H. MATSUNAGA
KODAI MATH. J.
5 (1982), 495—502

GROUP ACTIONS ON SPHERE BUNDLES OVER SPHERES

BY HIROMICHI MATSUNAGA

Introduction.

In a previous paper [8], we studied group actions on S^k-bundles over S^n in cases of (1) $k<n\leqq 8$, (2) $k\geqq n$ and (3) $k=n-1$, $n\equiv 0 \bmod 4$. It is the purpose of this paper to estimate degrees of symmetry of bundle spaces and to construct group actions on semistable bundles over spheres.

Using Ku-Mann-Sicks-Su's theorems [5], [6], we shall give some estimation for upper bounds on degrees of symmetry in § 1. In § 2, we shall construct generators of stable groups $\pi_{4s-1}(SO)$ which yield group actions on bundle spaces. Some theorems due to Kervaire [4] provide an information on generators of semistable groups $\pi_i(SO(n))$. In § 3 we shall obtain some group actions on semistable sphere bundles over spheres. I would like to thank Professor S. Sasao for helpful conversations.

§ 1. Degree of symmetry.

For a closed connected smooth manifold M, the degree of symmetry of M denoted by $N(M)$ is defined as the upper bound of the dimensions of all compact Lie groups which act effectively and smoothly on M. Then we have next propositions.

PROPOSITION 1. *Let B be an S^k-bundle over S^n, where $n\geqq 9$. Then B can not be a homotopy sphere.*

Proof. If B is a sphere, then by 28.2 and 28.6 in [9], we have $k=n-1$, and $n=1, 2$ or $n\equiv 0 \bmod 4$, where the key point is that a fibre S^k is contractible to a point in B, then we can replace the standard sphere in [9] by a homotopy sphere. Since $n=4s$ and $s\geqq 3$, the space B has a cell complex structure $B=S^{n-1}\cup_{2m\iota_{n-1}}\cup e^n\cup e^{2n-1}$ (cf. 3 in [7]), which is a contradiction.

PROPOSITION 2. *Suppose that an S^k-bundle over S^n admits a cross section or $k<n-1$, then we have*

$$N(B)\leqq\frac{1}{2}n(n+1)+\frac{1}{2}k(k+1) \qquad \textit{for } n+k\geqq 19\,.$$

Received January 8, 1982

Proof. If the bundle B admits a cross section or $k<n-1$, then the cohomology group $H^n(B, Q)$ with rational coefficient is non zero. Hence by Theorem 1 in [5] we obtain the proposition.

PROPOSITION 3. *Let B be an S^{n-1}-bundle over S^n, where $n\geqq 10$. Suppose that B is not homotopically equivalent to the product $S^{n-1}\times S^n$, then we have*

$$N(B)\leqq n^2-1\,.$$

Proof. Suppose that $N(B)>n^2-1$, then $N(B)>(1/4)(2n-1)^2+(1/2)(2n-1)$. By theorem 2 in [6], $B=\partial(D^k\times X)$, $k\geqq n+1$, where D^k denotes a k-disk and X is a compact manifold with possibly boundary.

Case of $\partial X\neq\emptyset$.

Consider the homology exact sequence of the pair $(D^k\times X, B)$:

$$\longrightarrow H_{i+1}(D^k\times X)\longrightarrow H_{i+1}(D^k\times X, B)\longrightarrow H_i(B)\longrightarrow H_i(D^k\times X)\longrightarrow .$$

We have isomorphisms $H_{i+1}(D^k\times X, B)\approx H_{i+1}(S^k\wedge(X/\partial X))\approx H_{i-k+1}(X/\partial X)$ and $H_i(D^k\times X)\approx H_i(X)$. When $i=n-1$, we have an exact sequence

$$\longrightarrow H_{n-k}(X/\partial X)\longrightarrow H_{n-1}(B)\longrightarrow H_{n-1}(X)\longrightarrow H_{n-k-1}(X/\partial X)\longrightarrow .$$

Since the dimension of the manifold X is less than or equal to $n-1$, we have $H_{n-1}(X)=0$, and $H_{n-k}(X/\partial X)\approx H_{n-k-1}(X/\partial X)=0$, because $k\geqq n+1$. Thus $H_{n-1}(B)=0$. On the other hand, if B yields a cross section then $H_{n-1}(B)\approx Z$, and by the proposition 1, B can not be a sphere, hence $H_{n-1}(B)\neq 0$ if B does not yield any cross section. Thus any way we have $H_{n-1}(B)\neq 0$, which is a contradiction.

