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THE SECTIONAL CURVATURE OF A 5-DIMENSIONAL HARMONIC RIEMANNIAN MANIFOLD

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In 1939, E. T. Copson and H. S. Ruse [4] initiated to study harmonic Riemannian manifolds. And, in 1944, A. Lichnerowicz [13] proved the curvature identities (cf. § 2) in a harmonic Riemannian manifold and gave the following

CONJECTURE. *If a Riemannian manifold M with positive definite metric is harmonic, then M is locally symmetric.*

For Riemannian manifolds of dimension 2 or 3 the conjecture is trivially affirmative, because a harmonic manifold is Einsteinian and therefore of constant curvature. A. G. Walker [22] showed that the conjecture is affirmative for Riemannian manifolds of dimension 4 (cf. [16], [2]). Later, harmonic Riemannian manifolds were studied by T. J. Willmore [28], [29], A. J. Ledger [11], [12], A. Allamigeon [1], S. Tachibana [19], Y. Watanabe [23], [24], [25], [26], [27], A. Besse [2], L. Vanhecke [20], [21], M. Kôzaki [9], K. Sakamoto [18] and others. But it is an open problem to show that the Lichnerowicz's conjecture is affirmative for Riemannian manifolds of dimension >4, and no counterexample is known up to now. The main purpose of this paper is to prove the main Theorem 3.3 giving a sufficient condition, by pinching the sectional curvature at a point, for a harmonic Riemannian manifold of dimension 5 to be locally symmetric, i. e. of constant curvature. In § 3, we prove Lemma 3.2, which implies immediately the main theorem, because a locally symmetric harmonic manifold is locally flat or locally isometric to a rank one symmetric space (cf. [12], [6], [3]), i. e., because it is of constant curvature in the case where it is odd-dimensional.

In § 1, we give some preliminaries concerning Riemannian manifolds. In § 2, we give definitions and curvature conditions concerning harmonic Riemannian manifolds. The last section § 3 is devoted to the proof of the main theorem and another.

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§ 1. Preliminaries.

First, we shall recall in Riemannian manifolds some curvature identities which will be useful in the sequel. In the present paper, every Riemannian manifold we consider is assumed to be of class C^ω and connected. Let M be an n-dimensional Riemannian manifold with positive definite metric g and ∇ be the Levi-Civita connection. The Riemannian curvature tensor R is defined by

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

for any vector fields X, Y and Z. With local components the curvature tensor can be written as $R=(R_{kji}{}^h)$. Let $R_1=(R_{aji}{}^a)=(R_{ji})$ and $S=g^{ji}R_{ji}$ be the Ricci tensor and the scalar curvature, respectively. Denote by T_xM the tangent space to M at a point x of M. The sectional curvature $\kappa_x(X, Y)$ for an orthonormal pair $\{X, Y\}$, where X and Y belonging to T_xM, is given by

$$\kappa_x(X, Y)=-g(R(X, Y)X, Y).$$

We recall the well known Bianchi's identities

$$\text{(1.1)} \qquad \text{(a)}\quad R_{kji}{}^h+R_{jik}{}^h+R_{ikj}{}^h=0,$$

$$\text{(b)}\quad \nabla_l R_{kji}{}^h+\nabla_k R_{jli}{}^h+\nabla_j R_{lki}{}^h=0,$$

where ∇_l denotes the covariant differentiation with respect to the Levi-Civita connection. Generally speaking, we put $|T|^2=T_{kji}T^{kji}$ for any tensor field of any type, say $T=(T_{kji})$. Then from (1.1) we get the following well known formulas (cf. [17])

$$\text{(1.2)} \qquad \nabla^u R^{abcd}\nabla^u R_{adcb}=\nabla_u R^{abcd}\nabla_u R_{cbad}=\nabla^u R^{abcd}\nabla_c R_{abud}$$
$$=\nabla^u R^{abcd}\nabla_d R_{abcu}=\nabla^u R^{abcd}\nabla_a R_{ubcd}$$
$$=\nabla^u R^{abcd}\nabla_c R_{ubad}=\frac{1}{2}|\nabla R|^2.$$

