D'H'$, $A_H = A_{H'} + \varepsilon I$.

Proof. Obvious from (2.8), (2.14) and Lemma 1.

In the following, by a *hyperplane section* N of S_s^m (or of H_{s-1}^m) we mean the intersection of S_s^m (or of H_{s-1}^m) and a hyperplane L of E_s^{m+1}.

By applying Lemma 2, we have the following.

PROPOSITION 1. *Let M be a submanifold of the pseudo-Riemannian sphere S_s^m (respectively, of the pseudo-hyperbolic space H_{s-1}^m.) If M is a pseudo-umbilical submanifold with parallel mean curvature vector, then M is a minimal submanifold of a hyperplane section of S_s^m (respectively, of H_{s-1}^m).*

Proof. Under the hypothesis, the mean curvature vector H' of M in S_s^m (or in H_{s-1}^m) satisfies

(3.7) $\langle H', H' \rangle \neq 0$, $A'_{H'} = \mu I$ and $D'H' = 0$,

where A' and D' denote the Weingarten map and the normal connection of M in S_s^m (or in H_{s-1}^m). Let A be the Weingarten map of M in E_s^{m+1}. Then, by (3.7) and Lemma 3, we know that M has parallel mean curvature vector H in E_s^{m+1}, too. Since H is parallel, $\langle H, H \rangle$ is constant.

If $\langle H, H \rangle = 0$, then $H = H' - \varepsilon x$ is a light-like vector. Thus, we have $\langle H', H' \rangle = -\varepsilon$. Because $A'_{H'} = \mu I$, we find $\mu = \langle H', H' \rangle = -\varepsilon$. Thus, we get

$$(3.8) \qquad A_{H'} = A'_{H'} = -\varepsilon I.$$

Applying (3.7), (3.8) and Lemma 3, we find

$$(3.9) \qquad \tilde{\nabla}_X H = \tilde{\nabla}_X H' - \varepsilon \tilde{\nabla}_X x = 0.$$

This shows that H is a constant vector in E_s^{m+1}. Let $H = -\varepsilon c$. Then we have

$$(3.10) \qquad \langle x-c, x-c \rangle = -\varepsilon.$$

Since $\langle x, x \rangle = \varepsilon$, we obtain $\langle x, c \rangle = \varepsilon - 1/2 \langle c, c \rangle$. This shows that M lies in the hyperplane section N given by

$$N = \{ x \in E_s^{m+1} \mid \langle x, x \rangle = \varepsilon \quad \text{and} \quad \langle x, c \rangle = \varepsilon - 1/2 \langle c, c \rangle \}$$

for some constant c. Since $H = -\varepsilon c$ is normal to the hyperplane $\{ x \in E_s^{m+1} \mid \langle x, c \rangle = \varepsilon - 1/2 \langle c, c \rangle \}$, the mean curvature vector of M in N vanishes. Thus, M is a minimal submanifold of N.

If $\langle H, H \rangle \neq 0$, then M is pseudo-umbilical in E_s^{m+1} by Lemma 2. Thus, Lemma 2 implies that M is a minimal submanifold of a $S_s^m(c, r)$ or of a $H_{s-1}^m(c, r)$ for some $c \in E_s^{m+1}$ and $r > 0$. Thus, we have

$$(3.11) \qquad \langle x-c, x-c \rangle = \bar{\varepsilon} r^2,$$

where $\bar{\varepsilon} = 1$ or -1 according to M is a minimal submanifold of $S_s^m(c, r)$ or of $H_{s-1}^m(c, r)$. Since $\langle x, x \rangle = \varepsilon$, (3.11) gives $2\langle x, c \rangle = \varepsilon - \bar{\varepsilon} r^2 + \langle c, c \rangle$. Thus, M lies in the hyperplane section given by

$$N = \{ x \in E_s^{m+1} \mid \langle x, x \rangle = \varepsilon \quad \text{and} \quad \langle x-c, x-c \rangle = \bar{\varepsilon} r^2 \}.$$

Since M is minimal in $\{ x \in E_s^{m+1} \mid \langle x-c, x-c \rangle = \bar{\varepsilon} r^2 \}$, M is minimal in N, too.

(Q. E. D.)