Case of $\partial X=\emptyset$.

In this case $B=S^{k-1}\times X$ and $k-1\geqq n$. By the proposition 1, B can not be a sphere, then X is a positive dimensional manifold. Thus $k-1=n$ and X is an $n-1$ dimensional manifold. We have $H_{n-1}(X)\approx Z$. Since $H_i(B)=0$ for $0<i<n-1$, by Künneth formula, $H_i(X)=0$ for $0<i<n-1$. Hence X is a homology sphere. Since X is simply connected it is a homotopy sphere, which contradicts the assumption.

§ 2. Stable bundles.

In the previous paper [8], we studied group actions on S^k-bundles over S^n for $k\geqq n$. Now using Barratt-Mahowald's formula, we can construct large group actions on these bundle spaces

PROPOSITION 4. *Any S^{q+1}-bundle over S^{q+1} admits an $SO((1/2)(q+1))$-action if q is odd and an $SO((1/2)q)$-action if q is even, where $q\geqq 23$.*

Proof. By Theorem 2 in [2], the homomorphism $\pi_q(SO(n))\rightarrow\pi_q(SO(q+2))$ is epimorphic if $q\leqq 2(n-1)-1$ and $n\geqq 13$, where the homomorphism is the one induced by the inclusion map $SO(n)\subset SO(q+2)$. The structure group of the bundle can be reduced to $SO((1/2)(q+3))$ if q is odd and $SO((1/2)q+2)$ if q is even and $q\geqq 23$. By the inclusion maps

$$SO\Big(\frac{1}{2}(q+3)\Big)\times SO\Big(\frac{1}{2}(q+1)\Big)\subset SO(q+2) \qquad \text{for odd } q\,,$$

$$SO\Big(\frac{1}{2}q+1\Big)\times SO\Big(\frac{1}{2}q\Big)\subset SO(q+2) \qquad \text{for even } q\,,$$

we have required actions.

Next we consider stable sphere bundles over S^{4k} for $k\geqq 3$. Here we shall take an analogy of [1] with a view to construct actions.

Let $\varepsilon_7\colon S^7\to SO(8)$ be the map defined by $\varepsilon_7(x)(y)=xy$ for $x, y\in S^7$, where the multiplication in S^7 is that of Cayley numbers. Further, let $\varepsilon_{k-1}\colon S^{k-1}\to SO(k)$, $k\equiv 1, 2, 4 \bmod 8$ be the map which gives a respresentative of a generator of the stable group $\pi_{k-1}(SO)$ by the inclusion map $SO(k)\to SO$. Clearly the bundle $S^{k-1}\to B\to S^k$ corresponding to ε_{k-1} admits an S^1-action for $k=1, 2$, an S^3-action for $k=4$ and a G_2-action for $k=8$ [8]. The map ε_{k-1} defines a complex over the k-disk D^k,

$$E(\varepsilon_{k-1})\colon 0\longrightarrow D_1^k\times R^k \overset{\varepsilon_{k-1}}{\longrightarrow} D_2^k\times R^k\longrightarrow 0 \quad (\text{cf. Lemma 10.1 in [1]})\,.$$

