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ON THE GROWTH OF NON-ADMISSIBLE SOLUTIONS OF THE DIFFERENTIAL EQUATION $(w')^n=\sum_{j=0}^{m} a_j w^j$

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1. Introduction.

Let $a_0, \cdots, a_m$ be meromorphic in the complex plane and $a_m \neq 0$. We consider the differential equation

$$(1) \qquad (w')^n=\sum_{j=0}^{m} a_j w^j \qquad (m \geqq 1).$$

It is said ([1]) that any meromorphic solution $w(z)$ of (1) in the complex plane is admissible when it satisfies the condition

$$T(r, a_j)=o(T(r, w)) \qquad (j=0, 1, \cdots, m)$$

for $r \to \infty$ possibly outside a set of r of finite linear measure.

In this paper we will denote by E any set of r of finite linear measure and the term "meromorphic" will mean meromorphic in the complex plane.

A few years ago, Gackstatter and Laine ([1], 3) investigated the differential equation (1) in many cases. One of their results is

THEOREM A. *When m-$n=k \geqq 1$ and k is not a divisor of n, the differential equation* (1) *does not have any admissible solutions.*

It is well-known that this theorem is true when $k \geqq n+1$.

They also gave the conjecture that, when $1 \leqq m \leqq n-1$, the differential equation (1) does not possess any admissible solutions. With respect to this conjecture, we have recently proved the following theorems in [7].

THEOREM B. *When $1 \leqq m \leqq n-1$, the differential equation* (1) *has no admissible solutions, except when n-m is a divisor of n and* (1) *has the form*:

$$(w')^n=a_m(w+\alpha)^m \qquad (\alpha : \text{constant}).$$

THEOREM C. *When $1 \leqq m \leqq n-1$, any meromorphic solution of the differential equation* (1) *is of order at most ρ, where $\rho=\max(\rho_0, \cdots, \rho_m)$, ρ_j=the order of $a_j<\infty$.*

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These were first proved by Ozawa ([6]) when $m=1$, 2 and 3.

The purpose of this paper is to give some improvements of Theorems A, B and C by estimating $T(r, w)$ with $T(r, a_0), \cdots, T(r, a_m)$ and to prove a result when $m=n$. It is assumed that the reader is familiar with the notation of Nevanlinna theory ([3], [5]).

2. Lemmas.

We shall give some lemmas for later use first.

LEMMA 1. *Let g_0 and g_1 be meromorphic functions which are linearly independent over C and put*

$$g_0+g_1=\phi . \tag{2}$$

Then, we have

$$T(r, g_0)\leqq T(r, \phi)+\bar{N}(r, \phi)+\bar{N}(r, 0, g_0)+\bar{N}(r, g_0)+\bar{N}(r, 0, g_1)+2\bar{N}(r, g_1)+S(r),$$

where

$$S(r)=\begin{cases} O(1) & (\textit{when } g_0 \textit{ and } g_1 \textit{ are rational}); \\ O(\log^+T(r, g_0)+\log^+T(r, g_1))+O(\log r) & (r\notin E, \textit{ the other cases}). \end{cases}$$

Proof. From (2) and $g_0'+g_1'=\phi'$, we have

$$g_0=(\phi g_1'/g_1-\phi')/(g_1'/g_1-g_0'/g_0),$$

so that we obtain

$$\begin{aligned} m(r, g_0)&\leqq m(r, \phi g_1'/g_1-\phi')+m(r, (g_1'/g_1-g_0'/g_0)^{-1})+O(1) \\ &\leqq m(r, \phi g_1'/g_1-\phi')+m(r, g_1'/g_1-g_0'/g_0)+N(r, g_1'/g_1-g_0'/g_0) \\ &\quad -N(r, 0, g_1'/g_1-g_0'/g_0)+O(1) \end{aligned} \tag{3}$$

and

$$N(r, g_0)\leqq N(r, \phi)+\bar{N}(r, \phi)+N(r, 0, g_1'/g_1-g_0'/g_0)+\bar{N}(r, g_1). \tag{4}$$