Remark 2. If M is a minimal submanifold of a hyperplane section N of S_s^m (or of H_{s-1}^m), then either N is totally geodesic in S_s^m (or in H_{s-1}^m) or N is a pseudo-umbilical submanifold with parallel mean curvature vector.

Let M be a pseudo-Riemannian submanifold with orthonormal local basis $E_1, \cdots, E_n$. For any real function f on M, the Laplacian Δf of f is defined by

$$\Delta f = -\sum_{i=1}^n \varepsilon_i \{ E_i E_i f - \nabla_{E_i} E_i f \}.$$

We mention the following lemma for later use.

LEMMA 4. *Let M be an n-dimensional submanifold of E_s^{m+1}. Then we have*

(3.12) $$\Delta x = -nH.$$

Proof. Let a be any fixed vector in E_s^{m+1} and p a point in M. Let $E_1, \cdots, E_n$ be an orthonormal local basis about p such that $(\nabla_{E_i} E_j)(p) = 0$; $i, j = 1, \cdots, n$. Then we have

$$(\Delta \langle x, a \rangle)_p = -\sum_{i=1}^{n} \varepsilon_i (E_i)_p \langle E_i, a \rangle$$

$$= -\sum \varepsilon_i \langle \tilde{\nabla}_{E_i} E_i, a \rangle (p)$$

$$= -\sum \varepsilon_i \langle h(E_i, E_i), a \rangle_p$$

$$= -n \langle H, a \rangle_p.$$

Since both Δx and H are independent of the choice of the local basis, we have $\langle \Delta x, a \rangle = -\langle nH, a \rangle$. Because the inner product $\langle , \rangle$ is nondegenerate, this implies equation (3.12).   (Q. E. D.)

Combining Lemmas 1 and 4 we have the following.

PROPOSITION 2. *There exist no compact space-like minimal submanifolds in any pseudo-hyperbolic space H_s^m.*

Proof. If M is a compact space-like minimal submanifold of H_s^m, then Lemma 1 gives $H = x$. Thus, Lemma 4 implies $\Delta x = -nx$. This shows that $-n$ is an eigenvalue of M. Since M is a compact Riemannian manifold, eigenvalues of Δ on M are non-negative.   (Q. E. D.)

Remark 3. In contrast to Proposition 2, there exist compact minimal space-like submanifolds in pseudo-Riemannian spheres.

LEMMA 5. *If M is compact space-like submanifold of E_s^{m+1}, then we have*

(3.13) $$\int_M H \, dV = 0,$$

(3.14) $$\int_M \langle H, x \rangle dV + \int_M dV = 0.$$

Proof. Since M is compact, Lemma 4 and Hopf's Lemma implies (3.13).

By using (3.12) we have $\Delta \langle x, x \rangle = -2n(1 + \langle H, x \rangle)$. Thus, we also have (3.14).   (Q. E. D.)

In the following, a compact submanifold M of $S_s^m(c, r)$ (or of $H_{s-1}^m(c, r)$) in E_s^{m+1} is called *mass-symmetric* if the center of mass of M in E_s^{m+1} is just the center c of $S_s^m(c, r)$ (or of $H_{s-1}^m(c, r)$). Lemmas 1 and 4 imply the following.

LEMMA 6. *If M is a compact, space-like, minimal submanifold of the pseudo-Riemannian sphere S_s^m, then M is mass-symmetric in E_s^{m+1}.*

Proof. Under the hypothesis, we have $\Delta x=-nH=nx$. Thus $\int_M x\,dV=0$.

(Q. E. D.)

Remark 4. From the proof of Lemma 6, we see that if M is an n-dimensional, compact, space-like, minimal submanifold of $S_s{}^m$, then the first non-zero eigenvalue λ_1 of Δ on M satisfies $\lambda_1\leq n$.

§4. Finite-type submanifolds in $E_s{}^{m+1}$.

Let M be a compact, space-like submanifold of $E_s{}^{m+1}$. Then M with the induced metric is a Riemannian manifold. Thus, the Laplacian Δ of M is an elliptic differential operator and it has infinite sequence of eigenvalues:

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty.$$

Let $V_k=\{f\in C^\infty(M)\mid \Delta f=\lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k. Then each V_k is finite-dimensional. If we define an inner product on $C^\infty(M)$ by $(f,g)=\int_M fg\,dV$, then the decomposition $\sum_{k=0}^\infty V_k$ is orthogonal and dense in $C^\infty(M)$ (in L^2-sense). V_0 is 1-dimensional and it consists of constant functions.