Denote by $\varepsilon_{k-1}^*\colon S^{k-1}\to SO(k)$ the map given by $\varepsilon_{k-1}^*(x)=$the transposed matrix of $\varepsilon_{k-1}(x)$ for $x\in S^{k-1}$. Let

$$E(\tilde{\varepsilon}_{8+k-1})\colon 0\longrightarrow F_1\longrightarrow F_0\longrightarrow 0\,,$$

be the complex over $D^8\times D^k$, which is defined by

$$F_1=D^8\times D^k\times(R^8\otimes R^k\oplus R^8\otimes R^k)\,,\qquad F_0=D^8\times D^k\times(R^8\otimes R^k\oplus R^8\otimes R^k)$$

and

$$\tilde{\varepsilon}_{8+k-1}(x_1, x_2)=\frac{1}{2}\begin{pmatrix}1\otimes\varepsilon_{k-1}(x_2) & \varepsilon_7(x_1)\otimes 1\\ \varepsilon_7^*(x_1)\otimes 1 & -1\otimes\varepsilon_{k-1}^*(x_2)\end{pmatrix}\colon S^{8+k-1}\longrightarrow SO(16k)\,,$$

$$\text{for } (x_1, x_2)\in D^8\times S^{k-1}\cup S^7\times D^k=S^{8+k-1}\,.$$

By (10.4) in [1], we have $\chi(E(\tilde{\varepsilon}_{8+k-1}))=\chi(E(\varepsilon_7))\cdot\chi(E(\varepsilon_{k-1}))$, where χ is the Euler characteristic of a complex. Due to Bott periodicity $\chi(E(\tilde{\varepsilon}_{8+k-1}))$ gives a generator of the group $KO(S^{8+k})$ and determines an S^{16k-1}-bundle over S^{8+k}, say $B^{(8+k,16k-1)}$. Then we have

THEOREM 5. *The bundle $B^{(8+k,16k-1)}$ admits a G_2-action for $k=1$, a $G_2\times S^1$-action for $k=2$, a $G_2\times S^3$-action for $k=4$ and a $G_2\times G_2$-action for $k=8$.*

Proof. Consider the case $k=4$. For $(x_1, x_2)\in S^7\times S^3$, $(y_0\otimes y_1'\oplus y_1\otimes y_0')\in R^8\otimes R^4\oplus R^8R^4$, we define an action of $(g, q)\in G_2\times S^3$ by

$$(g, q)\{(x_1, x_2), (y_0\otimes y_1'\oplus y_1\otimes y_0')\}$$
$$=\{(g(x_1), qx_2q^{-1}), (g(y_0)\otimes qy_1'q^{-1}\oplus g(y_1)\otimes qy_0'q^{-1})\},$$

then the clutching map $\tilde{\varepsilon}_{11}$ is compatible with the action (cf. § 1 in [8]). Remainder cases $k=1$, 2 and 8 are treated similarly.

COROLLARY 6. *The generator of* $\pi_{8l+k-1}(SO(16^l\cdot k))$ *gives a stable sphere bundle admitting a* $(G_2)^l$*-action for* $k=1$, *a* $(G_2)^l\times S^1$*-action for* $k=2$, *a* $(G_2)^l\times S^3$*-action for* $k=4$ *and a* $(G_2)^{l+1}$*-action for* $k=8$.

Proof. These generators are given inductively by the map

$$\frac{1}{2}\begin{pmatrix} 1\otimes\tilde{\varepsilon}_{8(l-1)+k-1} & \varepsilon_7\otimes 1 \\ \varepsilon_7^*\otimes 1 & -1\otimes\tilde{\varepsilon}^*_{8(l-1)+k-1}\end{pmatrix}: S^{8l+k-1}\longrightarrow SO(16^l\cdot k).$$

Then we can construct desired actions analogously as the theorem.

§ 3. Semistable bundles.

The group structure of $\pi_{8s+r}(SO(8s+r-k))$ has given by the table in [4] for $s\geqq 1$ and $-1\leqq k\leqq 4$. Here we shall give generators of these groups in order to obtain some information for the proof of the existence of group actions. We use the surjectivity of $j_*: \pi_i(SO(n))\to\pi_i(SO)$ by Barratt-Mahowald, then the restriction $s\geqq 4$ is required. In order to obtain generators, we shall depend on the followings:

