A CERTAIN PROPERTY OF GEODESICS OF THE
FAMILY OF RIEMANNIAN MANIFOLDS O_n^2 (V)

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Introduction.

This is exactly a continuation of Part (IV) ([14]) with the same title written by the present author. We shall use the same notation in it.

The period T of any non-constant solution $x(t)$ of the non-linear differential equation of order 2:

$$(E) \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$(0.1) \qquad T=\sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

We proved in the last 10th section of Part (IV) that the following conjecture is true for $n\geq84$ and shall show that it is also true for $16\leq n<84$ in the present paper.

CONJECTURE C. The period function T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to $n(>2)$ for any fixed τ $(0<\tau<1)$.

The section numbers of this paper will start from 11 for convenience sake and so the section numbers from 1 to 10 mean the ones in Part (IV).

§11. The fundamental principle to attain the purpose.

Setting $T=\Omega(\tau, n)$, we have the two formulas

$$(11.1) \qquad \frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2b^2\sqrt{n}}\int_{x_0}^{1}\frac{(1-x)W(x, x_1)dx}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}$$

where $b=\sqrt{B-c}$, $B=(n-1)^{n-1}$ and

$$W(x, x_1) := \frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\}.$$

$$+\left[\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3}-\frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]\left\{\frac{n}{n-1}\frac{(x_1-1)^2}{x_1(n-x_1)}-\lambda(x_1)+\lambda(X)\right\}$$

$$-\left[\frac{X^2 f_1(X)}{(1-X)^3\sqrt{n-X}}-\frac{x^2 f_1(x)}{(1-x)^3\sqrt{n-x}}\right]\frac{\phi(x)-\phi(x_1)}{n\phi(x)},$$

where $X=X(x)$, $\phi(x)=x(n-x)^{n-1}$ (Lemma 3.1 in (III)) and

(11.2) $$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c/n}}{2b^2 n}\int_{x_0}^{1}\frac{(1-x)\sqrt{x(n-x)^{n-1}-c}}{x^2(n-x)^n}V(x, x_1)dx$$

((7.4) and Proposition 3 in (III) and (1.1)).

In order to prove $\dfrac{\partial\Omega(\tau, n)}{\partial n}<0$, we have tried to prove $V(x, x_1)>0$ for $X(x)$ $<x_1<n$ and succeeded in it in the case $n\geqq84$.

Using the notation

(6.1) $$\alpha_0=\alpha_0(n):=X_n^{-1}(2), \qquad \alpha_1=\alpha_1(n):=X_n^{-1}\Big(\frac{2n+10}{2n+1}\Big),$$

we have the following.

LEMMA 11.1. *When $n\geqq6$, we have*

(11.3) $$V(x, x_1)>0 \quad for \ 0<x\leqq\alpha_0 \quad and \quad X(x)\leqq x_1<n.$$

Proof. Since $V(x\,; x)$ is increasing with respect to x_1 by Lemma 8.1 in (III) for $0<x<1$ and

(11.2) $$V(x, X(x))=U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x),$$

where $U_i(x)$, $i=0, 1, 2, 4, 5, 6$, are all positive for $0<x<1$, by Proposition 1 in (II), Lemma 2.1 in (II) and Lemma 7.1 in (III), it is sufficient to prove that

$$V(x, X(x))>0 \quad for \ 0<x\leqq\alpha_0.$$

When $n>2$, we have by means of Lemma 2.5

(11.4) $$-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)$$
$$>\frac{nX}{(n-1)(X-1)^3\sqrt{n-X}}\{G_2(X)+3(X-1)f_0(X)\} \quad for \ 0<x<1,$$

and when $n\geqq6$, we have by means of Lemma 3.1

$$G_2(x)+3(x-1)f_0(x)>0 \quad for \ 2\leqq x<n.$$

Thus, we obtain the inequality (11.3). Q. E. D.

LEMMA 11.2. *When $n>13$, we have*

(11.5) $V(x, x_1) > 0$ for $\alpha_0 < x \leq \alpha_1$ and $X(x) \leq x_1 < n$.

Proof. We obtain this from (11.2), (11.4) and Proposition 3 in (IV).

