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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A CERTAIN SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

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§1. Introduction.

Consider the following ordinary differential operator of second order:

$$P=\partial_t^2+k^2(t^2+y^2)+kc. \tag{1.1}$$

Here k belongs to the set $\boldsymbol{R}_+$ of the positive numbers, c to the complex number field $\boldsymbol{C}$, and (t, y) lies in the Euclid plane $\boldsymbol{R}^2$.

The purpose of the present article is to give a fundamental pair of the solutions to the equation:

$$Pu=0 \tag{1.2}$$

with detailed asymptotic properties as $k\to+\infty$. The novelty we claim here is its derivation as we will roughly sketch immediately after the statement of our Main Theorem (Theorem 1.1) below. Our asymptotic expansions are in fact different from usually given ones (see Nishimoto [5]). We expect that our results will be extended to partial differential operators such as

$$\partial_t^2-\partial_x^2+k^2(t^2+x^2+y^2)+kc.$$

Details on the latter case will be discussed elsewhere.

Now we explain what will be required in our formulation of asymptotics. Let

$$T_\varepsilon(t, y)=\varepsilon t+\sqrt{t^2+y^2}, \qquad \varepsilon\in\{+, -\}, \tag{1.3}$$

and denote by D_ε the set of (t, y, k) such that $k>0$, $T_\varepsilon(t, y)>0$, y running on the real line $\boldsymbol{R}$. Thus, D_ε is the portion of the half space $\boldsymbol{R}_+^3=\{(t, y, k); k>0\}$ obtained by deleting the quarter plane $\{(t, y, k); \varepsilon t\leqq 0, y=0, k>0\}$. D_ε and the quarter space $\boldsymbol{R}_+\times\boldsymbol{R}\times\boldsymbol{R}_+$ are diffeomorphic by the bijection:

$$\Phi_\varepsilon(t, y, k)=(T_\varepsilon(t, y), y, k), \qquad (t, y, k)\in D_\varepsilon.$$

We will consider everything in the Fréchet space $\mathcal{E}(D_\varepsilon)$ of the infinitely differen-

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tiable functions on D_ε. Several particular classes of subspaces of $\mathcal{E}(D_\varepsilon)$ will play essential roles in our asymptotic expansions. Denote by $\boldsymbol{R}_+$ the multiplicative group of the positive numbers. The $\boldsymbol{R}_+$-action in D_ε:

(1.4) $$g_\rho: (t, y, k) \longrightarrow (\rho^{-1/2}t, \rho^{-1/2}y, \rho k), \qquad \rho>0,$$

and

(1.5) $$h_\rho: (t, y, k) \longrightarrow (t, y, \rho k), \qquad \rho>0,$$

respectively induce the differentiable $\boldsymbol{R}_+$-action in $\mathcal{E}(D_\varepsilon)$:

(1.4′) $$(g_\rho u)(t, y, k)=u(g_\rho(t, y, k)),$$

and

(1.5′) $$(h_\rho u)(t, y, k)=u(h_\rho(t, y, k)),$$

$u\in\mathcal{E}(D_\varepsilon)$ (cf. Yoshikawa [9]).

Let $\mu\in\boldsymbol{C}$. We denote by $\Gamma^\mu(D_\varepsilon; h_\rho)$ the set of the h_ρ-homogeneous elements of degree μ:

$$\Gamma^\mu(D_\varepsilon; h_\rho)=\{u\in\mathcal{E}(D_\varepsilon);\ h_\rho u=\rho^\mu u \text{ for all } \rho>0\}.$$

$\Gamma^\mu(D_\varepsilon; g_\rho)$ is defined analogously. On the other hand, for any $s\in\boldsymbol{R}$, we denote by $\mathcal{B}^s(D_\varepsilon; h_\rho)$ the totality of those $u\in\mathcal{E}(D_\varepsilon)$ such that for any $\rho_0>0$ the set $\{\rho^{-s}h_\rho u;\ \rho\geqq\rho_0\}$ is bounded in $\mathcal{E}(D_\varepsilon)$. We will also use the spaces

$$\mathcal{B}^{s+0}(D_\varepsilon; h_\rho)=\bigcap_{\tau>0}\mathcal{B}^{s+\tau}(D_\varepsilon; h_\rho)$$

and

$$\mathcal{B}^{-\infty}(D_\varepsilon; h_\rho)=\bigcap_{-\infty<s<\infty}\mathcal{B}^s(D_\varepsilon; h_\rho).$$

Note $T_\varepsilon(t, y)\in\Gamma^{-1/2}(D_\varepsilon; g_\rho)$. The operator P is g_ρ-homogeneous of degree 1 in the sense

(1.6) $$g_\rho P g_\rho^{-1}=\rho P, \qquad \rho>0.$$

However, we take as the symbol $\sigma_P(t, y, \tau, \eta)$ of the operator P the function

$$\sigma_P(t, y, \tau, \eta)=-\tau^2+t^2+y^2.$$

Let

(1.7) $$\Lambda_\varepsilon^\delta=\{(t, y, k\tau, k\eta, k);\ \tau=\varepsilon\delta\sqrt{t^2+y^2},\ \eta=\delta y\log\{\sqrt{k}\,T_\varepsilon(t, y)\},\ (t, y, k)\in D_\varepsilon\},$$

