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ON SUBMANIFOLDS WITH FLAT NORMAL CONNECTION IN A CONFORMALLY FLAT SPACE

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1. Introduction.

In this paper we construct Gauss maps with respect to non-degenerate parallel normal unit vector fields on an n-dimensional submanifold N which has flat normal connection in an m-dimensional conformally flat space M $(2\leqq n<m)$. A relation between the Riemannian curvatures of N, M and the Gauss images of N is obtained in theorem 1. We also find a result about the metric tensors of the Gauss images, which is in the case of a space form M closely related to a formula of Obata.

2. Preliminaries.

We always suppose that all manifolds, vector fields, etc. are differentiable of class C^{∞}. Assume that $\bar{\nabla}$ (resp. ∇) is the Riemannian connection of M (resp. N) and that X and Y are vector fields of N. Then

$$\bar{\nabla}_X Y=\nabla_X Y+h(X, Y),$$

and h is the vector valued second fundamental tensor of N in M. Let ξ be a normal vector field on N. Decomposing $\bar{\nabla}_X\xi$ in a tangent and a normal component we find

$$\bar{\nabla}_X\xi=-A_{\xi}(X)+\nabla_X^{\perp}\xi .$$

A_{ξ} is a self-adjoint linear map $N_p\rightarrow N_p$ at each point p and $\nabla^{\perp}$ is a metric connection in the normal bundle $N^{\perp}$. We have also, if g denotes the metric tensor of M and the induced metric tensor on N,

$$g(h(X, Y), \xi)=g(A_{\xi}(X), Y).$$

M is said to be conformally flat if for each point p we have a neighbourhood U and a diffeomorfism $\varphi: U\rightarrow R^m$, where R^m is the euclidean m-space, such that the metric tensor g of $\varphi(U)$ (identified with U) is obtained from the standard metric tensor of R^m by a conformal change of this tensor. Equivalently, g is locally of the form $g=\rho^2 g'$, where ρ is a strict positive function and g' is a flat metric tensor. The normal curvature tensor $R^{\perp}$ of N in M is given by

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$$R^{\perp}(X, Y)=\nabla_X^{\perp}\nabla_Y^{\perp}-\nabla_Y^{\perp}\nabla_X^{\perp}-\nabla_{[X,Y]}^{\perp}.$$

N has flat normal connection in M if $R^{\perp}$ vanishes everywhere. It is wellknown that in this case there is in a neighbourhood of each point p of N an orthonormal base field $\eta_1, \cdots, \eta_{m-n}$ of $N^{\perp}$ such that each η_i is parallel in $N^{\perp}$, that is, such that $\nabla_X^{\perp}\eta_i=0$ for each vector field of N. Moreover, if M is conformally flat, then $R^{\perp}=0$ iff all the second fundamental tensors A_{ξ} are simultaneously diagonalizable ([2], theorem 4).

3. The Gauss maps of non-degenerate parallel unit normal vector fields.

Suppose that η is a parallel unit normal vector field on N with domain U, then we say that η is non-degenerate if det $A_{\eta}\neq 0$ everywhere in U. In this case we define a new metric tensor $\tilde{g}$ on U by $\tilde{g}(X, Y)=g(\overline{\nabla}_X\eta, \overline{\nabla}_Y\eta)$ for all vectors X and Y at each point p of U (cf. [1]).

With this new metric tensor the differentiable manifold U becomes a new Riemannian manifold $\tilde{U}$ which is called the Gauss image of U with respect to η. The Gauss map of η is then simply the natural bijection $i: U\rightarrow\tilde{U}$. In the following we identify vector fields and tensor fields on U and $\tilde{U}$, so we do not use the Jacobian i_* and the dual linear map i^*.

Remark that we also have, since η is parallel, $\tilde{g}(X, Y)=g(A_{n_p}(X), A_{n_p}(Y))$.

Recall that we always suppose that N is an n-dimensional submanifold of the m-dimensional conformally flat space M.