(1) the homotopy exact sequence associated to the fibering $SO(n-1)\to SO(n)\to S^{n-1}$,
(2) boundary formulas of Theorems 1, 2 and 3, and Lemma 1 in [4]. Relations for generators of stable groups given by Lemma 2 in [4],
(3) the properties of characteristic maps given by 23.4, 24.3 and 24.5 in [9], and the structure of tangent bundles which is given by 27.8-27.11 in [9],
(4) the distributive law in homotopy groups of spheres,

$$(\beta_1+\beta_2)\circ(E\alpha)=\beta_1\circ(E\alpha)+\beta_2\circ(E\alpha).$$

Let $j^{(n,n-l)}: SO(n-l)\to SO(n)$ be the inclusion map. If $x\in\pi_j(SO(n))$ yields $y\in\pi_i(SO(n-l))$ such that $j_*^{(n,n-l)}(y)=x$, then we use a symbol $x^{(-l)}$ for y. Further we use the following symbols:

τ_{n-1}; the characteristic map of the tangent bundle of S^n,
ε_{4s-1}; one of the generators of $\pi_{4s-1}(SO(4s))$ such that $j_*^{(4s+1,4s)}(\varepsilon_{4s-1})$ gives the generator of the stable group $\pi_{4s-1}(SO(4s+1))$,
η_n; the generator of the stable group $\pi_{n+1}(S^n)$,
θ_n; the generator of the stable group $\pi_{n+2}(S^n)$,
ν_n; the generator of the stable group $\pi_{n+3}(S^n)$.

Now the generators of $\pi_{8s+r}(SO(8s+r-k))$, for $-1\leqq k\leqq 4$, $0\leqq r\leqq 7$, $s\geqq 4$ are given by the following table (1).

Table (1)

$r\backslash k$	-1	0	1
0	$Z_2 \quad +Z_2$ $\tau_{8s}, \quad \varepsilon_{8s-1}\eta_{8s-1}$	$Z_2 \quad +Z_2 \quad +Z_2$ $\tau_{8s}^{(-1)}, \varepsilon_{8s-1}^{(-1)}\eta_{8s-1}, \tau_{8s-1}\eta_{8s-1}$	$Z_2 \quad +Z_2$ $\varepsilon_{8s-1}^{(-2)}\eta_{8s-1}, (\tau_{8s-1}\eta_{8s-1})^{(-1)}$
1	$Z \quad +Z_2$ $\tau_{8s+1}, \quad \varepsilon_{8s-1}\theta_{8s-1}$	$Z_2 \quad +Z_2$ $\varepsilon_{8s-1}^{(-1)}\theta_{8s-1}, \quad \tau_{8s}\eta_{8s}$	$Z_2 \quad +Z_2 \quad +Z_2$ $\tau_{8s-1}\theta_{8s-1}, \tau_{8s}^{(-1)}\eta_{8s}, \varepsilon_{8s-1}^{(-2)}\theta_{8s-1}$
2	Z_2 τ_{8s+2}	Z_4 $\tau_{8+2}^{(-1)}$	Z_8 $\tau_{8s+2}^{(-2)}$
3	$Z \quad +Z$ $\varepsilon_{8s+3}, \quad \tau_{8s+3}$	Z $\varepsilon_{8s+3}^{(-1)}$	Z $\varepsilon_{8s+3}^{(-2)}$
4	Z_2 τ_{8s+4}	$Z_2 \quad +Z_2$ $\tau_{8s+4}^{(-1)}, \quad \tau_{8s+3}\eta_{8s+3}$	Z_2 $(\tau_{8s+3}\eta_{8s+3})^{(-1)}$
5	Z τ_{8s+5}	Z_2 $\tau_{8s+4}\eta_{8s+4}$	$Z_2 \quad +Z_2$ $\tau_{8s+3}\theta_{8s+3}, \tau_{8s+4}^{(-1)}\eta_{8s+4}$
6	Z_2 τ_{8s+6}	Z_4 $\tau_{8s+6}^{(-1)}$	Z_8 $\tau_{8s+6}^{(-2)}$
7	$Z \quad +Z$ $\varepsilon_{8s+7}, \quad \tau_{8s+7}$	Z $\varepsilon_{8s+7}^{(-1)}$	Z $\varepsilon_{8s+7}^{(-2)}$

Table (1)