Q. E. D.

LEMMA 11.3.   *When* $n \geq 12$, *we have* $\dfrac{2n+10}{2n+1} < \beta$, *where* β *is the root of the quadratic equation*: $(n-1)x^2 + 2x - 2n = 0$, *which is given by*

(11.6) $$\beta := \frac{-1 + \sqrt{2n^2 - 2n + 1}}{n-1}.$$

Proof. Since we have

$$[(n-1)x^2 + 2x - 2n]_{x = \frac{2n+10}{2n+1}} = \frac{4(n-1)(n+5)^2}{(2n+1)^2} + \frac{4(n+5)}{2n+1} - 2n$$

$$= -\frac{2}{(2n+1)^2} \cdot (2n^3 - 18n^2 - 51n + 40),$$

it is sufficient to prove

$$2n^3 - 18n^2 - 51n + 40 > 0 \quad \text{for} \quad n \geq 12 .$$

Now, setting $n = m + 11$, we have easily

$$2n^3 - 18n^2 - 51n + 40$$
$$= 2m^3 + 48m^2 + 279m - 37 > 0 \quad \text{for} \quad m \geq 1 . \qquad \text{Q. E. D.}$$

Remark. We know the fact on β as follows: When $n > 1$, we have

$$\sqrt{\frac{2n}{n+1}} < \beta < \sqrt{2}$$

by Lemma 3.6 in (III). Since we have

and
$$[2m^3 + 48m^2 + 279m - 37]_{m=0.13} = 0.085594 > 0$$
$$[2m^3 + 48m^2 + 279m - 37]_{m=0.12} = -2.825344 < 0 ,$$

we can insist the inequality in the above lemma for $n \geq 11.13$ and not do it when $n = 11.12$.
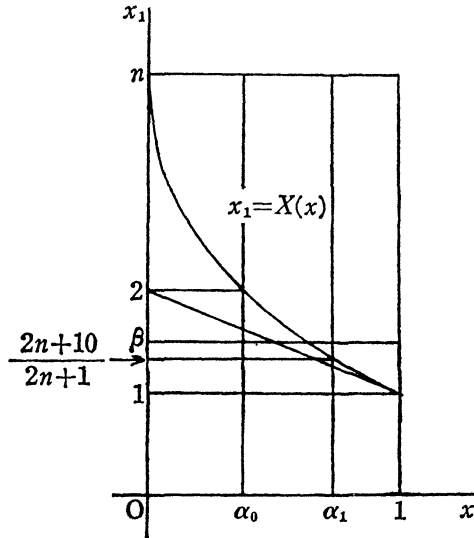
LEMMA 11.4.   *When* $n \geq 11$, *we have*

$$\frac{\partial \Omega(\tau, n)}{\partial n} < 0 \quad \text{for} \quad 1 < x_1 \leq \beta .$$

Proof. By means of Proposition 1 in (III) we have

$$W(x, x_1) > 0 \quad \text{for} \quad X^{-1}(\beta) \leq x < 1 \quad \text{and} \quad X(x) \leq x_1 < n$$

when $n \geq 11$. Hence, we obtain this lemma from (11.1).   Q. E. D.

By virtue of Lemma 11.1~Lemma 11.4 and the property of $V(x, x_1)$ that it is increasing with respect to x_1, we can obtain the inequality:

$$\frac{\partial \Omega(\tau, n)}{\partial n} < 0 \qquad \text{for} \quad \beta \leqq x_1 < n,$$

when $n \geqq 13$, if we can prove the inequality:

(11.7) $V(x, \beta) \geqq 0$ for $\alpha_1 \leqq x < 1$.

Finally, we express the function $V(x, \beta)$ in a convenient form for our purpose. From (1.2)~(1.8) we have

$$V(x, x_1) = \frac{x^2 N(x, x_1)}{(1-x)^5 \sqrt{n-x}} + \frac{X^2 N(X, x_1)}{(X-1)^5 \sqrt{n-X}}$$

$$= \frac{x^2 \sqrt{n-x}\, F_2(x)\{\lambda(x) - \tilde{\lambda}(x_1)\}}{(1-x)^5} - \frac{3x^2 f_0(x)}{(x-1)^3 \sqrt{n-x}} + \frac{2nx^2\{B - \psi(x)\}}{(1-x)^2 \sqrt{n-x}}$$

$$+ \frac{X^2 \sqrt{n-X}\, F_2(X)\{\lambda(X) - \tilde{\lambda}(x_1)\}}{(X-1)^5} + \frac{3X^2 f_0(X)}{(X-1)^3 \sqrt{n-X}} - \frac{2nX^2\{B - \psi(X)\}}{(X-1)^2 \sqrt{n-X}}$$

and by the definitions of $U_i(x)$, (1.9)~(1.13), we obtain

$$= U_0(x) - U_1(x) + U_2(x) + \frac{x^2 \sqrt{n-x}\, F_2(x)}{(1-x)^5}\{\tilde{\lambda}(X) - \tilde{\lambda}(x_1)\}$$

$$+ U_4(x) + U_5(x) - U_6(x) + \frac{X^2 \sqrt{n-X}\, F_2(X)}{(X-1)^5}\{\tilde{\lambda}(X) - \tilde{\lambda}(x_1)\},$$

hence

(11.8) $V(x, \beta) = V(x, X(x))$

$$+ \left\{ \frac{x^2\sqrt{n-x}\,F_2(x)}{(1-x)^5} + \frac{X^2\sqrt{n-X}\,F_2(X)}{(X-1)^5} \right\} \{\tilde{\lambda}(X) - \tilde{\lambda}(\beta)\}.$$

§ 12. Some evaluations related with $\tilde{\lambda}(x)$ and $F_2(x)$.