THEOREM 1. *Suppose that N has flat normal connection in M and that $e_1, \cdots, e_n$ is an orthonormal base field with domain U of N which diagonalizes simultaneously all the second fundamental tensors A_{ξ}. Let $\eta_1, \cdots, \eta_{m-n}$ be an orthonormal base field of $N^{\perp}$ with domain U such that each η_r is parallel in $N^{\perp}$ and non-degenerate and K_{ij} (resp. $\overline{K}_{ij}$) and $\tilde{K}_{ij}^r$ be the Riemannian curvature of N (resp. M) and of the Gauss image $\tilde{U}_r$ of η_r in the plane direction (e_i, e_j) $i\neq j$ $i, j=1, \cdots, n$. If N is invariant and $K_{ij}\neq 0$, then*

$$\sum_{r=1}^{m-n}\frac{1}{\tilde{K}_{ij}^r}=\frac{K_{ij}-\overline{K}_{ij}}{K_{ij}}.$$

For a surface N we have $\sum_{r=1}^{m-n}\frac{1}{\tilde{K}^r}=\frac{K-\overline{K}}{K}$

Proof. First let r be fixed $1\leqq r\leqq m-n$. There are non-zero real valued functions λ_h^r in U such that $A_{\eta_r}(e_h)=\lambda_h^r e_h$ $h=1, \cdots, n$. Let $a_h=e_h/\lambda_h^r$ then $a_1, \cdots, a_n$ is an orthonormal base field of $\tilde{U}_r$. We prove that the Riemannian connection $\tilde{\nabla}$ of $\tilde{U}_r$ is given by

$$\tilde{\nabla}_X Y=\sum_{h=1}^{n} g(\overline{\nabla}_X\overline{\nabla}_Y\eta_r, \overline{\nabla}_{a_h}\eta_r)a_h \qquad \text{for any two vector fields } X \text{ and } Y \text{ of } U .$$

It is not difficult to see that $\tilde{\nabla}$ is indeed a connection. It is a metric connection for the metric tensor $\tilde{g}_r$ of $\tilde{U}_r$, because, if Z is an other vector field of U, then a straightforward calculation gives

$$Z\tilde{g}_r(X, Y)=Zg(\bar{\nabla}_X\eta_r, \bar{\nabla}_Y\eta_r)=g(\bar{\nabla}_Z\bar{\nabla}_X\eta_r, \bar{\nabla}_Y\eta_r)+g(\bar{\nabla}_X\eta_r, \bar{\nabla}_Z\bar{\nabla}_Y\eta_r)$$
$$=\tilde{g}_r(\tilde{\nabla}_Z X, Y)+\tilde{g}_r(X, \tilde{\nabla}_Z Y).$$

Next we prove that the torsion tensor of $\tilde{\nabla}$ vanishes:

$$\tilde{\nabla}_X Y-\tilde{\nabla}_Y X=\sum_{h=1}^{n} g(\bar{\nabla}_X\bar{\nabla}_Y\eta_r-\bar{\nabla}_Y\bar{\nabla}_X\eta_r, \bar{\nabla}_{a_h}\eta_r)a_h,$$

and because of the equation of Ricci, we know that if $\bar{R}$ is the curvature tensor of M, $R^{\perp}$ the normal curvature tensor of N in M and ξ any normal vector field on N, then

$$g(\bar{R}(X, Y)\eta_r, \xi)=g(R^{\perp}(X, Y)\eta_r, \xi)+g(A_\xi A_{\eta_r}(X)-A_{\eta_r}A_\xi(X), Y).$$

N has flat normal connection in M, thus $R^{\perp}=0$ and since M is conformally flat we have $A_\xi A_{\eta_r}=A_{\eta_r}A_\xi$. Since N is invariant, i, e., $\bar{R}(X, Y)N_p\subset N_p$, we get

$$\bar{R}(X, Y)\eta_r=\bar{\nabla}_X\bar{\nabla}_Y\eta_r-\bar{\nabla}_Y\bar{\nabla}_X\eta_r-\bar{\nabla}_{[X,Y]}\eta_r=0,$$

and thus

$$\tilde{\nabla}_X Y-\tilde{\nabla}_Y X=\sum_{h=1}^{n} g(\bar{\nabla}_{[X,Y]}\eta_r, \bar{\nabla}_{a_h}\eta_r)a_h$$
$$=\sum_{h=1}^{n}\tilde{g}_r([X, Y], a_h)a_h=[X, Y].$$

Next, the Riemannian curvature of $\tilde{U}_r$ in the plane direction (e_i, e_j) is given by