$\varepsilon, \delta\in\{+, -\}$. Here we adopt the convention: $\varepsilon\delta=+$ if $\varepsilon=\delta$, and $\varepsilon\delta=-$ if $\varepsilon\neq\delta$ for $\varepsilon, \delta\in\{+, -\}$. The symbol σ_P of P vanishes on $\Lambda_\varepsilon^\delta$. For each ε, δ, $\Lambda_\varepsilon^\delta$ is invariant under the $\boldsymbol{R}_+$-actions g_ρ and h_ρ. $\Lambda_\varepsilon^\delta$ is interpreted as a Hamilton flow associated to the Hamiltonian $\tau-\varepsilon\delta\sqrt{t^2+y^2}$. Although

$$d\tau\wedge dt+d\eta\wedge dy=-\delta d\Big(\frac{1}{4}y^2\Big)\wedge\frac{dk}{k} \tag{1.8}$$

along $\Lambda_\varepsilon^\delta$, we have a generating function:

$$S_\varepsilon^\delta(t,\ y,\ k)=\frac{\varepsilon\delta}{2}t\sqrt{t^2+y^2}+\frac{\delta}{2}y^2\log\{\sqrt{k}\,T_\varepsilon(t,\ y)\}-\frac{\delta}{4}y^2 \tag{1.9}$$

in the sense

$$\Lambda_\varepsilon^\delta=\{(t,\ y,\ k\partial_t S_\varepsilon^\delta(t,\ y,\ k),\ k\partial_y S_\varepsilon^\delta(t,\ y,\ k),\ k);\ (t,\ y,\ k)\in D_\varepsilon\}. \tag{1.10}$$

Note that $kS_\varepsilon^\delta(t, y, k)$ is a primitive with respect to t of $\varepsilon\delta k\sqrt{t^2+y^2}$ and is g_ρ-homogeneous of degree 0. The requirement on g_ρ-homogeneity determines S_ε^δ uniquely from (1.8) and (1.10). By the way, $S_\varepsilon^\delta(t, 0, k)=(\delta/2)t^2$ since $\varepsilon t>0$ when $y=0$.

Now we are ready for stating our main result:

THEOREM 1.1. *Let* $\varepsilon\in\{+, -\}$ *be fixed. There are a fundamental pair* $v_\varepsilon^+(t, y, k)$ *and* $v_\varepsilon^-(t, y, k)$ *of the solutions to the equation* (1.2) *in* $\boldsymbol{R}^2\times\boldsymbol{R}_+$ *which enjoy the following properties:*

(i) $v_\varepsilon^+(t,y,k)$ *and* $v_\varepsilon^-(t,y,k)$ *are infinitely differentiable in* $(t,y,k)\in\boldsymbol{R}^2\times\boldsymbol{R}_+$.

(ii) $v_\varepsilon^\delta(t,y,k)\exp\{-\sqrt{-1}\,S_\varepsilon^\delta(t,y,k)k\}\in\mathcal{E}(D_\varepsilon)$, $\delta\in\{+,-\}$.

(iii) *Let*

$$\mu(\delta)=\frac{1}{4}-\delta\frac{\sqrt{-1}}{4}c, \qquad \delta\in\{+,\ -\}. \tag{1.11}$$

There are families of functions

$$u_{\varepsilon,j}^\delta(t,\ y,\ k)\in\Gamma^{\mu(\delta)}(D_\varepsilon;\ g_\rho)\cap\Gamma^{-j}(D_\varepsilon;\ h_\rho), \tag{1.12}$$

$j=0, 1, 2, \cdots$, *such that*

$$u_{\varepsilon,0}^\delta(t,\ y,\ k)=(t^2+y^2)^{-1/4}T_\varepsilon(t,\ y)^{\delta\sqrt{-1}c/2} \tag{1.13}$$

and that

$$\begin{aligned}v_\varepsilon^\delta(t,\ y,\ k)\exp\{-\sqrt{-1}\,S_\varepsilon^\delta(t,\ y,\ k)k\}-\sum_{j<N}u_{\varepsilon,j}^\delta(t,\ y,\ k)\\ \in\Gamma^{\mu(\delta)}(D_\varepsilon;\ g_\rho)\cap\mathcal{B}^{-N}(D_\varepsilon;\ g_\rho)\end{aligned} \tag{1.14}$$

for any positive integer N *(See Lemma 2.2 to decipher function spaces).*

(iv) *The wronskian is given by*

$$v_\varepsilon^+(t,\ y,\ k)\partial_t v_\varepsilon^-(t,\ y,\ k)-v_\varepsilon^-(t,\ y,\ k)\partial_t v^+(t,\ y,\ k)=-2\sqrt{-1}\,k. \tag{1.15}$$

Remark. (1.12) implies $u_{\varepsilon,j}^\delta(t, y, k)=u_{\varepsilon,j}^\delta(t, y, 1)k^{-j}$, $k>0$. Note also

$$u_{\varepsilon,0}^\delta(t,\ y,\ k)=|\partial_t T_\varepsilon(t,\ y)|^{1/2}T_\varepsilon(t,\ y)^{-2\mu(\delta)}. \tag{1.13'}$$

