S. KOBAYASHI
KODAI MATH. J.
8 (1985), 163—170

IMAGE AREAS AND BMO NORMS OF ANALYTIC FUNCTIONS

Dedicated to Professor Yukio Kusunoki on his 60th birthday

BY SHŌJI KOBAYASHI

1. Introduction. Let R be a nonparabolic open Riemann surface and $g(z, a)$ denote its Green's function with logarithmic singularity at $a \in R$. For a function f analytic on R, we define

$$A(f)=\frac{1}{\pi}\operatorname{area}\{f(R)\},$$

$$B(f)=\sup_{a\in R}\frac{2}{\pi}\iint_R |f'(z)|^2 g(z, a)\,dx\,dy \tag{1.1}$$

and

$$D(f)=\frac{1}{\pi}\iint_R |f'(z)|^2\,dx\,dy,$$

where $z=x+iy$ denotes a local coordinate on R. We consider following spaces of analytic functions on R:

$$BMOA(R)=\{f : B(f)<+\infty\}, \tag{1.2}$$

$$AD(R)=\{f : D(f)<+\infty\}.$$

Metzger [10] introduced $BMOA(R)$ by (1.1) and (1.2) and showed the inclusion relation $AD(R)\subset BMOA(R)$ by using a celebrated result of Hayman and Pommerenke [3]. Stegenga [13] independently obtained a similar result as theirs and remarked as an easy consequence that the inequality

$$B(f)\leqq cA(f) \tag{1.3}$$

with some constant c holds for functions f analytic in the unit disc U of the complex plane $\boldsymbol{C}$. Recently the author [8] showed that (1.3) holds with $c=1$, that is, the inequality

$$B(f)\leqq A(f) \tag{1.4}$$

holds for functions f analytic on R, which obviously implies

Received February 7, 1984

$$B(f) \leqq D(f), \tag{1.5}$$

and left open problems as conjectures on equality conditions of (1.4) and (1.5). In the present paper we offer a rather simple proof of (1.4) and settle the conjectures, one negatively and the other affirmatively.

In Section 2 we restate our results in the form of a theorem. In Section 3 we obtain an expression of $B(f)$ employing least harmonic majorants, from which we see the invariance of $B(f)$ under the pull-back by a universal covering map. In Section 4 we prepare preliminary lemmas, which we use in Section 5 for the proof of our main results. In Section 6 we deal with the conjectures offered by the author [8] on equality conditions of (1.4) and (1.5).

I would like to express my deep gratitude to Professor N. Suita for his constant encouragement and helpful comments on the present paper, especially, thanks to his suggestion, the proof of Lemma 4.2 was made considerably short and simple, although the original one was somewhat long and complicated.

2. Main results.

THEOREM 2.1. *If f is analytic function on R, then*

$$B(f) \leqq A(f). \tag{2.1}$$

COROLLARY 2.2.

$$B(f) \leqq D(f). \tag{2.2}$$

Corollary 2.2 is an easy consequence of Theorem 2.1, since the inequality

$$A(f) \leqq D(f) \tag{2.3}$$

is obvious, where equality occurs if and only if f is univalent on R.

3. BMO norms and least harmonic majorants. We denote by $h_a(z)$ the *least harmonic majorant* of $|f(z)-f(a)|^2$ on R for every $a \in R$. Let $p: U \to R$ be a *universal covering map* of R.

Proposition 3.1 is easily proved by a routine way using Green's formula, which is similar to that of obtaining a formula for the solution of Dirichlet problem in terms of the normal derivative of Green's function (see, for example, [5, pp. 399-405] or [6, Lemma 1]).