Using the following inequalities:

$$m(r, \phi g_1'/g_1-\phi')\leqq m(r, \phi)+m(r, \phi'/\phi)+m(r, g_1'/g_1)+O(1),$$

$$m(r, g_1'/g_1-g_0'/g_0)\leqq m(r, g_1'/g_1)+m(r, g_0'/g_0)+O(1),$$

$$N(r, g_1'/g_1-g_0'/g_0)\leqq \bar{N}(r, 0, g_0)+\bar{N}(r, g_0)+\bar{N}(r, 0, g_1)+\bar{N}(r, g_1),$$

we have from (3) and (4)

$$T(r, g_0) \leqq T(r, \psi)+\bar{N}(r, \psi)+\bar{N}(r, 0, g_0)+\bar{N}(r, g_0)+\bar{N}(r, 0, g_1)+2\bar{N}(r, g_1)+S(r),$$

where

$$S(r)=m(r, \psi'/\psi)+m(r, g_0'/g_0)+m(r, g_1'/g_1)+O(1)$$

$$=\begin{cases} O(1) & \text{(when } g_0 \text{ and } g_1 \text{ are rational)};\\ O(\log^+ T(r, g_0)+\log^+ T(r, g_1))+O(\log r) & (r \notin E, \text{ the other cases}). \end{cases}$$

Remark 1. This is an improvement of Lemma 1 in [8]. Using this lemma, we can improve Theorem 1 in [8].

LEMMA 2. *Let $f, a_0, \cdots, a_k$ be meromorphic, then we have the following inequalities*:

(i) $m(r, \sum_{j=0}^{k} a_j f^j) \leqq km(r, f)+\sum_{j=0}^{k} m(r, a_j)+O(1)$,

(ii) $T(r, \sum_{j=0}^{k} a_j f^j) \leqq kT(r, f)+\sum_{j=0}^{k} T(r, a_j)+O(1)$

(*see* [2], *p.* 46).

We can easily prove (i) and (ii) by the mathematical induction.

3. Theorems.

We shall give an improvement of Theorem A first.

THEOREM 1. *When $m-n=k\geqq 1$ and k is not a divisor of n, any nonconstant meromorphic solution $w=w(z)$ of the differential equation* (1) *satisfies the following inequality*:

$$T(r, w) \leqq K_1 \sum_{j=0}^{m} T(r, a_j)+nm(r, w'/w)+O(1),$$

where K_1 is a constant independent of r.

Proof. From (1), we have

$$w^k=a_m^{-1}((w'/w)^n-\sum_{j=0}^{m-1} a_j w^{j-n}). \tag{5}$$

For an arbitrarily fixed $r>0$, let M_r be the set of θ for which $|w(re^{i\theta})| \geqq 1$ and $0 \leqq \theta \leqq 2\pi$. Then, from (5)

$$k \log^+|w(re^{i\theta})| \leqq n \log^+|w'(re^{i\theta})/w(re^{i\theta})|+\log^+|\sum_{j=n}^{m-1} a_j(w(re^{i\theta}))^{j-n}|$$

$$+\sum_{j=0}^{n-1}(n-j)\log^+|1/w(re^{i\theta})|+\sum_{j=0}^{n-1}\log^+|a_j|+\log^+|1/a_m|+O(1).$$

Integrating both sides of this inequality with respect to θ in M_r and dividing by 2π, we obtain

$$km(r, w)\leqq nm(r, w'/w)+m(r, \sum_{j=n}^{m-1} a_j w^{j-n})+\sum_{j=0}^{n-1} m(r, a_j)+m(r, 1/a_m)+O(1)$$

and using Lemma 2(i) we have

$$(6) \qquad m(r, w)\leqq nm(r, w'/w)+\sum_{j=0}^{m-1} m(r, a_j)+m(r, 1/a_m)+O(1).$$

On the other hand, as k is not a divisor of n, $w(z)$ does not have any poles other than those of a_j or zeros of a_j $(j=0, \cdots, m)$, so that we have

$$\bar{N}(r, w)\leqq \sum_{j=0}^{m} (\bar{N}(r, a_j)+\bar{N}(r, 0, a_j)).$$

Using this inequality and applying the method used in [1], p. 265, which is also valid for $k\geqq n+1$, we have the inequality:

$$(7) \qquad N(r, w)\leqq K\sum_{j=0}^{m} (N(r, a_j)+N(r, 0, a_j))$$

for a constant K. Adding (6) and (7), we have

$$T(r, w)\leqq K_1 \sum_{j=0}^{m} T(r, a_j)+nm(r, w'/w)+O(1),$$

where K_1 is a constant smaller than $2K$.