For each $f\in C^\infty(M)$, let f_t be the projection of f onto V_t. Then we have the following decomposition:

(4.2) $$f=\sum_{t=0}^\infty f_t \quad \text{(in L^2-sense)}.$$

If f is a non-constant function on M, there is a positive integer $p\geq 1$ such that $f_p\neq 0$ and $f=f_0+\sum_{t\geq p}f_t$. If there are infinite f_t's which are nonzero, we put $q=\infty$. Otherwise, there is an integer $q\geq p$ such that $f_q\neq 0$ and $f=f_0+\sum_{t=p}^q f_t$. Thus, in both cases, we have

(4.3) $$f-f_0=\sum_{t=p}^q f_t,$$

where q is either ∞ or an integer $\geq p$.

For the compact, space-like submanifold M in $E_s{}^{m+1}$, we put

(4.4) $$x=(x_1,\cdots,x_{m+1}),$$

where x_A is the A-th rectangular coordinate function of M in $E_s{}^{m+1}$. For each x_A, we have

(4.5) $$x_A-(x_A)_0=\sum_{t=p_A}^{q_A}(x_A)_t, \qquad A=1,\cdots,m+1.$$

We put

(4.6) $$p=\inf\{p_A\}, \qquad q=\sup\{q_A\},$$

where A ranges among all A such that $x_A\neq(x_A)_0$. It is clear that p is an

integer ≥ 1 and q is either ∞ or an integer $\geq p$. The pair of these invariants $[p, q]$ is called the *order of M* in E_s^{m+1} (cf. [1]). The submanifold M in E_s^{m+1} is said to be *of finite type* if q is finite. Otherwise, M is of infinite type. The submanifold M in E_s^{m+1} is said to be *of k-type* if there exist exactly k nonzero x_t's in the following decomposition:

$$(4.7) \qquad x = x_0 + \sum_{t=p}^{q} x_t , \qquad \Delta x_t = \lambda_t x_t .$$

We need the following.

PROPOSITION 3. *Let M be a compact, space-like submanifold of E_s^{m+1}. Then M is of finite type if and only if there exists a non-trivial polynomial P such that $P(\Delta)H = 0$ (or equivalently, $P(\Delta)(x - x_0) = 0$).*

Proof. Consider the decomposition (4.7). If M is of finite type, q is finite. Thus, we have from Lemma 4,

$$(4.8) \qquad -n\Delta^i H = \sum_{t=p}^{q} \lambda_t^{i+1} x_t , \qquad i = 0, 1, 2, \cdots .$$

Let $c_1 = -\sum_{t=p}^{q} \lambda_t$, $c_2 = \sum_{t<s} \lambda_t \lambda_s$, $\cdots$, $c_{q-p+1} = (-1)^{q-p+1} \lambda_p \cdots \lambda_q$. Then we find

$$(4.9) \qquad \Delta^k H + c_1 \Delta^{k-1} H + \cdots + c_k H = 0 , \qquad k = q - p + 1 .$$

Conversely, if H satisfies (4.9) for some $k \geq 0$, then $k \geq 1$ by Lemma 4. Thus, by (4.7) and (4.9), we find

$$(4.10) \qquad \sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_k) x_t = 0 .$$

Let

$$x_t = ((x_t)_1, \cdots, (x_t)_{m+1}) .$$

Then (4.10) gives

$$(4.11) \qquad \sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_k)(x_t)_A = 0 , \qquad A = 1, \cdots, m+1 .$$

If $x_s \neq 0$, then

$$(x_s)_A \neq 0$$

for some A. Thus, we find

$$(4.12) \qquad 0 = \sum_{t=1}^{\infty} \lambda_t (\lambda_t^k + c_1 \lambda_t^{k-1} + \cdots + c_k)((x_t)_A, (x_s)_A) = 0 ,$$

where $(f, g) = \int_M fg \, dV$. Since

$$\lambda_t((x_t)_A, (x_s)_A) = (\Delta(x_t)_A, (x_s)_A)$$

$$= ((x_t)_A, \Delta(x_s)_B) = \lambda_s((x_t)_A, (x_s)_A) ,$$

we find

$$((x_t)_A,\ (x_s)_A)=0$$

if $t\neq s$. Therefore, (4.12) implies

(4.13) $\qquad \lambda_t{}^k+c_1\lambda_t{}^{k-1}+\ \cdots\ +c_k=0 \qquad$ whenever $\ x_t\neq 0$.