As far as the equation (1.2) is concerned, there might be several proofs to Theorem 1.1. One such proof might be based on Laplace's method by giving an integral representation of a solution of (1.2). In fact, there have been closely related studies by such an approach (Alinhac [1], Sibuya [6], cf. Yoshikawa [10]). Another proof would be done by relating solutions of (1.2) to particular confluent hypergeometric functions (e.g., Erdelyi et al. [3]). Namely,

$$U^{\pm}(t, y, k)=\Psi\Big(\frac{1}{4}\mp\frac{\sqrt{-1}}{4}c\mp\frac{\sqrt{-1}}{4}ky^2, \frac{1}{2}; \pm\sqrt{-1}\,kt^2\Big)\cdot\exp(\pm\sqrt{-1}\,kt^2/2) \tag{1.16}$$

make up fundamental pair of the solutions of (1.2). Here, for complex a, $\Psi(a, 1/2; z)$ is a confluent hypergeometric function, and

$$\Psi\Big(a, \frac{1}{2}; \mp\sqrt{-1}\,s^2\Big)=\frac{\Gamma(1/2)}{\Gamma(a+1/2)}\,{}_1F_1\Big(a, \frac{1}{2}; \mp\sqrt{-1}\,s^2\Big)+\frac{\Gamma(-1/2)}{\Gamma(a)}e^{\mp\sqrt{-1}\,\pi/4}\,{}_1F_1\Big(a+\frac{1}{2}, \frac{3}{2}; \mp\sqrt{-1}\,s^2\Big)s$$

are entire functions of s and a (cf. Yoshikawa [8]). However, through these methods, detailed asymptotic properties of the solutions of (1.2) might not be easily obtained.

On the other hand, our present approach is based on classical Frobenius' method (compare, however, with Taylor [7]). It makes visible how each term in the asymptotic expansions (1.14) is determined from the equation (1.2). Thus, determination of $u^{\delta}_{\varepsilon,j}(t, y, k)$ is rather obvious. The only non-trivial part in our approach is to show that there are actually solutions with given asymptotic expansions. We do this essentially following Erdelyi [2] by solving Volterra type integral equations. Here lie some difficulties, though. For we will be working on the infinite interval $(0, +\infty)$ and in the Fréchet space $\Gamma^0(D_\varepsilon; g_\rho)\cap\mathscr{B}^{-\infty}(D_\varepsilon; h_\rho)$. This means that the customary successive approximation is not recommended, but, as we will expound below, a variant of the contraction principle will do the work.

Some further remarks are due now. Firstly, note the operator (1.1) is invariant under the reflections $t\to -t$ or $y\to -y$. Also $T_\varepsilon(t, y)=T_{\varepsilon'}(-t, y')$, $\varepsilon\neq\varepsilon'$, $y'=\pm y$. Therefore, Theorem 1.1 provides two sets of fundamental pairs of the solutions of (1.2):

$$v^{\delta}_{+}(t, y, k), \qquad \delta\in\{+, -\}$$

and

$$v^{\delta}_{-}(t, y, k), \qquad \delta\in\{+, -\}.$$

However, we do not know the explicit form of the transformation matrix $S=(S^{\delta\delta'}(y, k))$:

$$v^{\delta}_{+}(t, y, k)=\sum_{\delta'\in\{+,-\}} S^{\delta\delta'}(y, k)v^{\delta'}_{-}(t, y, k),$$

$\delta\in\{+,-\}$.

Secondly, our discussions are valid even if we replace D_ε by the set $\tilde{D}_\varepsilon$ of $(t, y, k)\in \boldsymbol{C}\times\boldsymbol{R}\times\boldsymbol{R}_+$ such that $T_\varepsilon(t, y)$ does not touch the non-positive real axis. We then have to replace $\mathcal{E}(D_\varepsilon)$ by the Fréchet space $\mathcal{O}(\tilde{D}_\varepsilon)$ of the smooth functions on $\tilde{D}_\varepsilon$, holomorphic in t, infinitely differentiable in (y, k). Details are left to the reader, including complex extensions of y and k.

Finally we recall that a result closely related to Theorem 1.1 has played an essential role in constructing parametrices for a certain class of non-strictly hyperbolic partial differential operators (See [1], [10]).

§ 2. Formal construction.

In the present section, we show the formal part of Theorem 1.1. We take $\varepsilon=+$ and omit the reference to ε in what follows. So we simply write D, $T(t, y)$, $v^{\pm}(t, y, k)$, $S^{\delta}(t, y, k)$, or $u_j^{\delta}(t, y, k)$ instead of D_+, $T_+(t, y)$, $v_+^{\pm}(t, y, k)$, $S_+^{\delta}(t, y, k)$ or $u_{+,j}^{\delta}(t, y, k)$.