PROPOSITION 3.1.

$$h_a(a) = \frac{2}{\pi} \iint_R |f'(z)|^2 g(z, a)\, dx\, dy. \tag{3.1}$$

COROLLARY 3.2.

$$B(f)=\sup_{a\in R} h_a(a)\,. \tag{3.2}$$

It is well known that a universal covering map preserves the least harmonic majorant of any subharmonic function (see [11, p. 50] or [9, Lemma 1, p. 316]), so we easily see that the BMO norm $B(\cdot)$ is invariant under the pull-back by a universal covering map, that is

COROLLARY 3.3. $\quad B(f)=B(f\circ p)\,.$

COROLLARY 3.4. $\quad BMOA(R)=\{f : f\circ p\in BMOA(U)\}\,.$

Corollaries 3.3 and 3.4 were essentially obtained by Metzger [10, p. 1257], whose proof, however, heavily depends on a Myrberg's theorem on Green's function of a covering surface.

4. Preliminary lemmas. In this section we prepare two preliminary lemmas. Lemma 4.1 is easily derived from Proposition 3.1 and the subordination principle (see, for example, [5, p. 422]).

LEMMA 4.1. *Let R_1 and R_2 be Riemann surfaces and $\phi : R_1\to R_2$ be an analytic map from R_1 into R_2, then*

$$B(f\circ\phi)\leqq B(f) \tag{4.1}$$

for any $f\in BMOA(R_2)$.

Let $p_j : U\to R_j$ denotes a universal covering map of R_j for $j=1, 2$. By the monodromy theorem, we can define an analytic function ψ in U bounded by 1 for which $\phi\circ p_1=p_2\circ\psi$. We call ϕ an *inner map* when ψ is an *inner function*, i.e. $|\psi^*(e^{i\theta})|=1$ a.e. on the boundary of U, where ψ^* denotes the *Fatou's boundary function* of ψ. It is known that ϕ preserves the least harmonic majorant of a nonnegative subharmonic function if and only if ϕ is an inner map (see [9, Theorem 1, p. 316] or [12]). Therefore, on noting Corollary 3.2, we easily see that equality occurs in (4.1) if ϕ is an inner map. On the other hand, the converse is not valid. Indeed, in the last section we will offer a counterexample, which shows that equality can occur in (4.1) even if ϕ is not inner (see Corollary 6.5 in Section 6).

The next lemma was claimed by the author [7] in order to give another proof of an inequality on image areas and H_2 norms obtained by Alexander, Taylor and Ullman [1]. Here we state a simple proof, so as to make the present paper self-contained and to make use of the argument for considering the equality conditions of (2.1) and (2.2) in Section 6.

LEMMA 4.2. *Let D be a plane domain of finite area and $\chi_a(z)$ denote the least harmonic majorant of $|z-a|^2$ in D for $a\in D$, then*

$$\chi_a(a) \leqq \frac{1}{\pi}\operatorname{area}(D), \tag{4.2}$$

and equality occurs in (4.2) *if and only if* D *is a domain of the form* $D=\{z : |z-a|<r\}-E$, *where* $r>0$ *and* E *is a closed set of capacity zero.*

Proof. First we assume that D is a domain with smooth boundary ∂D. If $u(z)$ is a C^2 function on the closure $\bar{D}$ of D, then by a Green's formula

$$\iint_D \Delta u \, dx dy = \int_{\partial D} \frac{\partial u}{\partial n} ds, \tag{4.3}$$

where Δ denotes the Laplacian, $\frac{\partial}{\partial n}$ the differentiation in the outer normal direction and ds the arc length measure on ∂D. Let $v(z)=|z-a|^2$ and apply (4.3) with $u=v$ in D, then we see

$$4\operatorname{area}(D) = \int_{\partial D} \frac{\partial v}{\partial n} ds, \tag{4.4}$$

since $\Delta v=4$. Next let $w(z)=|z-a|^2 e^{2g(z,a)} \equiv v e^{2g}$ and again apply (4.3) with $u=w$ in D, then we see

$$\begin{aligned} \iint_D \Delta w \, dx dy &= \int_{\partial D} \frac{\partial w}{\partial n} ds \\ &= \int_{\partial D} \frac{\partial v}{\partial n} ds + 2\int_{\partial D} v \frac{\partial g}{\partial n} ds, \end{aligned} \tag{4.5}$$

since $\frac{\partial w}{\partial n} = \frac{\partial v}{\partial n} + 2v\frac{\partial g}{\partial n}$ on ∂D. Combining (4.4) and (4.5), we obtain

$$\chi_a(a) + \frac{1}{4\pi}\iint_D \Delta w \, dx dy = \frac{1}{\pi}\operatorname{area}(D), \tag{4.6}$$

since $\int_{\partial D} v \frac{\partial g}{\partial n} ds = -2\pi \chi_a(a)$.