Since equation (4.13) has at most k real solutions, at most k of the x_t's in (4.7) are nonzero. Thus, M is of finite-type. Similar argument applies to $P(\Delta)(x-x_0)$ $=0$. $\hfill$ (Q. E. D)

From the proof of Proposition 3, we also have the following.

PROPOSITION 4. *Let M be a compact, space-like submanifold of $E_s{}^{m+1}$. If M is of k-type, then there exists a polynomial P of degree k such that $P(\Delta)H=0$ (or $P(\Delta)(x-x_0)=0$.) Conversely, if $P(\Delta)H=0$ (or $P(\Delta)(x-x_0)=0$) for a polynomial P of degree $d\geqq1$, then M is of k-type for some k; $d\geqq k\geqq1$.*

Now, we give the following Lemmas for later use.

LEMMA 7. *Let M be a compact, space-like submanifold of $E_s{}^{m+1}$. If M is not of 1-type and if H satisfies $\Delta^2H+b\Delta H+cH=0$ for some constant b and c, then $b=-(\lambda_p+\lambda_q)<0$ and $c=\lambda_p\lambda_q>0$ where $[p, q]$ is the order of M in $E_s{}^{m+1}$.*

Proof. Under the hypothesis, Proposition 4 shows that M is of 2-type. Thus, we have $x=x_0+x_p+x_q$. Therefore, we find

$$(\lambda_p{}^3+b\lambda_p{}^2+c\lambda_p)x_p+(\lambda_q{}^3+b\lambda_q{}^2+c\lambda_q)x_q=0\,.$$

Thus, we have $\lambda_p{}^2+b\lambda_p+c=0$, $\lambda_q{}^2+b\lambda_q+c=0$. Since λ_p and λ_q are positive real roots of $t^2+bt+c=0$, we have $b=-(\lambda_p+\lambda_q)<0$ and $c=-\lambda_p\lambda_q>0$. $\hfill$ (Q. E. D)

LEMMA 8. *Let M be a compact, space-like submanifold of $E_s{}^{m+1}$. Then x_0 is the center of mass in $E_s{}^{m+1}$.*

Proof. Since $\displaystyle\int_M x_t\,dV=1/\lambda_t\int_M \Delta x_t\,dV=0$, (4.9) gives to $x_0=\displaystyle\int_M x\,dV/\int_M dV$. $\hfill$ (Q. E. D)

LEMMA 9. *Let M be a space-like submanifold of $E_s{}^{m+1}$. Then we have*

(4.15) $\qquad \displaystyle\Delta H=\Delta^D H+\sum_{i=1}^{n}\{(\nabla_{E_i}A_H)E_i+A_{D_{E_i}H}E_i+h(E_i,\ A_HE_i)\}\,,$

where Δ^D is the Laplacian associated with D and $E_1,\ \cdots,\ E_n$ an orthonormal local basis of M.

Proof. Let a be any vector in $E_s{}^{m+1}$ and X, Y any vector fields tangent to M. We have

(4.16) $\qquad YX\langle H,\ a\rangle=\langle D_YD_XH,\ a\rangle-\langle\nabla_Y(A_HX),\ a\rangle$

$$-\langle A_{D_XH}Y,\ a\rangle-\langle h(Y,\ A_HX),\ a\rangle\,.$$

Since $\langle\, ,\,\rangle$ is non-degenerate, this gives (4.15) where

$$\Delta^D H = -\sum_{i=1}^{n} \{D_{E_i}D_{E_i}H - D_{\nabla_{E_i}E_i}H\}\,.$$

(Q. E. D)

§5. 1-type Submanifolds in E_s^{m+1}.

In this section, we give the following *characterization theorem* for 1-type submanifolds.