On putting (cf. [2])

$$\hat{R}=R^{abcd}R_{ab}{}^{uv}R_{cduv} \quad\text{and}\quad \mathring{R}=R^{abcd}R_a{}^u{}_c{}^v R_{budv},$$

we have the following formulas (cf. T. Sakai [17]):

$$\text{(1.3)} \qquad \text{(a)}\quad R^{abcd}R_{ab}{}^{uv}R_{cudv}=R^{abcd}R_a{}^u{}_b{}^v R_{cduv}=R^{abcd}R_{ac}{}^{uv}R_{bduv}=\frac{1}{2}\hat{R},$$

$$\text{(b)}\quad R^{abcd}R_a{}^u{}_b{}^v R_{cudv}=R^{abcd}R_a{}^u{}_b{}^v R_{cvud}=R^{abcd}R_a{}^u{}_c{}^v R_{bduv}$$
$$=R^{abcd}R_{ac}{}^{uv}R_{budv}=\frac{1}{4}\hat{R},$$

$$\text{(c)}\quad R^{abcd}R_a{}^u{}_c{}^v R_{bvdu}=R^{abcd}R_a{}^v{}_c{}^u R_{budv}=\mathring{R}-\frac{1}{4}\hat{R}.$$

A. Lichnerowicz [14] gave the following identity

$$\frac{1}{2}\Delta|R|^2=|\nabla R|^2-4R^{jihk}\nabla_j\nabla_h R_{ik}+2R_{ij}R^{ihkl}R^j{}_{hkl}+\hat{R}+4\mathring{R}\,, \tag{1.4}$$

where Δ is the Laplace-Beltrami operator acting on differentiable functions on M (cf. [30]).

If M is Einsteinian and $|R|^2=$constant, then (1.4) reduces to

$$|\nabla R|^2+\frac{2}{n}S|R|^2+\hat{R}+4\mathring{R}=0\,. \tag{1.5}$$

§ 2. Harmonic Riemannian manifolds.

Let M be a Riemannian manifold of dimension n. Take a normal neighborhood N centered at a point x_0 of M. Denoting by $s(x)$ the geodesic distance measured from x_0 to a point x of N, we define in N a function s by $s: x\rightarrow s(x)(x\in N)$. Given a fixed point x_0 of M, the Riemannian manifold M is said to be harmonic at the point x_0, if there is a normal neighborhood U centered at x_0 in such a way that there is in $U-\{x_0\}$ a nontrivial solution u, analytically depending only on $\Omega=(1/2)s^2$, of the Laplace equation $\Delta u=0$. When M is harmonic at every point of M, it is called a *harmonic Riemannian manifold.* It is well known (cf. [13], [16]) that in a harmonic Riemannian manifold the local function $\Delta\Omega$ has the form $f(\Omega)$ in each normal neighborhood U, where $f(\Omega)$ is a function analytically depending on Ω, and the function $f(\Omega)$ is independent of the choice of the center x_0. The function $f(\Omega)$ is called the *characteristic function* of the harmonic Riemannian manifold.

Let M be a harmonic Riemannian manifold of dimension n and $\{y^i\}$ be a normal coordinate system, covering a sufficient small normal neighborhood U centered at a point x_0 of M. Then $\Delta\Omega=f(\Omega)$ and $f(\Omega)$ thus admits the Maclaurin expansion

$$\begin{aligned} f(\Omega)&=f(0)+\dot{f}(0)\Omega+\frac{1}{2!}\ddot{f}(0)\Omega^2+\frac{1}{3!}\dddot{f}(0)\Omega^3+\cdots \\ &=f(0)+\frac{1}{2}\dot{f}(0)s^2+\frac{1}{2!\,2^2}\ddot{f}(0)s^4+\frac{1}{3!\,2^3}\dddot{f}(0)s^6+\cdots\,, \end{aligned} \tag{2.1}$$

taking account of $\Omega=(1/2)s^2$, where $(\dot{\ })$ means the operator taking the derivative with respect to Ω. On the other hand, in any Riemannian manifold the formula