$$\tilde{K}^r_{ij}=-\tilde{g}_r(\tilde{\nabla}_{a_i}\tilde{\nabla}_{a_j}a_i-\tilde{\nabla}_{a_j}\tilde{\nabla}_{a_i}a_i-\tilde{\nabla}_{[a_i,a_j]}a_i, a_j)$$
$$=-a_i g(\bar{\nabla}_{a_j}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_j}\eta_r)+\sum_{h=1}^{n} g(\bar{\nabla}_{a_j}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_h}\eta_r)\tilde{g}_r(a_h, \tilde{\nabla}_{a_i}a_j)$$
$$+a_j g(\bar{\nabla}_{a_i}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_j}\eta_r)-\sum_{h=1}^{n} g(\bar{\nabla}_{a_i}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_h}\eta_r)\tilde{g}_r(a_h, \tilde{\nabla}_{a_j}a_j)$$
$$+g(\bar{\nabla}_{[a_i,a_j]}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_j}\eta_r)$$
$$=-g(\bar{\nabla}_{a_i}\bar{\nabla}_{a_j}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_j}\eta_r)-g(\bar{\nabla}_{a_j}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_i}\bar{\nabla}_{a_j}\eta_r)$$
$$+\sum_{h=1}^{n} g(\bar{\nabla}_{a_j}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_h}\eta_r)g(\bar{\nabla}_{a_i}\bar{\nabla}_{a_j}\eta_r, \bar{\nabla}_{a_h}\eta_r)+g(\bar{\nabla}_{a_j}\bar{\nabla}_{a_i}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_j}\eta_r)$$
$$+g(\bar{\nabla}_{a_i}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_j}\bar{\nabla}_{a_j}\eta_r)-\sum_{h=1}^{n} g(\bar{\nabla}_{a_i}\bar{\nabla}_{a_i}\eta_r, \bar{\nabla}_{a_h}\eta_r)g(\bar{\nabla}_{a_j}\bar{\nabla}_{a_j}\eta_r, \bar{\nabla}_{a_h}\eta_r)$$

$$+g(\overline{\nabla}_{[a_i, a_j]}\overline{\nabla}_{a_i}\eta_r, \overline{\nabla}_{a_j}\eta_r) \tag{2}$$

Recall that $\overline{K}_{ij}$ and K_{ij} are connected by.

$$K_{ij}=\overline{K}_{ij}-g(h(e_i, e_j), h(e_i, e_j))+g(h(e_i, e_i), h(e_j, e_j))$$

$$=\overline{K}_{ij}+\sum_{q=1}^{m-n} g(A_{\eta_q}(e_i), e_i)g(A_{\eta_q}(e_j), e_j)=\overline{K}_{ij}+\sum_{q=1}^{m-n}\lambda_i^q\lambda_j^q. \tag{3}$$

Next, because of the definition of a_h, we have

$$\overline{\nabla}_{a_h}\eta_r=-A_{\eta_r}(a_h)=-e_h \qquad h=1, \cdots, n,$$

and thus, the sum of the first, the fourth and the last term of (2) is equal to

$$-g(\overline{R}(a_i, a_j)e_i, e_j)=\frac{\overline{K}_{ij}}{\lambda_i^r\lambda_j^r}.$$

The sum of the second and the third term of (2) is given by

$$-g(h(a_j, e_i), h(a_i, e_j))=0$$

and the sum of the fifth and the sixth term becomes

$$g(h(a_i, e_i), h(a_j, e_j)).$$

From all this we get

$$\tilde{K}^r_{ij}=\frac{K_{ij}}{\lambda_i^r\lambda_j^r} \tag{4}$$

and finally because of (3) we find

$$\sum_{r=1}^{m-n}\frac{1}{\tilde{K}^r_{ij}}=\frac{\sum_{r=1}^{m-n}\lambda_i^r\lambda_j^r}{K_{ij}}=\frac{K_{ij}-\overline{K}_{ij}}{K_{ij}},$$

which completes the proof.
Because of (4) we have immediately the following:

COROLLARY. *If $K_{ij}=0$ then each $\tilde{K}^r_{ij}=0$ $r=1, \cdots, m-n$. If N is in particular a flat surface then each Gauss image $\tilde{U}_r$ is a flat Riemannian space.*

If X and Y are vector fields, $e_1, \cdots, e_n$ is an orthonormal base field of N and if R is the curvature tensor of N, then the Riccitensor of N is given by

$$\mathrm{Ric}(N)(X, Y)=\sum_{h=1}^{n} g(R(e_h, X)Y, e_h).$$

Define a new symmetric two-covariant tensor $\mathrm{Ric}_N(M)$ on N by ($\overline{R}$ is again the curvature tensor of M):

$$\mathrm{Ric}_N(M)(X, Y)=\sum_{h=1}^{n} g(\bar{R}(e_h, X)Y, e_h).$$

This is independent of the choice of the orthonormal base field $e_1, \cdots, e_n$ of N and for a unit vector e of N_p, $\mathrm{Ric}_N(M)(e, e)$ is equal to the sum of the Riemannian curvatures of M in $n-1$ mutually orthogonal plane directions of N_p containing e. If N is a surface, then $\mathrm{Ric}(N)=Kg$ and $\mathrm{Ric}_N(M)=\bar{K}g$.