In order to deal with the case where D is a general domain, let $\{D_m\}$ be a smooth exhaustion of D such that $a \in D_m$ for $m=1, 2, \cdots$. We denote by $\chi_{a,m}$ and w_m, respectively, the functions for D_m which correspond to χ_a and w for D, then (4.6) for D_m is

$$\chi_{a,m}(a) + \frac{1}{4\pi}\iint_{D_m} \Delta w_m dx dy = \frac{1}{\pi}\operatorname{area}(D_m).$$

On letting $m \to \infty$, we see by Lebesgue's monotone convergence theorem and Fatou's lemma

$$\chi_a(a) + \frac{1}{4\pi}\iint_D \Delta w \, dx dy \leqq \frac{1}{\pi}\operatorname{area}(D), \tag{4.7}$$

since w_m converges to w uniformly on compact subsets of D. Note that $\Delta w \geqq 0$ in D since w is subharmonic, so (4.7) implies (4.2).

Suppose that equality occurs in (4.2), then we see by (4.7) that $\Delta w \equiv 0$ in D, which means that $g(z, a)=\log(r/|z-a|)$ for some positive constant r. Therefore D must be a domain as mentioned in the lemma. Conversely, if D is such a domain, then equality evidently occurs in (4.2), since in that case $\chi_a(z)\equiv r^2$ in D and a set of capacity zero is of area zero.

COROLLARY 4.3. *If $I(z)\equiv z$ for $z\in D$, then*

$$B(I)\leqq \frac{1}{\pi}\operatorname{area}(D), \tag{4.8}$$

where equality occurs if and only if D is a domain of the form $D=\{z:|z-c|<r\}-E$ with $c\in \boldsymbol{C}$, $r>0$ and $Cap(E)=0$.

Proof. On noting Corollary 3.2, (4.8) immediately follows from (4.2). We must prove the equality condition. For this we first assume that equality occurs in (4.8). Take a sequence $\{a_n\}$ of points in D such that

$$B(I)=\lim_{n\to\infty}\chi_{a_n}(a_n). \tag{4.9}$$

We may assume, if necessary by taking a subsequence, that a_n converges to some point $c\in\bar{D}$ and that $g(z, a_n)$ converges uniformly on compact subsets of $D-\{c\}$. Write $w_n(z)=|z-a_n|^2 e^{g(z,a_n)}$. Applying Fatou's lemma, we see by (4.7) and (4.9)

$$B(I)+\frac{1}{4\pi}\iint_D \lim_{n\to\infty}\Delta w_n\, dx\, dy \leqq \frac{1}{\pi}\operatorname{area}(D),$$

and hence $\lim_{n\to\infty}\Delta w_n\equiv 0$, since we assumed the equality in (4.8). Therefore we see

$$\lim_{n\to\infty} g(z, a_n)=\log(r/|z-c|) \tag{4.10}$$

for some positive constant r. Since Green's functions are positive, (4.10) implies that D is contained in the disc $W=\{z:|z-c|<r\}$. Let $E=W-D$ and suppose that E is of positive capacity, then we can take a $\delta>0$ such that $E_\delta= E\cap\{z:|z-c|\geqq\delta\}$ is of positive capacity. Let $D_\delta=D\cup\{z:|z-c|<\delta\}$ and $g_\delta(z, a)$ be Green's function of D_δ. Since $D\subset D_\delta\subset W$, we see by (4.10)

$$\begin{aligned}\log(r/|z-c|)&=\lim_{n\to\infty} g(z, a_n)\\ &\leqq \lim_{n\to\infty} g_\delta(z, a_n)\\ &=g_\delta(z, c)\\ &\leqq \log(r/|z-c|),\end{aligned}$$

and hence $g_\delta(z, a)=\log(r/|z-c|)$, which means $\mathrm{Cap}(E_\delta)=0$. This is a contradiction. Therefore we see $\mathrm{Cap}(E)=0$, and hence that D must be a domain as mentioned in the corollary.