THEOREM 1. *Let M be a compact, space-like submanifold of E_s^{m+1}. Then M is of 1-type in E_s^{m+1} if and only if M is a minimal submanifold in a pseudo-Riemannian sphere $S_s^m(c, r)$ for some $c \in E_s^{m+1}$ and $r > 0$.*

Proof. If M is of 1-type in E_s^{m+1}, we have $\Delta x = \lambda_p(x - x_0)$. Thus, by Lemma 4, we find

(5.1) $$nH = \lambda_p(x_0 - x)\,.$$

This shows that $n\tilde{\nabla}_X H = -\lambda_p X$. Thus, we have

(5.2) $$A_H = \frac{\lambda_p}{n}I \quad \text{and} \quad DH = 0\,.$$

Moreover, by (5.1), we also have

$$X\langle x - x_0,\ x - x_0\rangle = 2\langle X,\ x - x_0\rangle = 0$$

for any X tangent to M. Thus, $\langle x - x_0,\ x - x_0\rangle$ is constant.

If $\langle x - x_0,\ x - x_0\rangle = 0$, then M lies in the light cone $C = \{x \in E_s^{m+1} \mid \langle x - x_0,\ x - x_0\rangle = 0\}$ with vertex at x_0. Since M lies in C and x_0 is the center of mass of M in E_s^{m+1}, M must lies in both parts of the light cone C. This is impossible since M is assumed to be a space-like submanifold. Thus, $\langle H, H\rangle \neq 0$. Consequently, M is pseudo-umbilical in E_s^{m+1} with parallel mean curvature vector. Thus, by Lemma 2, M is either a minimal submanifold of a pseudo-Riemannian sphere or a minimal submanifold of a pseudo-hyperbolic space. Since M is compact, Proposition 2 shows that the second case cannot occurs.

Conversely, if M is a minimal submanifold of a pseudo-Riemannian sphere, then Lemmas 1 and 4 imply that $\Delta H + \lambda H = 0$ for some constant λ. Thus, by Proposition 4, M is of 1-type in E_s^{m+1}.   (Q. E. D.)

In the following, a compact hypersurface N of the pseudo-hyperbolic space H_{s-1}^m is called a *small hypersphere* of H_{s-1}^m if N is the intersection of H_{s-1}^m with a pseudo-Riemannian sphere $S_s^m(c, r)$ in E_s^{m+1}. A small hypersphere of H_{s-1}^m is a totally umbilical submanifold in both H_{s-1}^m and E_s^{m+1}. Moreover, a small hypersphere of H_{s-1}^m is a totally umbilical hypersurface of a linear hyperplane L in E_s^{m+1} with c as its hyperplane normal.

By applying Theorem 1, we may obtain the following.

COROLLARY 1. *Let M be a compact, space-like submanifold of a pseudo-*

hyperbolic space $H_{s-1}{}^m$ in $E_s{}^{m+1}$. Then M is of 1-type in $E_s{}^{m+1}$ if and only if M is a minimal submanifold of a small hypersphere of $H_{s-1}{}^m$.

In particular, if $E_s{}^{m-1}$ is the Lorentz-Minkowski space-time, Corollary 1 reduces to the following.

COROLLARY 2. *A compact submanifold M of the hyperbolic space $H^m \subset E_1{}^{m+1}$ is of 1-type in the Lorentz-Minkowski world if and only if M is a minimal submanifold of a small hypersphere of H^m.*

§ 6. Hypersurfaces in Anti-de Sitter World.

It is known that there exist abundant examples of compact hypersurfaces with constant mean curvature and constant scalar curvature in a Riemannian sphere S^m. In this section, by applying the theory of finite-type submanifolds, we prove the following non-existence theorem in the anti-de Sitter world.

THEOREM 2. *There exist no compact space-like hypersurfaces with constant mean curvature and constant scalar curvature in the anti-de Sitter space-time $H_1{}^{n+1}$.*

Proof. We regard the anti-de Sitter space-time $H_1{}^{n+1}$ as a hypersurface of $E_2{}^{n+2}$ defined by

$$(6.1) \qquad H_1{}^{n+1} = \{x \in E_2{}^{n+2} \mid \langle x, x \rangle = -1\}$$