$$\Delta\Omega=n+\left\{ {i \atop i\,k} \right\}y^k \tag{2.2}$$

holds with respect to normal coordinates $\{y^k\}$, where $\left\{ {i \atop j\,k} \right\}$ denotes the Christoffel symbols. If $\left\{ {i \atop i\,k} \right\}$ are expanded in Taylor expansion with respect to y^h, then using (2.1) and (2.2), the following curvature conditions are obtained (cf. [13], [10], [16]):

$$R_{ji}=-\frac{3}{2}\dot{f}(0)g_{ji}, \qquad S=-\frac{3n}{2}\dot{f}(0), \tag{2.3}$$

$$\mathfrak{S}\Big(R_{pij}{}^{q}R_{qkl}{}^{p}+\frac{45}{8}\ddot{f}(0)g_{ij}g_{kl}\Big)=0, \tag{2.4}$$

$$\mathfrak{S}(9\nabla_k R_{pij}{}^{q}\nabla_l R_{qmn}{}^{p}-32R_{pij}{}^{q}R_{qkl}{}^{r}R_{rmn}{}^{p}-315\dddot{f}(0)g_{ij}g_{kl}g_{mn})=0, \tag{2.5}$$

where $\mathfrak{S}$ means the summation taken over all permutation of the free indices appearing incide the parenthesis ( ). By the definition of harmonicity we see that $\dot{f}(0)$, $\ddot{f}(0)$ and $\dddot{f}(0)$ are absolute constants, i.e. that they are independent of choice of the center x_0. Then transvecting $g^{ij}g^{kl}$ with (2.4) and using (2.3), we obtain

$$|R|^2=-\frac{3n}{2}\Big\{\dot{f}(0)^2+\frac{5(n+2)}{2}\ddot{f}(0)\Big\} \tag{2.6}$$

and see that $|R|^2$ is constant.

We now need the following two lemmas for computing $\dddot{f}(0)$ in terms of the scalar functions constructed by the curvature tensors.

LEMMA 2.1. *For a tensor (field) $T=(T_{ijklmn})$ of type (0, 6), we have*

$$\begin{aligned} g^{ij}g^{kl}g^{mn}\mathfrak{S}(T_{ijklmn})=48(&T^{i}{}_{i}{}^{j}{}_{j}{}^{k}{}_{k}+T^{j}{}_{i}{}^{jk}{}_{jk}+T^{i}{}_{i}{}^{jk}{}_{kj}+T^{ij}{}_{ij}{}^{k}{}_{k}+T^{ij}{}_{i}{}^{k}{}_{jk}\\ &+T^{ij}{}_{i}{}^{k}{}_{kj}+T^{ij}{}_{ji}{}^{k}{}_{k}+T^{ij}{}_{j}{}^{k}{}_{ik}+T^{ij}{}_{jk}{}^{k}{}_{i}+T^{ijk}{}_{ijk}\\ &+T^{ijk}{}_{ikj}+T^{ijk}{}_{jik}+T^{ijk}{}_{jki}+T^{ijk}{}_{kij}+T^{ijk}{}_{kji}). \end{aligned} \tag{2.7}$$

Proof. See [23].

LEMMA 2.2. *In a harmonic Riemannian manifold, we have*

(2.8) (a) $g^{ij}g^{kl}g^{mn}\mathfrak{S}(A_{ijklmn})=144|\nabla R|^2$,

(b) $g^{ij}g^{kl}g^{mn}\mathfrak{S}(B_{ijklmn})=48\Big(\frac{S^3}{n^2}+\frac{9}{2n}S|R|^2-\frac{7}{2}\hat{R}+\mathring{R}\Big)$,

(c) $g^{ij}g^{kl}g^{mn}\mathfrak{S}(g_{ij}g_{kl}g_{mn})=48n(n+2)(n+4)$,

where

$$A_{ijklmn}=\nabla_k R_{pij}{}^{q}\nabla_l R_{qmn}{}^{p} \quad\text{and}\quad B_{ijklmn}=R_{pij}{}^{q}R_{qkl}{}^{r}R_{rmn}{}^{p}.$$