Let $S^{\delta}(t, y, k)$, $\delta\in\{+,-\}$, be defined by (1.9), $\varepsilon=+$. Then $S^{\delta}(t, y, k)k\in \Gamma^0(D; g_\rho)\cap\mathcal{B}^{1+0}(D; h_\rho)$. Let

$$P^{\delta}u=\exp\{-\sqrt{-1}\,S^{\delta}(t, y, k)k\}P[\exp\{\sqrt{-1}\,S^{\delta}(t, y, k)k\}u], \tag{2.1}$$

$u\in\mathcal{E}(D)$, $\delta\in\{+,-\}$. The operators P^{δ} are g_ρ-homogeneous of degree 1, but

$$h_\rho P^{\delta}h_\rho^{-1}=\rho P_1^{\delta}+P_0^{\delta}, \tag{2.2}$$

where

$$P_1^{\delta}=\delta 2k\sqrt{-1}\sqrt{t^2+y^2}\,\partial_t+\frac{\delta k\sqrt{-1}\,t}{\sqrt{t^2+y^2}}+kc \tag{2.3}$$

and

$$P_0^{\delta}=\partial_t^2 \tag{2.4}$$

The relation (2.2) is a decomposition of the operators P^{δ} into h_ρ-homogeneous parts.

PROPOSITION 2.1. *There are a pair of formal series*

$$u_*^{\delta}(t, y, k)=\sum_{j=0}^{\infty}u_j^{\delta}(t, y, k), \qquad \delta\in\{+,-\}, \tag{2.5}$$

such that $v_*^{\delta}(t, y, k)=u_*^{\delta}(t, y, k)\exp\{\sqrt{-1}\,S^{\delta}(t, y, k)k\}$, $\delta\in\{+,-\}$, *are formal solutions of the equation* $Pv=0$. *Furthermore, for* $j\geqq 0$,

$$u_j^{\delta}(t, y, k)\in\Gamma^{\mu(\delta)}(D; g_\rho)\cap\Gamma^{-j}(D; h_\rho), \tag{2.6}$$

where $\mu(\delta)$ *are given by* (1.11), $\delta\in\{+,-\}$. $u_j^{\delta}(t, y, k)$ *are determined from the equations:*

$$P_1^\delta u_0^\delta = 0 \tag{2.7}$$

and

$$P_1^\delta u_j^\delta + P_0^\delta u_{j-1}^\delta = 0 \tag{2.8}$$

for $j \geqq 1$. *In particular, we can take*

$$u_0^\delta(t, y, k) = (t^2+y^2)^{-1/4} T(t, y)^{\delta\sqrt{-1}c/2}. \tag{2.9}$$

The proof of this proposition is based on two facts. One is the following observation:

$$\sqrt{t^2+y^2}\,\partial_t = r\partial_r, \qquad r = T(t, y), \tag{2.10}$$

and another is the following

LEMMA 2.2. *Let* $u(t, y, k) \in \Gamma^\nu(D; g_\rho) \cap \mathcal{B}^s(D; h_\rho)$, $s \in \boldsymbol{R}$, $\nu \in \boldsymbol{C}$. *Then for any* $k_0 > 0$, $y_0 > 0$, $r_0 > 0$ *and any integer* $i, j, m \geqq 0$,

$$|\partial_t^i \partial_y^j \partial_k^m u(t, y, k)| \leqq C T(t, y)^{-2\operatorname{Re}\nu - i - j + 2s} k^{s-m} \tag{2.11}$$

for $T(t, y) \geqq r_0$, $|y| \leqq y_0 T(t, y)$, $k \geqq k_0$, *with a positive constant* C *depending on* k_0, y_0, r_0, i, j, m.

Let us show (2.11) for $i=j=m=0$. Since $u \in \Gamma^\nu(D; g_\rho)$, $u(t, y, k) = r^{-2\nu} u(r^{-1}t, r^{-1}y, r^2k)$, $r = T(t, y)$. Note $T(r^{-1}t, r^{-1}y) = 1$ then. Since $u \in \mathcal{B}^s(D; h_\rho)$, $|u(r^{-1}t, r^{-1}y, r^2k)| \leqq C(r^2k)^s$ if $r^2k \leqq r_0^2 k_0$ and $|r^{-1}y| \leqq y_0$, $T(r^{-1}t, r^{-1}y) = 1$, with an appropriate constant $C > 0$.

Returning to the proof of Proposition 2.1, we observe

$$P_1^\delta = 2\delta k (t^2+y^2)^{-1/4} T(t, y)^{(\delta\sqrt{-1}/2)c} \sqrt{t^2+y^2}\,\partial_t \{(t^2+y^2)^{1/4} T(t, y)^{-(\delta\sqrt{-1}/2)c}\cdot\}. \tag{2.12}$$

(2.9) is then immediate from (2.7). Putting

$$U_j^\delta(t, y, k) = u_j^\delta(t, y, k)/u_0^\delta(t, y, k), \qquad j \geqq 0, \tag{2.13}$$

we obtain from (2.8) and (2.10),

$$r\partial_r u_j^\delta + \delta(2\sqrt{-1}\,k)^{-1}(u_0^\delta)^{-1}\partial_t^2(u_0^\delta U_{j-1}^\delta) = 0, \tag{2.14}$$