$r\backslash k$	2	3	4
0	$Z_{12} \quad +Z_2$ $\tau_{8s-3}\nu_{8s-3}, \quad \varepsilon_{8s-1}^{(-3)}\eta_{8s-1}$	Z_2 $\varepsilon_{8s-1}^{(-4)}\eta_{8s-1}$	Z_2 $\varepsilon_{8s-1}^{(-5)}\eta_{8s-1}$
1	$Z_2 \quad +Z_2$ $(\tau_{8s-1}\theta_{8s-1})^{(-1)}, \quad \varepsilon_{8s-1}^{(-3)}\theta_{8s-1}$	Z_2 $\varepsilon_{8s-1}^{(-4)}\theta_{8s-1}$	Z_2 $\varepsilon_{8s-1}^{(-5)}\theta_{8s-1}$
2	$Z_{24} \quad +Z_8$ $\tau_{8s-1}\nu_{8s-1}, \quad \tau_{8s+2}^{(-3)}$	Z_8 a	Z_8 $a^{(-1)}$
3	$Z \quad +Z_2$ $\varepsilon_{8s+3}^{(-3)}, \quad \tau_{8s}\nu_{8s}$	$Z \quad +Z_2$ $\varepsilon_{8s+3}^{(-4)}, \quad (\tau_{8s}^{(-1)}\nu_{8s})$	$Z \quad +Z_2$ $\varepsilon_{8s+3}^{(-5)}, \quad (\tau_{8s}^{(-1)}\nu_{8s})^{(-1)}$
4	Z_{12} $\tau_{8s+1}\nu_{8s+1}$	0	0
5	$Z_2 \quad +Z_2$ $\tau_{8s+2}\nu_{8s+2} \quad (\tau_{8s+3}\eta_{8s+3})^{(-1)}\eta_{8s+4}$	Z_2 $((\tau_{8s+3}\eta_{8s+3})^{(-1)}\eta_{8s+4})^{(-1)}$	Z_2 $((\tau_{8s+3}\eta_{8s+3})^{(-1)}\eta_{8s+4})^{(-2)}$
6	$Z_4 \quad +Z_{24d}$ $(b) \quad (c)$	Z_8 $\tau_{8s+6}^{(-4)}$	Z_8 $\tau_{8s+6}^{(-5)}$
7	$Z \quad +Z_2$ $\varepsilon_{8s+7}^{(-3)}, \quad \tau_{8s+4}\nu_{8s+4}$	$Z \quad +Z_2$ $\varepsilon_{8s+7}^{(-4)}, \quad \tau_{8s+4}^{(-1)}\nu_{8s+4}$	$Z \quad +Z_2$ $\varepsilon_{8s+7}^{(-5)}, \quad (\tau_{8s+4}^{(-1)}\nu_{8s+4})^{(-1)}$

We have some relations on these generators:

$$\tau_{8s}^{(-1)} \simeq T'_{4s+1}\colon S^{8s} \longrightarrow U(4s) \qquad ((5) \text{ of } 24.2 \text{ in } [9]),$$

$$\tau_{8s+2}^{(-1)} \simeq T'_{4s+3}\colon S^{8s+2} \longrightarrow U(4s+1),$$

$$\tau_{8s}\circ\theta_{8s} = 4\tau_{8s+2}^{(-2)},$$

$$\tau_{8s+2}^{(-3)} \simeq T''_{2s+1}\colon S^{8s+2} \longrightarrow Sp(2s) \qquad (24.11 \text{ in } [9]),$$

$$j_*^{(8s,8s-1)}(a) = 6\tau_{8s-1}\circ\nu_{8s-1} + \tau_{8s+2}^{(-3)},$$

$$\tau_{8s}^{(-1)}\circ\nu_{8s} = T'_{4s+1}\circ\nu_{8s}\colon S^{8s+4} \longrightarrow U(4s),$$

$$\tau_{8s+4}^{(-1)} \simeq T'_{4s+5}\colon S^{8s+4} \longrightarrow U(4s+2),$$

$$\tau_{8s+4}\circ\eta_{8s+4} \simeq T'_{4s+5}\circ\eta_{8s+4}\colon S^{8s+5} \longrightarrow U(4s+2),$$