Proof. Putting $T_{ijklmn}=g_{ij}g_{kl}g_{mn}$ in Lemma 2.1, the formula (c) follows immediately. Next putting $T_{ijklmn}=A_{ijklmn}$ in Lemma 2.1, (1.1), (1.2) and (2.3) imply that

$$A^{ijk}{}_{ijk}=\frac{1}{2}|\nabla R|^2, \quad A^{ijk}{}_{ikj}=\frac{1}{4}|\nabla R|^2, \quad A^{ijk}{}_{jik}=|\nabla R|^2,$$

$$A^{ijk}{}_{jki}=\frac{1}{2}|\nabla R|^2, \quad A^{ijk}{}_{kij}=\frac{1}{2}|\nabla R|^2, \quad A^{ijk}{}_{kji}=\frac{1}{2}|\nabla R|^2,$$

and all the others corresponding to the terms appearing in the right hand side of (2.7) vanish. Thus we obtain the formula (a). Lastly putting $T_{ijklmn}=B_{ijklmn}$ in Lemma 2.1, (1.1), (1.3) and (2.3) imply

$$B^{i}{}_{i}{}^{j}{}_{j}{}^{k}{}_{k}=\frac{1}{n^2}S^3, \quad B^{i}{}_{i}{}^{jk}{}_{jk}=\frac{1}{2n}S|R|^2, \quad B^{i}{}_{i}{}^{jk}{}_{kj}=\frac{1}{n}S|R|^2,$$

$$B^{ij}{}_{ij}{}^{k}{}_{k}=\frac{1}{2n}S|R|^2, \quad B^{ij}{}_{i}{}^{k}{}_{jk}=-\frac{1}{4}\hat{R}, \quad B^{ij}{}_{i}{}^{k}{}_{kj}=-\frac{1}{2}\hat{R},$$

$$B^{ij}{}_{ji}{}^{k}{}_{k}=\frac{1}{n}S|R|^2, \quad B^{ij}{}_{j}{}^{k}{}_{ik}=-\frac{1}{2}\hat{R}, \quad B^{ij}{}_{j}{}^{k}{}_{ki}=-\hat{R},$$

$$B^{ijk}{}_{ijk}=\mathring{R}-\frac{1}{4}\hat{R}, \quad B^{ijk}{}_{ikj}=-\frac{1}{4}\hat{R}, \quad B^{ijk}{}_{jik}=-\frac{1}{4}\hat{R},$$

$$B^{ijk}{}_{jki}=-\frac{1}{2}\hat{R}, \quad B^{ijk}{}_{kij}=\frac{1}{2n}S|R|^2, \quad B^{ijk}{}_{kji}=\frac{1}{n}S|R|^2.$$

Thus we obtain the formula (b).

Remark. Recently A. Gray and L. Vanhecke [7] has given in a Riemannian manifold many formulas which are useful in obtaining systematically scalar functions such as given in Lemma 2.2.

If we transvect $g^{ij}g^{kl}g^{mn}$ with (2.5), then taking account of the formulas (2.8), we have the following lemma (cf. Y. Watanabe [23]).

LEMMA 2.3. *In a harmonic Riemannian manifold of dimension n, we have*

$$27|\nabla R|^2-32\Big(\frac{S^3}{n^2}+\frac{9}{2n}S|R|^2-\frac{7}{2}\hat{R}+\mathring{R}\Big)=315n(n+2)(n+4)\dddot{f}(0). \tag{2.9}$$

§ 3. 5-dimensional harmonic Riemannian manifolds.

We shall now prove a curvature identity (3.4) in a harmonic Riemannian manifold M of dimension 5. To do so, we introduce in a Riemannian manifold of dimension n a function $G_{(m)}$ by

$$G_{(m)}=\delta^{j_1j_2\cdots j_{2m}}_{i_1i_2\cdots i_{2m}}R^{i_1i_2}{}_{j_1j_2}\cdots R^{i_{2m-1}i_{2m}}{}_{j_{2m-1}j_{2m}} \tag{3.1}$$

for any natural number $m\geqq 1$, where

$$\delta^{j_1j_2\cdots j_{2m}}_{i_1i_2\cdots i_{2m}}=|\delta^{j_a}_{i_b}| \qquad (a, b=1, 2, \cdots, 2m)$$