Remark 2. Naturally, this theorem contains the case $k\geqq n+1$.

COROLLARY 1. *Under the same condition as in Theorem* 1, *the differential equation* (1) *does not possess any admissible solution* ([1], *Satz* 6 *and* [4], *Theorem* 1).

COROLLARY 2. *Let* ρ_j $(<\infty)$ *be the order of* a_j *and* $\rho=\max(\rho_0, \cdots, \rho_m)$. *Under the same condition as in Theorem* 1, *the order of any meromorphic solution of* (1) *is at most* ρ.

Next, we consider the case $m=n$ in (1). As is noted in [1], p. 266, some differential equations of the type

$$(w')^n=\sum_{j=0}^{n} a_j w^j \qquad (a_n\neq 0)$$

can have an admissible solution. For example, $(w')^n=e^{nz}w^n$ has an admissible solution $w=\exp e^z$. But some of them cannot possess any admissible solution.

THEOREM 2. *Any meromorphic solution* $w=w(z)$ *of the differential equation*

$$(8) \qquad (w')^n=a_n w^n+\sum_{j=0}^{k} a_j w^j \qquad (0\leqq k\leqq n-3,\ a_n\neq 0 \text{ and } a_k\neq 0),$$

where a_j $(j=0, \cdots, k)$ *and* a_n *are meromorphic, satisfies the following inequality*:

$$T(r, w) \leqq K_2\Big(\sum_{j=0}^{k} T(r, a_j)+T(r, a_n)\Big)+O(\log r) \qquad (r \notin E)$$

for a constant K_2.

Proof. We have only to prove this theorem when $w=w(z)$ is not rational. Put

$$g_0(z)=-a_n(w(z))^n, \quad g_1(z)=(w'(z))^n, \quad \psi(z)=\sum_{j=0}^{k} a_j(w(z))^j.$$

(i) The case: $\psi=0$. As

$$a_k w^k=-\sum_{j=0}^{k-1} a_j w^j,$$

by Lemma 2(ii), we have

$$kT(r, w) \leqq (k-1)T(r, w)+\sum_{j=0}^{k} T(r, a_j)+O(1),$$

that is,

$$T(r, w) \leqq \sum_{j=0}^{k} T(r, a_j)+O(1).$$

(ii) The case: $\psi \neq 0$ and g_0, g_1 are linearly dependent over C. There are constants α, $\beta \in C$ such that

$$\alpha g_0+\beta g_1=0 \qquad (|\alpha|+|\beta| \neq 0).$$

β cannot be equal to zero. Therefore, we have

$$\frac{\alpha}{\beta} a_n w^n=a_n w^n+\sum_{j=0}^{k} a_j w^j,$$

that is,

$$\Big(\frac{\alpha}{\beta}-1\Big)a_n w^n=\sum_{j=0}^{k} a_j w^j. \tag{9}$$

As $\psi \neq 0$, $\alpha/\beta \neq 1$. By Lemma 2(ii), from (9) we have

$$nT(r, w) \leqq kT(r, w)+\sum_{j=0}^{k} T(r, a_j)+T(r, a_n)+O(1),$$

so that

$$T(r, w) \leqq \frac{1}{n-k}\Big(\sum_{j=0}^{k} T(r, a_j)+T(r, a_n)\Big)+O(1).$$

(iii) The case: $\psi \neq 0$ and g_0, g_1 are linearly independent over C. As $g_0+g_1=\psi$, we have by Lemma 1

$$\begin{aligned} T(r, g_0) \leqq\ & T(r, \psi)+\overline{N}(r, \psi)+\overline{N}(r, 0, g_0)+\overline{N}(r, g_0) \\ & +\overline{N}(r, 0, g_1)+2\overline{N}(r, g_1)+S(r). \end{aligned} \tag{10}$$