$$\tau_{8s+6}^{(-1)} \simeq T'_{4s+7}\colon S^{8s+6} \longrightarrow U(4s+3),$$

$$b = -\tau_{8s+6}^{(-3)} + 3\tau_{8s+3}\circ\nu_{8s+3}, \qquad c = \tau_{8s+2}^{(-3)} - \tau_{8s+3}\circ\nu_{8s+3}$$

$$\tau_{8s+6}^{(-3)} \simeq T''_{2s+2}\colon S^{8s+6} \longrightarrow Sp(2s+1),$$

$$\tau_{8s+4}^{(-1)}\circ\nu_{8s+4} \simeq T'_{4s+5}\circ\nu_{8s+4}\colon S^{8s+7} \longrightarrow U(4s+2).$$

Now we investigate group actions on a bundle space $D_1^n\times S^k\cup_\gamma D_2^n\times S^k$, where the action is to be compatible with the identification $(x, y)\equiv(x, \gamma(x)y)$ for $(x, y)\in S^{n-1}\times S^k$. We describe our results as a Table (2). In the table, each group denotes the one which is admitted by any bundle corresponding to a characteristic map in each block on the Table (1).

Proof of the results on the table (2).

We have a decomposition $\eta_{8s-1}=\eta_2*1(S^{8s-4})\colon S^{8s}=S^3*S^{8s-4}\to S^{8s-1}=S^2*S^{8s-4}$, where η_2 is the Hopf map $S^3\to S^2$ and $1(S^{8s-4})$ denotes the identity map of S^{8s-4}. η_2 is invariant under the principal S^1-action on S^3, then η_{8s-1} is also S^1-invariant. By Satz of 6.4 in [3] we have

$$\tau_{8s}(gx)(gy) = g\tau_{8s}(x)(y) \qquad \text{for } (x, y)\in S^{8s}\times S^{8s},\ g\in SO(2).$$

Since $\varepsilon_{8s-1}\circ\eta_{8s-1}$ is homotopic to $\varepsilon_{8s-1}^{(-2)}\circ\eta_{8s-1}$ in $SO(8s+1)$,

$$\begin{aligned}\varepsilon_{8s-1}\circ\eta_{8s-1}(gx)\tau_{8s}(gx)(gy) &= \varepsilon_{8s-1}^{(-2)}\circ\eta_{8s-1}(x)g\tau_{8s}(x)(y)\\ &= g\varepsilon_{8s-1}^{(-2)}\circ\eta_{8s-1}(x)\tau_{8s}(x)(y).\end{aligned}$$

Hence the bundle space with the characteristic map $\tau_{8s}+\varepsilon_{8s-1}\circ\eta_{8s-1}$ admits an $SO(2)$-action. This is the result for $(r, k)=(0, -1)$.

Similar argument is valid for $(r, k)=(0, 0), (0, 1), (0, 2), (1, -1), (1, 0), (1, 1)$.

By the surjectivity of the homomorphism $j_*\colon \pi_{8s-1}(SO(4s+1))\to \pi_{8s-1}(SO(8s-3))$, we have

Table (2)

$r\backslash k$	-1	0	1	2	3	4
0	$SO(2)$	$SO(2)$	$SO(2)$	$SO(2)$	$SO(4s-4)\times SO(2)$	$SO(4s-5)\times SO(2)$
1	$SO(2)$	$SO(2)$	$SO(2)$		$SO(4s-4)\times SO(2)$	$SO(4s-5)\times SO(2)$
2	$SO(8s+2)$	$U(4s)$	$Sp(2s)$	$Sp(2s)\cap SO(8s-6)$		
3		$SO(4s)$	$SO(4s-1)$			
4	$SO(8s+4)$	$U(4s)$		$SO(8s-5)\times S^3$	$SO(8s+6)\times SO(8s+1)$	$SO(8s+6)\times SO(8s)$
5	$SO(8s+5)$	$SO(8s+1)\times SO(2)$	$SO(2)$	$SO(2)$		
6	$SO(8s+6)$	$U(4s+2)$	$Sp(2s+1)$	$Sp(2s+1)\cap SO(8s-3)$	$SO(2)$	$SO(2)$
7		$SO(4s+2)$	$SO(4s+1)$			