is the so-called generalized Kronecker delta (cf. [15]). When M is compact and $n=2m$, the $G_{(m)}$ is, as is well known, the integrand of the Gauss-Bonnet theorem (see, for example, S. Kobayashi and K. Nomizu [8]). However, the function $G_{(m)}$ vanishes identically for a Riemannian manifold M of dimension $n<2m$, because in such a case $\delta^{j_1 j_2 \cdots j_{2m}}_{i_1 i_2 \cdots i_{2m}}$ is equal to zero. Therefore if the dimension of M is 5, then

$$G_{(3)}=0\,, \tag{3.2}$$

which will be used in the sequel.

On the other hand, the function $G_{(3)}$ has in a Riemannian manifold of dimension n the following form (cf. [17], [5]):

$$\begin{aligned} G_{(3)}=&8\mathring{R}-4\hat{R}-24R^{ab}R^{cd}R_{abcd}-24R^{uv}R_u{}^{abc}R_{vabc} \\ &+16R^{ab}R_a{}^cR_{bc}+S^3-12S|R_1|^2+3S|R|^2\,, \end{aligned} \tag{3.3}$$

as a consequence of (1.3). Thus using (3.3), we see that in an Einsteinian manifold of dimension 5 the formula (3.2) reduces to

$$4\mathring{R}-2\hat{R}=\frac{S}{10}(9|R|^2-S^2)\,. \tag{3.4}$$

The identity (3.4) is also obtained by transvecting R^{hijk} with the identity

$$\begin{aligned} &R_{hipq}R^{pq}{}_{jk}+2R_{hpqk}R_j{}^{pq}{}_i-2R_{hpqj}R_k{}^{pq}{}_i+\frac{3}{5}SR_{hijk} \\ &=\Big(\frac{3}{20}|R|^2-\frac{1}{20}S^3\Big)(g_{hj}g_{ik}-g_{hk}g_{ij})\,, \end{aligned}$$

proved by E. M. Patterson [15] in a harmonic Riemannian manifold of dimension 5.

From now on, let M be a 5-dimensional harmonic Riemannian manifold. We note here that any harmonic Riemannian manifold is necessarily Einsteinian (see § 2). Eliminating $\mathring{R}$ from (1.5) and (3.4), we have

$$|\nabla R|^2+\frac{13}{10}S|R|^2+3\hat{R}-\frac{S^3}{10}=0\,. \tag{3.5}$$

Eliminating $\mathring{R}$ from (2.9) and (3.4), we have

$$27|\nabla R|^2+96\hat{R}-36S|R|^2-\frac{12}{25}S^3=315^2\ddot{f}(0)\,. \tag{3.6}$$

Next eliminating $\hat{R}$ from (3.5) and (3.6), we have

$$5|\nabla R|^2+\frac{388}{5}S|R|^2-\frac{68}{25}S^3=-315^2\ddot{f}(0)\,. \tag{3.7}$$

Since $|R|^2$ and S are constant in a harmonic Riemannian manifold because of (2.3) and (2.6), we have from (3.7), (3.5) and (3.4)

PROPOSITION 3.1. *In a 5-dimensional harmonic Riemannian manifold, the* *scalar functions* $|\nabla R|^2$, $\hat{R}$ *and* $\mathring{R}$ *are all constant.*

In a harmonic Riemannian manifold M of dimension 5, we take a fixed point x_0 and a fixed unit vector X belonging to the tangent space T_0M to M at x_0. Transvecting $X^kX^iX^jX^lX^mX^n$ with (2.5), we have at the point x_0

$$9\nabla_k R_{pij}{}^q \nabla_l R_{qmn}{}^p X^k X^i X^j X^l X^m X^n = 32 R_{pij}{}^q R_{qkl}{}^r R_{rmn}{}^p X^k X^i X^j X^l X^m X^n + 315\dddot{f}(0)\,.$$