Here, we estimate each term of (10).

$$T(r, g_0) \geqq nT(r, w)-T(r, a_n)+O(1), \tag{11}$$

(12) $$T(r, \psi)\leqq kT(r, w)+\sum_{j=0}^{k} T(r, a_j)+O(1) \qquad \text{(by Lemma 2(ii))},$$

(13) $$\bar{N}(r, \psi)\leqq \bar{N}(r, w)+\sum_{j=0}^{k}\bar{N}(r, a_j),$$

(14) $$\bar{N}(r, 0, g_0)\leqq \bar{N}(r, 0, a_n)+\bar{N}(r, 0, w),$$

(15) $$\bar{N}(r, g_0)\leqq \bar{N}(r, a_n)+\bar{N}(r, w),$$

(16) $$\bar{N}(r, 0, g_1)=\bar{N}(r, 0, w'),$$

(17) $$\bar{N}(r, g_1)=\bar{N}(r, w),$$

(18) $$T(r, w')\leqq T(r, w)+\frac{1}{n}\Big(\sum_{j=0}^{k} T(r, a_j)+T(r, a_n)\Big)+O(1) \qquad \text{(from (8))},$$

(19) $$S(r)=O(\log^+ T(r, w)+\sum_{j=0}^{k}\log^+ T(r, a_j)+\log^+ T(r, a_n)+\log r) \qquad (r\notin E).$$

Further, w does not have any poles other than poles or zeros of $a_0, \cdots, a_k, a_n$. This can be easily seen from the equation (8). Therefore,

(20) $$\bar{N}(r, w)\leqq \sum_{j=0}^{k}(\bar{N}(r, 0, a_j)+\bar{N}(r, a_j))+\bar{N}(r, a_n)+\bar{N}(r, 0, a_n),$$

(21) $$\bar{N}(r, 0, w)\leqq T(r, w)+O(1),$$

(22) $$\bar{N}(r, 0, w')\leqq T(r, w')+O(1).$$

From (10)-(22), using $n-k-2\geqq 1$ and $\log^+ T(r, w)=o(T(r, w))$ $(r\to\infty)$, we have

$$T(r, w)\leqq (n-k-2)T(r, w)\leqq K_2'\Big(\sum_{j=0}^{k} T(r, a_j)+T(r, a_n)\Big)+O(\log r) \quad (r\notin E),$$

where K_2' is a constant.

Combining (i), (ii) and (iii), we have this theorem.

COROLLARY 3. *The differential equation* (8) *does not possess any admissible solution.*

COROLLARY 4. *The order of any meromorphic solution of* (8) *is at most equal to the maximum of the orders of* $a_0, \cdots, a_k$ *and* a_n *when they are finite.*

Remark 3. We cannot weaken the condition $k\leqq n-3$. In fact, the differential equation $(w')^2=-w^2+1$ has an admissible solution $w=\cos z$.

Next, we consider the case $m\leqq n-1$ in (1), that is, the differential equation

(23) $$(w')^n=\sum_{j=0}^{m} a_j w^j \qquad (1\leqq m\leqq n-1,\ a_m\neq 0).$$

As in [7], p. 241, we rewrite (23) as follows:

(23′) $$(w')^n = a_m(w+b)^m + \sum_{j=0}^{m-2} b_j w^j,$$

where $b = a_{m-1}/ma_m$, b_j is a rational function of a_j, a_{m-1} and a_m $(0 \leqq j \leqq m-2)$. Under these circumstances, we have the following theorem.

THEOREM 3. *Let $w = w(z)$ be any meromorphic solution of* (23′).
(I) *When there is at least one j such that $b_j \neq 0$,*

$$T(r, w) \leqq K_3 \sum_{j=0}^{m} T(r, a_j) + O(\log r) \qquad (r \notin E)$$

for some constant K_3.
(II) *When all $b_j = 0$ and $b \neq$ constant,*

$$T(r, w) \leqq K_3'(T(r, a_{m-1}) + T(r, a_m)) + O(\log r) \qquad (r \notin E)$$

for some constant K_3'.
(III) *When all $b_j = 0$, $b =$ constant such that $w(z) + b \not\equiv 0$ and $n - m$ is not a divisor of n,*