Conversely, if D is such a domain, then equality evidently occurs in (4.8), since $\lim_{a\to c}\chi_a(a)=r^2$.

Remark. The inequality (4.2) can be also deduced from an isoperimetric inequality on symmetrized Poisson problem [2, Theorem 2.8, p. 68] by setting $f\equiv -4$ there.

5. Proof of Theorem 2.1. Let $D=f(R)$ and $\rho: U\to D$ be a universal covering map of D. By the monodromy theorem, we can determine a single-valued branch of $\rho^{-1}\circ f\circ p$, which, denoted by ϕ, is an analytic map of U into itself such that

$$\rho\circ\phi=f\circ p\,, \tag{5.1}$$

where $p: U\to R$ denotes a universal covering map of R, as before. By Corollary 3.3, (5.1), Lemma 4.1 and Corollary 4.3, we see

$$\begin{aligned} B(f)&=B(f\circ p)\\ &=B(\rho\circ\phi)\\ &\leqq B(\rho)\\ &=B(I)\leqq\frac{1}{\pi}\mathrm{area}(D)\,, \end{aligned}$$

which completes the proof of the theorem.

6. Equality conditions. In this section we are concerned with the problem when equalities occur in (2.1) and (2.2). The author [8] presented the following conjectures on the problem and noted that the if parts are valid for both of them and that if the former is valid then so is the latter:

Conjecture 1. *Equality occurs in* (2.1) *if and only if* $f\circ p=c\phi+d$, *where* c *and* d *are constants and* ϕ *is an inner function.*

Conjecture 2. *Equality occurs in* (2.2) *if and only if* R *is a Riemann surface which is obtained from a simply-connected one* W *by deleting at most a set of capacity zero and* f *is* (*extended to*) *a conformal map of* W *onto a disc.*

In the following, we show that Conjecture 1 is not valid, by offering a counterexample (Example 6.4), while that Conjecture 2 is valid (Proposition 6.2).

PROPOSITION 6.1. *If equality occurs in* (2.1), *then* $f(R)=\{z: |z-c|<r\}-E$,

with $c\in\boldsymbol{C}$, $r>0$ *and* $\mathrm{Cap}(E)=0$.

Proof. The proposition easily follows from the proof of Theorem 2.1 and the equality condition of Corollary 4.3.

PROPOSITION 6.2. *Conjecture 2 is valid.*

Proof. Suppose that equality occurs in (2.2), then we see that equalities occur both in (2.1) and (2.3). Therefore we see by Proposition 6.1 that $f(R)=\{z:|z-c|<r\}-E$ with $c\in\boldsymbol{C}$, $r>0$ and $\mathrm{Cap}(E)=0$, and by the equality condition for (2.3) that f is univalent on R. Since a set of capacity zero is removable for H_p functions (see, for example, [4] or [11]), and hence for BMOA functions, we see that f and R must be as mentioned in Proposition 6.2.

Conversely, if f and R are as mentioned in the proposition, then it is easily seen that $h_a(a)$ approaches to r^2 if we take $a\to R$ for which $f(a)\to c$, and hence that equality occurs in (2.2), as asserted.

We conclude the present paper by giving two examples, one of which demonstrates that the condition mentioned in Proposition 6.1 is not a sufficient one for equality in (2.1), and the other demonstrates that equality can occur in (2.1) even if f is not an inner map, that is, the only if part of Conjecture 1 is not valid.

Let g be the conformal map of U onto $U\cap\{z:\mathrm{Re}\,z>-1/2\}$ with $g(0)=0$, and ψ be the singular inner function $\psi(z)=\exp\{-(1+z)/(1-z)\}$, $z\in U$.

EXAMPLE 6.3. *Let* $f(z)=\{g(z)\}^2$, *then* $f(U)=U$ *but equality does not occur in* (2.1).