Substituting $\dddot{f}(0)$ given by (3.7) into the equation above, we have at the point x_0 the following key equation

$$(3.8)\qquad 9\nabla_k R_{pij}{}^q \nabla_l R_{qmn}{}^p X^k X^i X^j X^l X^m X^n + \frac{1}{63}|\nabla R|^2$$
$$= 32 R_{pij}{}^q R_{qkl}{}^r R_{rmn}{}^p X^k X^i X^j X^l X^m X^n + \frac{1}{315}\Big(-\frac{388}{5}S|R|^2 + \frac{68}{25}S^3\Big).$$

We now define a linear transformation $\varPi_X$ in the tangent space T_0M by

$$(3.9)\qquad \varPi_X(Y) = -R(X, Y)X$$

for $Y \in T_0M$, which implies immediately

$$(3.10)\qquad \varPi_X(X) = 0\,.$$

For simplicity, we put $\varPi_X = \varPi$. Then X is obviously an eigen vector of $\varPi$ with eigen value 0, because of (3.10). Since the linear transformation $\varPi$ is symmetric because of (3.9), there is an orthonormal basis $\{X, e_1, e_2, e_3, e_4\}$ such that

$$(3.11)\qquad \varPi(e_\alpha) = \lambda_\alpha e_\alpha \qquad (\alpha = 1, 2, 3, 4)\,.$$

Then by (2.3) and (2.4) we get

$$(3.12)\qquad Tr(\varPi) = R_{ji}X^jX^i = \frac{S}{5} = -\frac{3}{2}\dot{f}(0)\,,$$

$$Tr(\varPi^2) = R_{pij}{}^q R_{qkl}{}^p X^i X^j X^k X^l = -\frac{45}{8}\ddot{f}(0)\,,$$

$$Tr(\varPi^3) = R_{pij}{}^q R_{qkl}{}^r R_{rmn}{}^p X^i X^j X^k X^l X^m X^n\,,$$

where Tr means the trace of each of linear transformations $\varPi$, $\varPi^2 = \varPi \circ \varPi$ and $\varPi^3 = \varPi \circ \varPi \circ \varPi$. Thus (3.8) implies the following equation (3.13) at the point x_0 because of (2.6) and (3.11).

$$(3.13)\qquad 9\nabla_k R_{pij}{}^q \nabla_l R_{qmn}{}^p X^k X^i X^j X^l X^m X^n + \frac{1}{63}|\nabla R|^2$$
$$= 32Tr(\varPi^3) + \frac{1}{315}\Big\{-388Tr(\varPi)\Big(-\frac{15}{2}\dot{f}(0)^2 - \frac{525}{4}\ddot{f}(0)\Big) + 340(Tr(\varPi))^3\Big\}\,.$$

Then arranging (3.13) and using (3.12), we get at the point x_0

$$(3.14)\qquad 9\nabla_k R_{pij}{}^q \nabla_l R_{qmn}{}^p X^k X^i X^j X^l X^m X^n + \frac{1}{63}|\nabla R|^2 = \frac{4}{27}F(\lambda),$$

where

$$F(\lambda)=216Tr(\Pi^3)+35(Tr(\Pi))^3-194Tr(\Pi)Tr(\Pi^2).$$

As a consequence of (3.11), we have

$$\begin{aligned}F(\lambda)&=216\sum_\alpha \lambda_\alpha^3+35(\sum_\alpha \lambda_\alpha)^3-194(\sum_\alpha \lambda_\alpha)\sum_\beta \lambda_\beta^2\\ &=54\{4\sum_\alpha \lambda_\alpha^3-(\sum_\alpha \lambda_\alpha)\sum_\beta \lambda_\beta^2\}+35\sum_\alpha \lambda_\alpha\{(\sum_\beta \lambda_\beta)^2-4\sum_\beta \lambda_\beta^2\}\\ &=54\sum_{\beta<\gamma}(\lambda_\beta-\lambda_\gamma)^2(\lambda_\beta+\lambda_\gamma)+35(\sum_\alpha \lambda_\alpha)\sum_{\beta<\gamma}(\lambda_\beta-\lambda_\gamma)^2\\ &=\sum_{\beta<\gamma}(\lambda_\beta-\lambda_\gamma)^2\{54(\lambda_\beta+\lambda_\gamma)-35\sum_\alpha \lambda_\alpha\},\end{aligned}$$