Group actions on an $S^{8s+r-k-1}$-bundle over S^{8s+r+1}.

$$\varepsilon_{8s-1}^{(-4)}\circ\eta_{8s-1}(g_1x)(g_2y)=g_2\varepsilon_{8s-1}^{(-4)}\eta_{8s-1}(x)(y) \qquad \text{for } (x,\ y)\in S^{8r}\times S^{8r-4},$$
$$(g_1,\ g_2)\in SO(2)\times SO(4s-4).$$

Then we have the result for $(r,\ k)=(0,\ 3)$. Similar argument is valid for $(r,\ k)=(0,\ 4),\ (1,\ 3),\ (1,\ 4),\ (4,\ 2)$.

Since we can obtain similar results for $\{T'\}$ and $\{T''\}$ to Satz of 6.4 in [3], using the relations on generators, group actions are obtained in the cases of $(r,\ k)=(2,\ -1),\ (2,\ 0),\ (2,\ 1),\ (6,\ -1),\ (6,\ 0),\ (6,\ 1)$.

By the decomposition $\tau_{8s-1}\circ\nu_{8s-1}=\tau_{8s-1}\circ(\nu_4*1(S^{8s-6}))$, and the relation $\tau_{8s+2}^{(-3)}\simeq T''_{2s+1}$, we have

$$\tau_{8s-1}\circ\nu_{8s-1}(gx)T''_{2s+1}(gx)(gy)=\tau_{8s-1}\circ\nu_{8s-1}(gx)(gT''_{2s+1}(x)(y))$$
$$=g\tau_{8s-1}\circ\nu_{8s-1}(x)(y),$$
$$\text{for } (x,\ y)\in S^{8s+2}\times S^{8s-1},\ g\in Sp(2s)\cap SO(8s-6),$$

where the action is given by $g(x_1,\ x_2;\ t)=(x_1,\ gx_2;\ t)$ for $x_1\in S^7$, $x_2\in S_8{}^{s-6}$, $0\leqq t\leqq 1$. Hence we have proved the cases of $(2,\ 2)$ and similarly $(6,\ 2)$.

Since $\tau_{8s+6}^{(-7)}\neq 0$, we have group actions in the cases of $(6,\ 3)$, $(6,\ 4)$.

We can obtain group actions for remainder cases. For empty blocks, I have not yet obtained group actions for general bundles (cf. § 3 in [8]).

References

[1] M.F. Atiyah, R. Bott and A. Shapiro, Clifford modules, Topology, 1. Suppl. **1** (1964), 3-38.

[2] M.G. Barratt and M.E. Mahowald, The meta stable homotopy of $O(n)$, Bull. Amer. Math. Soc. **70** (1964), 758-760.

[3] F. Hirzebruch and K.H. Mayer, $O(n)$-Mannigfaltigkeiten, exotische Sphären und Singularitäten, Springer, Lecture Notes in Math. **57** (1968).

[4] M.A. Kervaire, Some nonstable homotopy groups of Lie groups, Illinois J. Math. **4** (1960), 161-169.

[5] H.T. Ku, L.N. Mann, J.L. Sicks and J.C. Su, Degree of symmetry of a product manifolds, Tras. Amer. Math. Soc. **146** (1969), 133-149.

[6] H.T. Ku, L.N. Mann, J.L. Sicks and J.C. Su, Degree of symmetry of closed manifolds, Springer, Lecture Notes in Math. **298** (1972), 430-436.

[7] H. Matsunaga, A note on the differentable structures of total spaces of sphere bundles over spheres, Osaka J. Math. **2** (1965), 63-69.

[8] H. Matsunaga, S^1-actions on sphere bundles over spheres, Mem. Fac. Sci. Shimane Univ. **15** (1981), 9-16.

[9] N. Steenrod, The topology of fibre bundles, Princeton, 1951.

Department of Mathematics,
Shimane University