$r = T(t, y)$. Now the induction on j works. In fact, $U_0^\delta = 1$. If $U_{j-1}^\delta \in \Gamma^0(D; g_\rho) \cap \Gamma^{-j+1}(D; h_\rho)$, then

$$(2\sqrt{-1}\,k)^{-1}(u_0^\delta)^{-1}\partial_t^2(u_0^\delta \cdot U_{j-1}^\delta) \in \Gamma^0(D; g_\rho) \cap \Gamma^{-j}(D; h_\rho).$$

Thus, by Lemma 2.2, we can integrate $(2\sqrt{-1}\,k)^{-1} r^{-1} u_0^\delta \partial_t^2(u_0^\delta U_{j-1}^\delta)$, $r = T(t, y)$, from $r = +\infty$ to $r = T(t, y)$ parallel to the r-axis in the (r, y)-plane. We then have $U_j^\delta \in \Gamma^0(D; g_\rho) \cap \Gamma^{-j}(D; h_\rho)$, and the induction is complete.

PROPOSITION 2.3. *There are a pair of functions*

$$u_a^\delta(t, y, k)\in\Gamma^{\mu(\delta)}(D; g_\rho)\cap\mathscr{B}^0(D; h_\rho) \tag{2.15}$$

such that for each $N=0, 1, 2, \cdots$,

$$u_a^\delta(t, y, k)-\sum_{j=0}^{N}u_j^\delta(t, y, k)\in\Gamma^{\mu(\delta)}(D; g_\rho)\cap\mathscr{B}^{-N-1}(D; h_\rho). \tag{2.16}$$

Here is a standard proof. Let $\phi(\xi)\in C^\infty(\boldsymbol{R})$ such that $\phi(\xi)=1$ for $\xi>1$ and $\phi(\xi)=0$ for $\xi<1/2$. Let

$$\phi_n(t, y, k)=\phi(\rho_n T(t, y)\sqrt{k}),$$

$\rho_n>0$ to be chosen later. We have

$$\phi_n\in\Gamma^0(D; g_\rho)\cap\mathscr{B}^0(D; h_\rho)$$

and

$$1-\phi_n\in\Gamma^0(D; g_\rho)\cap\mathscr{B}^{-\infty}(D; h_\rho).$$

Let $K_{m,p}=\{(t, y, k)\in D; m^{-1}\leqq T(t, y)\leqq m, |y|\leqq mT(t, y), 2^{-p}\leqq k\leqq 2^p\}$ $m=1, 2, \cdots$, $p=1, 2, \cdots$. Then $K_{m,p}$ are compact and $\cup_{m,p}K_{m,p}=D$. For integers $i, j, l\geqq 0$, let

$$\Phi_{i,j,l}=\sup\left|\xi^{-2i}\left(\frac{1}{2\xi}\frac{\partial}{\partial\xi}\right)^j(\phi^{(l)}(\xi)\xi^l)\right|$$

and for integers $i, j, l, N\geqq 0$, $n, m, p\geqq 1$, $q=0, \pm 1, \pm 2, \cdots$,

$$\Psi_{n;i,j,l,m,p,q}=\sup_{K_{m,l}}|r^q\partial_r^i\partial_y^j\partial_k^l U_n^\delta(t, y, k)|$$

$(r=T(t, y))$, where $U_n^\delta(t, y, k)$ are those of (2.13). We now take ρ_n as follows. ρ_0 and ρ_1 may be arbitrary. For $n\geqq 2$, we take $0<\rho_n\leqq\min\{1, 2^{-n}/A^n\}$, where

$$A_n=\sup\sum_{i'\leqq i, j'\leqq j}\binom{i}{i'}\binom{j}{j'}\Phi_{j-j'+n-N, j', i'}\Psi_{n, i-i', l, j-j', m, p, 2j-i'+2n-2N},$$

supremum being taken over $i, j, l, N\geqq 0$, $m, p\geqq 1$ such that $i+j+l+p+m+N+1\leqq n$ (cf. Hörmander [4]). It then follows that the sums $\sum_{n=0}^\infty U_n^\delta(t, y, k)\phi_n(t, y, k)$ converges in $\Gamma^0(D; g_\rho)\cap\mathscr{B}^0(D; h_\rho)$. Then

$$u_a^\delta(t, y, k)=u_0^\delta(t, y, k)\sum_{n=0}^\infty U_n^\delta(t, y, k)\phi_n(t, y, k)$$

meet the requirements (2.15) and (2.16).

COROLLARY 2.4. *We have*

$$W_a^\delta(t, y, k)=P^\delta u_a^\delta(t, y, k)\in\Gamma^{1+\mu(\delta)}(D; g_\rho)\cap\mathscr{B}^{-\infty}(D; h_\rho), \tag{2.17}$$

$\delta\in\{+, -\}$, *and*

(2.18) $$R_a^\delta(t, y, k)=P(u_a^\delta(t, y, k)\exp\{\sqrt{-1}\,S^\delta(t, y, k)k\}) \in \Gamma^{1+\mu(\delta)}(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho).$$

Proof. (2.17) follows from (2.16), (2.12), (2.13), and (2.14). (2.18) is then im mediate from (2.1).