THEOREM 2. *Assume that N, η_r, $\tilde{U}_r$ are such as in the statement of theorem 1 and that $\tilde{g}_r$ denotes the metric tensor of $\tilde{U}_r$ $r=1, \cdots, m-n$. If H is the mean curvature vector field of N, we have*

$$\sum_{r=1}^{m-n} \tilde{g}_r = ng(H, h)-\mathrm{Ric}(N)+\mathrm{Ric}_N(M). \tag{5}$$

Proof. Let $e_1, \cdots, e_n$ be such as in theorem 1. Because of the definition (1) we have if

$$X=\sum_{i=1}^{n} x_i e_i \quad \text{and} \quad Y=\sum_{i=1}^{n} y_i e_i \quad \text{are vector fields of } N,$$

$$\tilde{g}_r(X, Y)=\sum_{i,j=1}^{n} x_i y_j g(A_{\eta_r}(e_i), A_{\eta_r}(e_j))=\sum_{i=1}^{n} (\lambda_i^r)^2 x_i y_i.$$

Next we find

$$(spA_{\eta_r})g(\eta_r, h(X, Y))=\Big(\sum_{j=1}^{n} \lambda_j^r\Big)\Big(\sum_{i=1}^{n} g(A_{\eta_r}(e_i), e_i)x_i y_i\Big)$$

$$=\tilde{g}_r(X, Y)+\sum_{\substack{i,j=1\\ i\neq j}}^{n} \lambda_i^r \lambda_j^r x_i y_i. \tag{6}$$

Finally, we have

$$\mathrm{Ric}(N)(X, Y)=\sum_{j=1}^{n} g(R(e_j, X)Y, e_j).$$

Because of the equation of Gauss this becomes

$$=\sum_{j=1}^{n} g(\bar{R}(e_j, X)Y, e_j)+\sum_{j=1}^{n} (g(h(e_j, e_j), h(X, Y))-g(h(e_j, Y), h(X, e_j)))$$

$$=\mathrm{Ric}_N(M)(X, Y)+\sum_{r=1}^{m-n}\Big(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \lambda_i^r \lambda_j^r x_i y_i\Big). \tag{7}$$

Since $\sum_{r=1}^{m-n} (spA_{\eta_r})\eta_r = nH$, formula (5) follows from (6) and (7). This completes the proof.

Remarks.

1. For a surface N, (5) becomes $\sum_{r=1}^{m-n} \tilde{g}_r = 2g(H, h) - (K - \bar{K})g$.

2. About theorem 1: if N is a submanifold of the euclidean m-space R^m, we have $\sum \frac{1}{\tilde{K}^r_{ij}} = 1$ and in this case the spaces $\tilde{U}_r$ are locally isometric with the Gauss images $\eta_r(N)$ of N which are generated by the endpoint of η_r after a parallel displacement of η_r in R^m to a fixed point 0. The submanifolds $\eta_r(N)$ $r=1, \cdots, m-n$ form a so-called "rectangular configuration" in the unit hypersphere with centre 0 ([4]). If N is a submanifold of a complete simply connected elliptic space E^m of curvature k (>0), then we have $\sum \frac{1}{\tilde{K}^r_{ij}} + \frac{k}{K_{ij}} = 1$ and we can in a somewhat analogous way also associate $m-n$ Gauss images of N which are locally isometric to $\tilde{U}_r$ and which form together with N a rectangular configuration in E^m ([4], [6]).

3. About theorem 2: if M is a space of constant curvature k, then (5) becomes

$$\sum_{r=1}^{m-n} \tilde{g}_r = ng(H, h) - \mathrm{Ric}\,(N) + k(n-1)g\,. \tag{8}$$

In [3] M. Obata constructed a generalized Gauss map $f: N \to Q$, where Q is the set of all the totally geodesic n-spaces in the complete simply connected space form M and he introduced a quadratic differential form $d\Sigma^2$ on Q, with respect to which Q (or in the euclidean case the natural projection of Q onto the Grassmann manifold $G_{n,m}$) becomes a symmetric (pseudo—if $k<0$) Riemannian space. From (8) and the formula of Obata ([3]): $f^*(d\Sigma^2) = ng(H, h) - \mathrm{Ric}\,(N) + k(n-1)g$, we get at once in this case $\sum_{r=1}^{m-n} \tilde{g}_r = f^*(d\Sigma^2)$.

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