Let M be a compact space-like hypersurface of $H_1{}^{n+1}$. Then the position vector x of M in $E_2{}^{n+2}$ is a time-like unit normal vector. From Lemma 1, we have

$$(6.2) \qquad H = H' + x,$$

where H' is the mean curvature vector of M in $H_1{}^{n+1}$. Since M is space-like, H' is either zero or time-like. We put

$$(6.3) \qquad \langle H', H' \rangle = -\alpha^2.$$

Then $H' = \alpha \xi$ for a unit time-like vector field ξ normal to M and tangent to $H_1{}^{n+1}$. Moreover, we have $\operatorname{tr} A_\xi = -n\alpha$. By applying (2.9), we find

$$(6.4) \qquad \sum_i \langle h(E_i, A_H E_i), x \rangle = \sum_i \langle A_H E_i, A_x E_i \rangle = -\sum_i \langle A_H E_i, E_i \rangle$$
$$= -n\langle H, H \rangle = n + n\alpha^2.$$

Similarly, applying (2.9) and (5.2) and Lemma 2, we also have

$$(6.5) \qquad \sum_i \langle h(E_i, A_H E_i), \xi \rangle = \alpha \|A_\xi\|^2 - \operatorname{tr} A_\xi$$
$$= \alpha \|A_\xi\|^2 + n\alpha.$$

Consequently, (6.4) and (6.5) give

(6.6) $\sum_i h(E_i, A_H E_i) = -(\|A_\xi\|^2 + n)H' - (n + n\alpha^2)x$.

Since A_ξ is self-adjoint and M is space-like, we may diagonalize A_ξ. Let $E_1, \cdots, E_n$ be an orthonormal local basis of M such that

(6.7) $A_\xi E_i = \mu_i E_i$, $i = 1, \cdots, n$.

If we put

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \omega_j{}^k(E_j) E_k \,,$$

then (6.7) implies

(6.8) $(\nabla_{E_i} A_H)E_j = (E_i\alpha)\mu_j E_j + \alpha(E_i\mu_j)E_j + \sum_k \alpha(\mu_j - \mu_k)\omega_j{}^k(E_i)E_k$.

Therefore, by the Codazzi equation, we obtain

(6.9) $\alpha\{(E_i\mu_j)E_j - (E_j\mu_i)E_i + \sum_k (\mu_j - \mu_k)\omega_j{}^k(E_i)E_k$

$$- \sum_k (\mu_i - \mu_k)\omega_i{}^k(E_j)E_k\} = 0 \,,$$

where we have used the fact $D\xi = 0$. If $j \neq i$, (6.9) gives

(6.10) $\alpha(E_i\mu_j) = \alpha(\mu_i - \mu_j)\omega_i{}^j(E_j)$, $j \neq i$.

Since $\omega_i{}^j = -\omega_j{}^i$, (6.8) and (6.10) imply

(6.11) $\sum_i (\nabla_{E_i} A_H)E_i = \sum_i [(E_i\alpha)\mu_i E_i + \alpha(E_i\mu_i)E_i] + \alpha \sum_{j \neq i} (E_j\mu_i)E_j$.

Since $\sum_i \mu_i = \operatorname{tr} A_\xi = -n\alpha$, (5.11) gives

(6.12) $\sum_i (\nabla_{E_i} A_H)E_i = \sum_i \left[A_{D'_{E_i}H'}E_i - \frac{n}{2}(E_i\alpha^2)E_i \right]$.

Consequently, by applying Lemma 2, Lemma 8, (6.6), (6.12), we find

(6.13) $\Delta H = \Delta^{D'} H' + 2\operatorname{tr} A_{D'H'} + \frac{n}{2}\operatorname{grad}\langle H', H'\rangle$

$$- (\|A_\xi\|^2 + n)H' - (n + n\alpha^2)x \,.$$

In particular, if M has constant mean curvature α in $H_1{}^{n+1}$, then $D'H' = 0$. Thus (6.13) reduces to

(6.14) $\Delta H = -(\|A_\xi\|^2 + n)H' - (n + n\alpha^2)x$.

On the other hand, since the second fundamental form h of M in $E_2{}^{n+2}$ is given by

(6.15) $h(X, Y) = -\langle A_\xi X, Y\rangle\xi + \langle X, Y\rangle x$,

equation (2.7) of Gauss gives

(6.16) $\tau = \|A_\xi\|^2 - n^2\alpha^2 - n(n-1)$,

where τ is the scalar curvature of M. Combining (6.14) with Lemma 1, we find

(6.17) $\Delta H + (\|A_\xi\|^2 + n)H + (n\alpha^2 - \|A_\xi\|^2)x = 0$.