§ 3. Exact solutions asymptotic to formal ones.

Corollary 2.4 means that

(3.1) $$v_a^\delta(t, y, k)=u_a^\delta(t, y, k)\exp\{\sqrt{-1}\,S^\delta(t, y, k)k\}, \qquad \delta\in\{+, -\},$$

are approximate solutions of the equation (1.2). We show in this section that we can find complementary functions

(3.2) $$z^\delta(t, y, k)\in\Gamma^{\mu(\delta)}(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$$

so that

(3.3) $$v^\delta(t, y, k)=(u_a^\delta(t, y, k)+z^\delta(t, y, k))\exp\{\sqrt{-1}\,S^\delta(t, y, k)k\}$$

are exact solutions of the equation (1.2), which admit the asymptotic expansions (1.14).

Let

(3.4) $$V_0^\delta(t, y, k)=u_0^\delta(t, y, k)\exp\{\sqrt{-1}\,S^\delta(t, y, k)k\}.$$

Then $V_0^\delta(t, y, k)\in\Gamma^{\mu(\delta)}(D; g_\rho)$ and

(3.5) $$|V_0^\delta(t, y, k)|>0.$$

Let

(3.6) $$Z^\delta(t, y, k)=z^\delta(t, y, k)/u_0^\delta(t, y, k).$$

We want to determine $Z^\delta(t, y, k)$ from

(3.7) $$P(V_a^\delta(t, y, k)+V_0^\delta(t, y, k)Z^\delta(t, y, k))=0$$

with

(3.8) $$Z^\delta(t, y, k)\in\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho).$$

It will often be more convenient to discuss in the arguments $r=T(t, y), y, k$ rather than in the original ones t, y, k. A given function $W(t, y, k)$ will be written $\hat{W}(r, y, k)$ if it is expressed in r, t, k. In other words. $\hat{W}(T(t, y), y, k) = W(t, y, k)$.

PROPOSITION 3.1. *Let*

(3.9) $$\hat{K}^{\delta}(r, r', y, k)=\frac{r'^2+y^2}{2r'^2}\int_{r'}^{r}\hat{V}_0^{\delta}(p, y, k)^{-2}\frac{p^2+y^2}{2p^2}dp\cdot(V_0^{\delta}PV_0^{\delta})^{\wedge}(r', y, k).$$

Then $\hat{Z}^{\delta}(r, y, k)$ *satisfy the integral equations:*

(3.10) $$\hat{Z}^{\delta}(t, y, k)+\int_{+\infty}^{r}\hat{K}^{\delta}(r, r', y, k)\hat{Z}^{\delta}(r', y, k)dr'=\hat{Q}^{\delta}(r, y, k)$$

in $\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$. *Here*

(3.11) $$\hat{Q}^{\delta}(r, y, k)=-\int_{\infty}^{r}\frac{r'^2+y^2}{2r'^2}\int_{r'}^{r}\hat{V}_0^{\delta}(p, y, k)^{-2}\frac{p^2+y^2}{2p^2}dp\cdot\hat{V}_0^{\delta}(r', y, k)R_a^{\delta}(r', y, k)dr'$$

$$\in\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho).$$

Proof. First observe that (3.7) (3.1) and (2.18) imply

$$\partial_t[(V_0^{\delta})^2\partial_t Z^{\delta}]+(V_0^{\delta}PV_0^{\delta})Z^{\delta}+V_0^{\delta}R_a^{\delta}=0,$$

or

$$\frac{2r^2}{r^2+y^2}\partial_r\left[(\hat{V}_0^{\delta})^2\frac{2r^2}{r^2+y^2}\partial_r\hat{Z}^{\delta}\right]+(V_0^{\delta}PV_0^{\delta})^{\wedge}\cdot\hat{Z}^{\delta}+\hat{V}_0^{\delta}\hat{R}_a^{\delta}=0,$$

because of (1.3) (2.10). By Lemma 2.2 and (3.8) we have

$$(\hat{V}_0^{\delta})^2\frac{2r^2}{r^2+y^2}\partial_r\hat{Z}^{\delta}+\int_{+\infty}^{r}\frac{r'^2+y^2}{2r'^2}(V_0^{\delta}PV_0^{\delta})^{\wedge}\cdot Z^{\delta}dr'+\int_{+\infty}^{r}\frac{r'^2+y^2}{2r'^2}V_0^{\delta}R_a^{\delta}dr'=0,$$

integrating along a line prallel to the r-axis in the (r, y)-plane. Integrating once more, and changing the order of integrations, we obtain (3.10). To verify (3.11) as well as for a later purpose we state the following

LEMMA 3.2. *For* $0<r\leqq r'$, $y\in\boldsymbol{R}$, $k>0$, *we have*

(3.12) $$\hat{K}^{\delta}(\rho^{-1/2}r, \rho^{-1/2}r', \rho^{-1/2}y, \rho k)=\rho^{1/2}\hat{K}^{\delta}(r, r', y, k),$$

$\rho>0$, $\delta\in\{+, -\}$. *For any non-negative integers* i, i', j, l,

(3.13) $$|\partial_r^i\partial_{r'}^{i'}\partial_y^j\partial_k^l\hat{K}^{\delta}(r, r', y, k)|\leqq Ck^{i+i'+j-l}$$

if $r_0\leqq r\leqq r'\leqq r_1$, $|y|\leqq y_0$, $k\geqq k_0$. *Here* r_0, r_1, y_0, k_0 *are arbitrary positive numbers and* C *is a positive constant depending on* $i, i', j, l, r_0, r_1, y_0, k_0$.