$$\frac{T(r, a_m)}{2n-m} - nm\left(r, \frac{w'}{w+b}\right) + O(1) \leqq T(r, w) \leqq K_3'' T(r, a_m) + nm\left(r, \frac{w'}{w+b}\right) + O(1)$$

for some constant K_3''.
(IV) *When all $b_j = 0$, $b =$ constant such that $w(z) + b \not\equiv 0$ and $n - m$ is a divisor of n, for any $\lambda > 1$,*

$$\frac{T(r, a_m)}{2n-m} - nm\left(r, \frac{w'}{w+b}\right) + O(1) \leqq T(r, w) \leqq K_3'''(\lambda) T(\lambda r, a_m)$$

for some $K_3'''(\lambda)$ depending only on λ.

Proof. (I) Let k be the largest number of j for which $b_j \neq 0$. Then (23)′ becomes

(24) $$(w')^n = a_m(w+b)^m + \sum_{j=0}^{k} b_j w^j \qquad (b_k \neq 0,\ 0 \leqq k \leqq m-2).$$

Let $w = w(z)$ be any meromorphic solution of (24) which is not equal to a constant and put

$$g_0 = -a_m(w+b)^m, \quad g_1 = (w')^n, \quad \phi = \sum_{j=0}^{k} b_j w^j.$$

(a) When $\phi = 0$, as in the case of Theorem 2(i), we have

$$T(r, w) \leqq \sum_{j=0}^{k} T(r, b_j) + O(1) \leqq K_{31} \sum_{j=0}^{m} T(r, a_j) + O(1)$$

for some constant K_{31} as b_j is a rational function of a_j, a_{m-1} and a_m.
(b) When $\phi \neq 0$ and g_0, g_1 are linearly dependent over C, as in the case of Theorem 2(ii), we have

$$T(r, w)\leqq\frac{1}{m-k}\left(mT(r, b)+T(r, a_m)+\sum_{j=0}^{k}T(r, b_j)\right)+O(1)$$

$$\leqq K_{32}\sum_{j=0}^{m}T(r, a_j)+O(1)$$

for some constant K_{32}.

(c) When $\phi\neq 0$ and g_0, g_1 are linearly independent over C, as in the case of Theorem 2(iii), applying Lemma 1 and using the inequality

$$T(r, w')\leqq\frac{m}{n}T(r, w)+\frac{1}{n}\sum_{j=0}^{m}T(r, a_j)+O(1)$$

we have

$$T(r, w)\leqq K_{33}\sum_{j=0}^{m}T(r, a_j)+O(\log r) \qquad (r\notin E).$$

Combining (a), (b) and (c), we obtain the case (I).

(II) Put $a_m=a$. From the inequality (18)′ in the proof of Theorem 2 ([7], p. 243):

$$N(r, 0, w')\leqq N(r, 0, b')+\bar{N}(r, 0, a)+N(r, 0, a)/n$$

and the estimate of $m(r, 1/w')$ in the proof of Theorem 3 ([7], p. 248):

$$m(r, 1/w')\leqq KT(r, b')+T(r, a)+O(\log^+T(r, w')+\log^+T(r, a)+\log r) \quad (r\notin E)$$

where K is a constant depending only on m, we obtain the inequality

$$(1-o(1))T(r, w')\leqq(K+1)T(r, b')+3T(r, a)+O(\log^+T(r, a)+\log r) \quad (r\notin E).$$

Here

$$T(r, b')\leqq(2+o(1))T(r, b)+O(\log r) \quad (r\notin E)$$

and using $b=a_{m-1}/ma_m$, we have

$$T(r, w')\leqq K'(T(r, a_m)+T(r, a_{m-1})+O(\log r) \quad (r\notin E)$$

for some constant K'. Further, as

$$nT(r, w')\geqq mT(r, w)-T(r, a_m)-mT(r, b)+O(1)$$

by

$$(w')^n=a_m(w+b)^m,$$

we arrive at the inequality:

$$T(r, w)\leqq K_3'(T(r, a_m)+T(r, a_{m-1}))+O(\log r) \quad (r\notin E).$$

(III) In this case, the differential equation has the form

$$(w')^n=a_m(w+b)^m \quad (b=\text{constant}).$$

Put $w+b=v$ and $a_m=a$, then the equation becomes

$$(v')^n=av^m.$$

Let $v=v(z)\equiv w(z)+b\not\equiv 0$ be a meromorphic solution of this equation, then