If the scalar curvature τ of M is also constant, then (6.16) shows that

(6.18) $b = \|A_\xi\|^2 + n$ and $c = n\alpha^2 - \|A_\xi\|^2$

are constant. Therefore, Lemmas 4 and 7 imply that M is of 1-type. Therefore, $x = x_0 + x_p$ for some integer $p \geq 1$. Combining this with Lemma 4, we obtain

(6.19) $nH = \lambda_p(x_0 - x)$.

On the other hand, (6.17) and Lemma 5 give

(6.20) $(n\alpha^2 - \|A_\xi\|^2)\int_M x \; dV = 0$.

If $n\alpha^2 = \|A_\xi\|^2$, then $\Delta H = -(n + \|A_\xi\|^2)H$ which is impossible. Therefore, we find $x_0 = 0$. Thus, (6.19) gives $nH = -\lambda_p x$. Since x and ξ are orthonormal, Lemma 1 implies that this case is also impossible.   (Q. E. D)

A space-like hypersurface in H_1^{n+1} (or in H^{n+1}) is said to be isoparametric if the Weingarten map A_ξ has constant eigenvalues. Since a space-like isoparametric hypersurface in H_1^{n+1} (or in H^{n+1}) has constant mean curvature and constant scalar curvature, Theorem 2 implies immediately the following.

COROLLARY 3. *There exist no compact space-like isoparametric hypersurfaces in the anti-de Sitter space-time.*

§7. Hypersurfaces in Hyperbolic Space.

In this section, we apply our previous results to give the following classification theorem in hyperbolic space.

THEOREM 3. *The only compact hypersurfaces with constant mean curvature and constant scalar curvature in the hyperbolic space H^{n+1} are small hyperspheres of H^{n+1}.*

Proof. We regard the hyperbolic space H^{n+1} as a space-like hypersurface of the Lorentz-Minkowski world E_1^{n+2} defined by

(7.1) $H^{n+1} = \{x \in E_1^{n+2} \mid \langle x, x \rangle = -1$ and $t > 0\}$.

Let M be a compact hypersurface of H^{n+1} with mean curvature vector H'. Let ξ be a unit vector field in the direction of H'. We put $H' = \alpha\xi$. Then, by an argument similar to that given in section 6, we may obtain

(7.2) $$\Delta H = \Delta^{D'} H' + 2 \operatorname{tr} A_{D' H'} - \frac{n}{2} \operatorname{grad} \langle H', H' \rangle$$

$$+ (\|A_\xi\|^2 - n)H' + (n\alpha^2 - n)x .$$

If M has constant mean curvature, (7.2) reduces to

(7.3) $$\Delta H = (\|A_\xi\|^2 - n)H' + (n\alpha^2 - n)x .$$

On the other hand, we have

(7.4) $$H = H' + x ,$$

(7.5) $$\tau = n^2\alpha^2 - \|A_\xi\|^2 - n(n-1) .$$

Thus, we obtain

(7.6) $$\Delta H + bH + cx = 0 ,$$

where $b = n - \|A_\xi\|^2$ and $c = \|A_\xi\|^2 - n\alpha^2$. If M has constant scalar curvature τ, too, then (7.5) shows that b and c are constants. Thus, by applying Lemma 5, we obtain

(7.7) $$c x_0 \int_M dV = c \int_M x \, dV = 0 .$$

Since M lies in H^{n+1}, $t > 0$. Therefore the center of mass cannot be the origin of E_1^{n+2}. Thus, $c = \|A_\xi\|^2 - n\alpha^2 = 0$. This implies that M is totally umbilical in H^{n+1}. Therefore, M is a small hypersphere of H^{n+1} (cf. [1, p. 129].)   (Q.E.D.)

Remark 4.   It is well-known that a small hypersphere of H^{n+1} has constant mean curvature and constant scalar curvature.

COROLLARY 4.   *The only compact isoparametric hypersurfaces of the hyperbolic space H^{n+1} are small hyperspheres.*

§8. Hypersurfaces in de Sitter World.

In this section, we study 2-type hypersurfaces in the de Sitter world.