Proof. Let $h_a(z)$ denotes the least harmonic majorant of $|f(z)-f(a)|^2$ in U as before, then we see by Corollary 3.2 and a simple calculation

$$\begin{aligned}B(f)&=\sup_{a\in U} h_a(a)\\&=\sup_{a\in U}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)-f(a)\right|^2 d\theta\\&=\sup_{a\in U}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)\right|^2 d\theta-|f(a)|^2\\&\leqq 1-\varepsilon,\end{aligned}$$

since for $1>\delta>0$ we can take an $\varepsilon>0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)\right|^2 d\theta\leqq 1-\varepsilon$$

if $|a|\leqq\delta$, and that $|f(a)|\geqq\varepsilon$ if $|a|>\delta$.

EXAMPLE 6.4. *Let* $f(z)=g(z)\psi(z)$, *then* f *is not an inner function but equality occurs in* (2.1).

Proof. Similarly as in the proof of Example 6.3, we see by a simple calculation

$$h_a(a)=\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)\right|^2 d\theta-|f(a)|^2$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left|g\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)\right|^2 d\theta-|f(a)|^2.$$

Noting that the integral $\frac{1}{2\pi}\int_0^{2\pi}\left|g\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right)\right|^2 d\theta$ coincides with the value at a of the harmonic function in U with boundary value $|g(\zeta)|^2$ on ∂U, we see that $h_a(a)\to 1$ as $a\to 1$, $a\in\boldsymbol{R}$, since $|f(a)|\leqq|\psi(a)|\to 0$ as $a\to 1$, $a\in\boldsymbol{R}$, and hence that $B(f)=1$. On the other hand evidently $A(f)\leqq 1$, since $|f(z)|\leqq 1$ for $z\in U$, so we see that equality must occur in (2.1), as asserted.

COROLLARY 6.5. *Let* $R_j=U$ *for* $j=1, 2$, *and consider the function* f *defined in Example* 6.4 *to be analytic map of* R_1 *into* R_2, *then* f *is not an inner map while* $B(I)=B(I\circ f)$ *holds.*

REFERENCES

[1] ALEXANDER, H., B. A. TAYLOR AND J. L. ULLMAN, Areas of projections of analytic sets, Invent. Math., **16** (1972), 335-341.

[2] BANDLE, C., Isoperimetric inequalities and applications, Pitman, 1980.

[3] HAYMAN, W. K. AND C. POMMERENKE, On analytic functions of bounded mean oscillation, Bull. London Math. Soc., **10** (1978), 219-224.

[4] HEJHAL, D. A., Classification theory for Hardy classes of analytic functions, Ann. Acad. Sci. Fenn. Ser. A I No. **566** (1973), 1-28.

[5] HILLE, E., Analytic function theory II, Chelsea, 1962.

[6] KOBAYASHI, S., On a classification of plane domains for BMOA, Kodai Math. J., **7** (1984), 111-119.

[7] KOBAYASHI, S., Image areas and H_2 norms of analytic functions, Proc. Amer. Math. Soc., **91** (1984), 257-261.

[8] KOBAYASHI, S., Range sets and BMO norms of analytic functions, to appear in Can. J. Math., **36** (1984), 747-755.

[9] KOBAYASHI, S. AND N. SUITA, On subordination of subharmonic functions, Kodai Math. J., **3** (1980), 315-320.

[10] METZGER, T. A., On BMOA for Riemann surfaces, Can. J. Math., **18** (1981), 1255-1260,

[11] RUDIN, W., Analytic functions of class H_p, Trans. Amer. Math. Soc., **78** (1955), 46-66.

[12] RYFF, J. V., Subordinate H^p functions, Duke Math. J., **33** (1966), 347-354.

[13] STEGENGA, D. A., A geometric condition which implies BMOA, Michigan Math. J., **27** (1980), 247-252.

DEPARTMENT OF ELECTRICAL ENGINEERING
TECHNOLOGICAL UNIVERSITY OF NAGAOKA