(25) $$mT(r,\ v)\leqq nT(r,\ v')+T(r,\ a)+O(1)\,,$$

(26) $$nT(r,\ v')\leqq mT(r,\ v)+T(r,\ a)+O(1)\,.$$

Further, from

$$(v')^{n-m}\left(\frac{v'}{v}\right)^m=a\,,$$

(27) $$m(r,\ a)\leqq(n-m)m(r,\ v')+mm(r,\ v'/v)+O(1)\,.$$

Let v have a pole of order $\mu\geqq 1$ at $z=z_0$ and ν be the order of pole of a at $z=z_0$. Then,

(28) $$n(\mu+1)=\nu+m\mu\,.$$

This shows that $\nu>0$; v has no poles other than those of a's. Now, from (28), as $\mu\geqq 1$,

$$\nu\geqq 2n-m$$

and

$$(n-m)(\mu+1)+\frac{m}{2n-m}\nu\geqq\nu\,.$$

This shows that

$$(n-m)N(r,\ v')+\frac{m}{2n-m}N(r,\ a)\geqq N(r,\ a)\,;$$

that is,

(29) $$N(r,\ a)\leqq\frac{2n-m}{2}N(r,\ v')\,.$$

From (27) and (29), making use of (26), we obtain

$$T(r,\ a)\leqq\left(n-\frac{m}{2}\right)T(r,\ v')+mm(r,\ v'/v)+O(1)$$

$$\leqq\left(n-\frac{m}{2}\right)\left(\frac{m}{n}T(r,\ v)+\frac{1}{n}T(r,\ a)\right)+mm(r,\ v'/v)+O(1)\,,$$

that is,

(30) $$\frac{T(r,\ a)}{2n-m}-nm\left(r,\ \frac{v'}{v}\right)+O(1)\leqq T(r,\ v)\leqq T(r,\ w)+O(1)\,.$$

We note that this is valid in the case (IV) because we did not use the condition that $n-m$ is not a divisor of n.

Next, put $u=1/v$, then the equation becomes

$$(-u')^n=au^{n+n-m}.$$

Now $n-m$ is not a divisor of n and applying Theorem 1 to this case we obtain for nonzero meromorphic solution of this equation $u=u(z)$

$$T(r, u) \leqq K_3'' T(r, a) + nm(r, u'/u) + O(1).$$

Using

$$u'/u = -v'/v \quad \text{and} \quad T(r, u) = T(r, v) + O(1)$$

for $v = v(z) \equiv 1/u(z)$, we have

$$T(r, v) \leqq K_3'' T(r, a) + nm(r, v'/v) + O(1). \tag{31}$$

Combining (30) and (31), we obtain the inequality in this case.

(IV) As in the case of (III), put $w+b=v$ and $a_m=a$, then $v=v(z)$ satisfies

$$(v')^n = av^m, \quad ((n-m) \mid m).$$

From this

$$\frac{n}{n-m}(v^{(n-m)/n})' = \frac{v'}{v^{m/n}} = a^{1/n}$$

and we have

$$\frac{1}{n} T(r. a) = T(r, (v^{(n-m)/n})') + O(1).$$

On the other hand, by a result of Valiron ([9], p. 33), for any constant $\lambda > 1$,

$$\frac{n-m}{n} T(r, v) = T(r, v^{(n-m)/n}) \leqq \Omega\left(\lambda, \frac{n-m}{n}\right) T(\lambda r, (v^{(n-m)/n})').$$

Therefore,

$$T(r, v) \leqq \frac{\Omega(\lambda, (n-m)/n)}{n-m} T(\lambda r, a) + O(1). \tag{32}$$

Putting $\Omega(\lambda, (n-m)/n)/(n-m) = K_3'''(\lambda)$ and combining (30) and (32), we obtain the result.

Remark 4. It is easily seen that this theorem contains Theorems B and C.

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