Proof. Obvious.

Let

(3.14) $$(K^{\delta}f)(t, y, k)=\int_{\infty}^{r}\hat{K}^{\delta}(r, r', y, k)\hat{f}(r', y, k)dr'|_{r=T(t,y)}.$$

(3.13) then implies that $K^{\delta}f$ belongs to $\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$ if so does f. Using a similar estimate to (3.13), we see (3.11) because of (2.18). This completes the proof of Proposition 3.1.

Thus to show the existence of $Z^{\delta}(t, y, k)$ with (3.7), (3.8), it is enough to show

PROPOSITION 3.3. *The operator $I+K^{\delta}$ in $\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$ is a bijection for each $\delta\in\{+, -\}$. Here I is the identity operator and K^{δ} the integral operator* (3.14).

Before giving a proof of this proposition, we rewrite the space $\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$ in a more convenient form. We also rewrite the integral operator (3.14) accordingly.

We begin by introducing the space $\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$ which consists of the functions $p(r, y)\in C^\infty(\boldsymbol{R}_+\times\boldsymbol{R})$ such that for each $r_0>0$, $y_0>0$ and non-negative integers i, j, N there is a positive constant C for which

$$|\partial_r^i\partial_y^j p(r, y)|\leqq Cr^{-N} \tag{3.15}$$

holds when $r\geqq r_0$, $|y|\leqq y_0 r$. Let $p(r, y)\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$. We define the mapping θ by

$$(\theta p)(t, y, k)=p(\sqrt{k}\,T(t, y), \sqrt{k}\,y), \qquad p\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R}). \tag{3.16}$$

PROPOSITION 3.4. *The spaces $\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$ and $\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$ are isomorphic via the mapping θ.*

Proof. $q(t, y, k)\in\mathcal{B}^{-\infty}(D, h_\rho)$ means that for any $y_0>0$, $r_1>r_0>0$, $k_0>0$ and any non-negative integers i, j, l, N, a positive constant C is chosen so that

$$|\partial_t^i\partial_y^j\partial_k^l q(t, y, k)|\leqq Ck^{-N} \tag{3.17}$$

holds when

$$|y|\leqq y_0, \quad r_0\leqq T(t, y)\leqq r_1, \quad k\geqq k_0. \tag{3.18}$$

If, furthermore, $q\in\Gamma^0(D; g_\rho)$, then

$$\begin{aligned} q(t, y, k)&=q(\sqrt{k}\,t, \sqrt{k}\,y, 1) \\ &=q(t/T(t, y), y/T(t, y), T(t, y)^2k). \end{aligned} \tag{3.19}$$

Now observe that

$$\sqrt{k}\,T(t, y)\geqq r_0', \qquad \sqrt{k}\,|y|\leqq y_0'\sqrt{k}\,T(t, y) \tag{3.20}$$

holds under (3.18) if $r_0'=\sqrt{k_0}\,r_0$ and $y_0'=y_0/r_0$. Therefore, if $p(r, y)=\hat{q}(r, y, 1)$, then the first equality of (3.19) implies $\theta p=q$ and (3.17) is true if $p\in\mathcal{S}(\boldsymbol{R}^+\times\boldsymbol{R})$. On the other hand, the second equality of (3.19) and (3.17) imply (3.15) since $T(t/T(t, y), y/T(t, y))=1$, $|y/T(t, y)|\leqq y_0'$, $T(t, y)^2k\geqq r_0'^2$ from (3.20). In particular, $q(t, y, k)\in\Gamma^0(D; g_\rho)\cap\mathcal{B}^{-\infty}(D; h_\rho)$ if and only if $\hat{q}(r, y, 1)\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$.

Recall (3.12). Let

$$K_\sharp^{\delta}(r, r', y)=\hat{K}^{\delta}(r, r', y, 1), \qquad \delta\in\{+, -\}. \tag{3.21}$$

LEMMA 3.5. *For any $r_0>0$, $y_0>0$ and any integers $i, i', j\geqq 0$, there is a positive constant C such that*

$$|\partial_r^i\partial_{r'}^{i'}\partial_y^j K_{\#}^{\delta}(r, r', y)|\leqq Cr'^{-1+i+i'+j}\{1+(r/r')^{\delta \operatorname{Im} c+2}\} \tag{3.22}$$

holds when $r_0\leqq r\leqq r'$, $|y|\leqq y_0 r$.