which implies the following formulas

$$\begin{aligned}(3.15)\qquad F(\lambda)=&(\lambda_1-\lambda_2)^2\{19(\lambda_1+\lambda_2)-35(\lambda_3+\lambda_4)\}+(\lambda_1-\lambda_3)^2\{19(\lambda_1+\lambda_3)-35(\lambda_2+\lambda_4)\}\\ &+(\lambda_1-\lambda_4)^2\{19(\lambda_1+\lambda_4)-35(\lambda_2+\lambda_3)\}+(\lambda_2-\lambda_3)^2\{19(\lambda_2+\lambda_3)-35(\lambda_1+\lambda_4)\}\\ &+(\lambda_2-\lambda_4)^2\{19(\lambda_2+\lambda_4)-35(\lambda_1+\lambda_3)\}+(\lambda_3-\lambda_4)^2\{19(\lambda_3+\lambda_4)-35(\lambda_1+\lambda_2)\}\end{aligned}$$

and

$$\begin{aligned}(3.16)\qquad F(\lambda)=&(\lambda_1-\lambda_2)^2\{19\sum_\alpha \lambda_\alpha-54(\lambda_3+\lambda_4)\}+(\lambda_1-\lambda_3)^2\{19\sum_\alpha \lambda_\alpha-54(\lambda_2+\lambda_4)\}\\ &+(\lambda_1-\lambda_4)^2\{19\sum_\alpha \lambda_\alpha-54(\lambda_2+\lambda_3)\}+(\lambda_2-\lambda_3)^2\{19\sum_\alpha \lambda_\alpha-54(\lambda_1+\lambda_4)\}\\ &+(\lambda_2-\lambda_4)^2\{19\sum_\alpha \lambda_\alpha-54(\lambda_1+\lambda_3)\}+(\lambda_3-\lambda_4)^2\{19\sum_\alpha \lambda_\alpha-54(\lambda_1+\lambda_2)\}.\end{aligned}$$

As a consequence of (3.14), we have the following inequality

$$(3.17)\qquad F(\lambda)\geqq 0.$$

We first note that λ_α is the sectional curvature for the orthonormal pair $\{X, e_\alpha\}$ ($\alpha=1, 2, 3, 4$), because of the definition (3.9) of $\Pi=\Pi_X$. Suppose that all λ_α satisfy $\delta\geqq\lambda_\alpha\geqq(19/35)\delta$ for some $\delta\geqq 0$. Then we see from (3.15) that the right hand side of (3.14) is non-positive. Consequently it follows from (3.14) and (3.17) that $|\nabla R|^2=0$ at the point x_0 of M. Since M is connected, Proposition 3.1 implies that $\nabla R=0$, i.e. that M is locally symmetric. Thus, we have the following

LEMMA 3.2. *Let M be a 5-dimensional harmonic Riemannian manifold all of whose sectional curvatures $\kappa_x(X, Y)$ at a point x satisfy $\delta\geqq\kappa_x(X, Y)\geqq(19/35)\delta$ for some $\delta\geqq 0$. Then M is locally symmetric.*

Since a locally symmetric harmonic manifold is locally flat or locally isometric to a rank one symmetric space (cf. [12], [6]), it is of constant curvature if it is odd-dimensional. Thus, Lemma 3.2 implies

THEOREM 3.3. *Let M be a* 5-*dimensional harmonic Riemannian manifold all of whose sectional curvatures* $\kappa_x(X, Y)$ *at a point x satisfy* $\delta \geqq \kappa_x(X, Y) \geqq (19/35)\delta$ *for some* $\delta \geqq 0$. *Then M is of constant curvature.*

Similarly using (3.16) and noting $S=5\sum_{\alpha}\lambda_{\alpha}$, we obtain

THEOREM 3.4. *Let M be a* 5-*dimensional harmonic Riemannian manifold. If all sectional curvatures* $\kappa_x(X, Y)$ *at a point x satisfy* $\kappa_x(X, Y) \geqq (19/540)S$, *where S is the scalar curvature, then M is of constant curvature.*

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