THEOREM 4.   *Let M be a compact space-like hypersurface in the de Sitter space-time $S_1^{n+1} \subset E_1^{n+2}$. Then M has nonzero constant mean curvature and constant scalar curvature in S_1^{n+1} if and only if M is mass-symmetric and of 2-type in the Lorentz-Minkowski world E_1^{n+2}.*

Proof.   We recall that the de Sitter space-time S_1^{n+1} is a hypersurface of E_1^{n+2} defined by

(8.1) $$S_1^{n+1} = \{x \in E_1^{n+2} \mid \langle x, x \rangle = 1\} .$$

Let H' be the mean curvature vector of the compact, space-like hypersurface M in S_1^{n+1}. Then either $H'=0$ or H' is time-like. Let ξ be the time-like unit vector in S_1^{n+1} normal to M. We put $H'=\alpha\xi$. Then we have

$$(8.2) \qquad\qquad H=H'-x, \qquad \operatorname{tr} A_\xi=-n\alpha .$$

By applying an argument similar to that given in section 6, we may obtain

$$(8.3) \qquad \Delta H=\Delta^{D'} H'+2\operatorname{tr} A_{D'H'}+\frac{n}{2}\operatorname{grad}\langle H', H'\rangle$$
$$+(n-\|A_\xi\|^2)H'-(n-n\alpha^2)x .$$

If M is mass-symmetric and of 2-type in E_1^{n+2}, then there exist two constants b and c such that (Proposition 3)

$$(8.4) \qquad\qquad \Delta H+bH+cx=0 .$$

Combining (8.2), (8.3) and (8.4) we find

$$(8.5) \qquad \Delta^{D'} H'+2\operatorname{tr} A_{D'H'}+\frac{n}{2}\operatorname{grad}\langle H', H'\rangle$$
$$=(\|A_\xi\|^2-n-b)H'+(b-c+n-n\alpha^2)x .$$

Since x is normal to S_1^{n+1} and other terms in (8.5) are tangent to S_1^{n+1}, we obtain $n\alpha^2=b-c+n$. Thus, M has constant mean curvature α in the de Sitter world. Therefore, (7.5) gives

$$(8.6) \qquad\qquad \|A_\xi\|^2=n+b$$

which is constant. From equation (2.7) of Gauss, we see that the scalar curvature τ of M satisfies

$$(8.7) \qquad\qquad \tau=\|A_\xi\|^2-n^2\alpha^2+n(n-1) .$$

Therefore, M has constant scalar curvature.

Conversely, if M has constant mean curvature in S_1^{n+1} and constant scalar curvature, then (8.3) implies

$$(8.8) \qquad\qquad \Delta H+bH+cx=0 ,$$

where $b=\|A_\xi\|^2-n$ and $c=\|A_\xi\|^2-n\alpha^2$ are constant. Therefore, by applying Lemma 5 and (8.8), we see that the center of mass of M in E_1^{n+2} is the origin. Thus, $M \subset S_1^{n+1}$ is mass-symmetric in E_1^{n+2}. Moreover, (8.8) and Proposition 3 show that M is either of 1-type or of 2-type. If M is of 1-type, Lemma 4 gives $nH=-\lambda_p x$. Thus, Lemma 1 implies that M is minimal in the de Sitter world S_1^{n+1}. This is a contradiction. (Q.E.D.)

COROLLARY 5. *Let M be a compact, space-like hypersurface in the de Sitter space-time S_1^{n+1}. If M has nonzero constant mean curvature and constant scalar*

curvature, then

(8.9) $$\alpha^2 = \left(1 - \frac{\lambda_p}{n}\right)\left(1 - \frac{\lambda_q}{n}\right),$$

(8.10) $$\tau = (n-1)(\lambda_p + \lambda_q) - \lambda_p \lambda_q > 0,$$

(8.11) $$\|A_\xi\|^2 = n - (\lambda_p + \lambda_q).$$

This corollary follows from Lemma 7 and the proof of Theorem 4. From Lemma 6 and Theorem 4 we also have the following.

COROLLARY 6. *Let M be a compact, space-like, isoparametric hypersurface of the de Sitter space-time $S_1^{n+1} \subset E_1^{n+2}$. Then M is mass-symmetric. Moreover, if M is not minimal in S_1^{n+1}, then M is of 2-type in E_1^{n+2}.*

Remark 5. We can prove that if M is a compact, space-like, 2-type hypersurface of S_1^{n+1}, then M is always mass-symmetric in S_1^{n+1}.

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