Proof. Since $K^{\delta}(r, r', y)$ contains exponential factors, the effect of one differentiation is a multiplication by a factor of order 1 with respect to r'. Thus, it is enough to show (3.22) for $i=i'=j=0$. If $\delta \operatorname{Im} c+1\neq -1$, then from (3.9), (3.21), (2.7), (2.9), (2.13) and (1.3), we have

$$|K_{\#}^{\delta}(r, r', y)|\leqq C\left|\int_{r'}^{r} p^{1+\delta \operatorname{Im} c}dp\right| r'^{-3-\delta \operatorname{Im} c}$$

if $r_0\leqq r\leqq r'$, $|y|\leqq y_0 r$ and C is an appropriate constant. (3.22) for $i=i'=j=0$ is then immediate. If $1+\delta \operatorname{Im} c=-1$, then use the relation:

$$\frac{p^2+y^2}{2p^2}\hat{V}_0^{\delta}(p, y, 1)^{-2}=\frac{\delta\sqrt{-1}}{2}p^{-\sqrt{-1}\delta c}\partial_p \exp\{-2\sqrt{-1}\hat{S}^{\delta}(p, y, 1)\},$$

which is a consequence if (3.4), (2.9) and (1.9). Integrating the integral $\int_{r'}^{r}\frac{p^2+y^2}{2p^2}\hat{V}_0^{\delta}(p, y, 1)^{-2}dp$ by parts, we obtain (3.22) for $i=i'=j=0$ even when $1+\delta \operatorname{Im} c=-1$.

Now we are ready for proving Proposition 3.3. In view of Proposition 3.4, we show the following

PROPOSITION 3.6. *Let*

$$(K_{\#}^{\delta}f)(r, y)=\int_{+\infty}^{r} K_{\#}^{\delta}(r, r', y)f(r', y)dr', \tag{3.23}$$

$f\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$. For any $g(r, y)\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$, there is a unique $h(r, y)\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$ such that $h+K_{\#}^{\delta}h=g$.

Proof. We introduce the following auxiliary Banach spaces. Let $r_0>0$, $y_0>0$ be arbitrarily given. Let N be a positive integer. We denote by $\mathcal{S}(r_0, y_0, N)$ the space of those $f(r, y)$ which are continuous when $r\geqq r_0$, $|y|\leqq y_0 r$, and $r^N|f(r, y)|$ are bounded from above when $r\geqq r_0$, $|y|\leqq y_0 r$. $\mathcal{S}(r_0, y_0, N)$ is a Banach space with the norm

$$\|f\|=\sup\{r^N|f(r, y)|\ ;\ r\geqq r_0,\ |y|\leqq y_0 r\}.$$

(3.22) and (3.23) imply that if $N+2+\delta \operatorname{Im} c>0$ then $K_{\#}^{\delta}f\in\mathcal{S}(r_0, y_0, N)$ when $f\in\mathcal{S}(r_0, y_0, N)$ and

$$\|K_{\#}^{\delta}f\|\leqq C\{N^{-1}+(N+2+\delta \operatorname{Im} c)^{-1}\}\|f\|.$$

Thus, if N is so large that

$$C\{N^{-1}+(N+2+\gamma \operatorname{Im} c)^{-1}\}<1,$$

then $I+K_{\#}^{\delta}$ is a bijection in $\mathcal{S}(r_0, y_0, N)$. Now if $g(r, y)\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$, then $g(r, y)\in\mathcal{S}(r_0, y_0, N)$ for any r_0, y_0, N. Therefore, we have $h(r, y)\in\mathcal{S}(r_0, y_0, N)$ solving $(I+K_{\#}^{\delta})h=g$ when $r\geqq r_0$, $|y|\leqq y_0 r$, and N is large enough. Uniqueness of the solution in each $\mathcal{S}(r_0, y_0, N)$ then implies that $h(r, y)$ is actually defined for $(r, y)\in\boldsymbol{R}_+\times\boldsymbol{R}$ and $h(r, y)\in\cap\mathcal{S}(r_0, y_0, N)$, where $\mathcal{S}(r_0, y_0, N)\subset\mathcal{S}(r_0', y_0', N')$ by the natural restriction when $r_0\leqq r_0'$, $y_0\leqq y_0'$, $N\geqq N'$. Using (3.22), we have similar estimates for the drivatives of $h(r, y)$. Thus, we have $h(r, y)\in\mathcal{S}(\boldsymbol{R}_+\times\boldsymbol{R})$ which solves $(I+K_{\#}^{\delta})h=g$.

§ 4. Completion of the proof of Theorem 1.1.

We have so far verified the statements (ii) and (iii) of Theorem 1.1. The remaining statements (i) and (iv) are rather obvious. From our discussions in § 3, $v^{\delta}(t, y, k)$, $\delta\in\{+, -\}$, are C^{∞} solutions of the equation (1.2) in D. $v^{\delta}(t, y, k)$ are extended to C^{∞} solutions of (1.2) in $\boldsymbol{R}^2\times\boldsymbol{R}_+$ because of (1.16) by solving the Cauchy problem with data at $t=1$. (Actually $v^{\delta}(t, y, k)$ are extended to entire analytic functions in t). The statement (iv) is also clear. For $w=v^+\partial_t v^- - v^-\partial_t v^+$ satisfies $\partial_t w=0$ so that (1.13) and (1.14) imply $w=-2\sqrt{-1}\,k$.

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