Regarding $V(x, X(x))$, we have the following evaluation: when $n \geqq 10$,

(10.18) $$V(x, X(x)) > \frac{sB\sqrt{n-1}}{100}(9.0651n - 379.1388 \times 2)$$

$$\text{for}\quad \alpha_1(n) \leqq x < 1,$$

where $s = X(x) - 1$.

LEMMA 12.1. *When $n \geqq 12$, we have*

$$\frac{\tilde{\lambda}(X) - \tilde{\lambda}(\beta)}{X - 1}$$

(12.1) $$> \frac{(2n^4 + 2n^3 - 38n^2 + 83n - 40)\beta - n(2n^3 + 4n^2 + 43n - 40)}{18(2n+1)\{n(n^2 - n + 2) - 2(n^2 - n + 1)\beta\}}$$

$$\text{for}\quad 1 < X \leqq \frac{2n+10}{2n+1}.$$

Proof. Since we have

$$\tilde{\lambda}'(x) = -\frac{(x-1)\{n + (n-1)x\}}{(n-1)x^2(n-x)}$$

by Lemma 7.1 in (III), we obtain

$$\tilde{\lambda}(X) - \tilde{\lambda}(\beta) = \int_\beta^x \tilde{\lambda}'(t)dt = \frac{1}{n-1}\int_x^\beta \frac{(t-1)\{n+(n-1)t\}\,dt}{t^2(n-t)} \ .$$

Now, we show that the function $\dfrac{n+(n-1)t}{t^2(n-t)}$ is decreasing with respect to t for $1 \leqq t \leqq 2$. In fact, we have

$$\frac{d}{dt}\frac{n+(n-1)t}{t^2(n-t)} = \frac{-1}{t^3(n-t)^2}\{2n^2 + n(n-4)t - 2(n-1)t^2\}$$

and

$$2n^2 + n(n-4)t - 2(n-1)t^2 > 0 \quad \text{for}\quad 0 < t \leqq 2$$

hence

$$\frac{d}{dt}\frac{n+(n-1)t}{t^2(n-t)} < 0 \quad \text{for}\quad 1 \leqq t \leqq 2 .$$

Thus we obtain

$$\tilde{\lambda}(X) - \tilde{\lambda}(\beta) > \frac{n+(n-1)\beta}{(n-1)\beta^2(n-\beta)}\int_x^\beta (t-1)dt \ .$$

Next, by Lemma 11.3 we obtain for $1 < X \leqq \dfrac{2n+10}{2n+1}$

$$\frac{1}{X-1}\int_x^\beta (t-1)dt = \frac{1}{2}\left\{\frac{(\beta-1)^2}{X-1}-(X-1)\right\}$$

$$> \frac{1}{2}\left\{\frac{(\beta-1)^2}{t-1}-(t-1)\right\}_{t=\frac{2n+10}{2n+1}}$$

$$= \frac{(2n+1)^2(\beta-1)^2-81}{18(2n+1)}$$

and

$$(2n+1)^2(\beta-1)^2-81 = (2n+1)^2\cdot\frac{2(n-\beta)}{n-1}-2(2n+1)^2\beta+(2n+1)^2-81$$

$$= \frac{2}{n-1}\cdot\{6n^3+4n^2-41n+40-n(2n+1)^2\beta\},$$

from which we get the inequality

$$\frac{1}{X-1}\int_x^\beta (t-1)dt > \frac{6n^3+4n^2-41n+40-n(2n+1)^2\beta}{9(n-1)(2n+1)}.$$

Therefore we have

$$\frac{\tilde{\lambda}(X)-\tilde{\lambda}(\beta)}{X-1} > \frac{n+(n-1)\beta}{2(n-\beta)^2}\cdot\frac{6n^3+4n^2-41n+40-n(2n+1)^2\beta}{9(n-1)(2n+1)}$$

On the other hand, using $(n-1)\beta^2=2(n-\beta)$, we obtain

$$(n-1)(n-\beta)^2 = n^2(n-1)-2n(n-1)\beta+2(n-\beta)$$

$$= n(n^2-n+2)-2(n^2-n+1)\beta$$

and

$$\{n+(n-1)\beta\}\{6n^3+4n^2-41n+40-n(2n+1)^2\beta\}$$

$$= n(6n^3+4n^2-41n+40)$$

$$\quad + \{(n-1)(6n^3+4n^2-41n+40)-n^2(2n+1)^2\}\beta-2n(2n+1)^2(n-\beta)$$

$$= (2n^4+2n^3-38n^2+83n-40)\beta-n(2n^3+4n^2+43n-40).$$

Thus, combining these with the above inequality we obtain easily (12.1).

Q. E. D.

Next, we study the first factor of the second term of the right hand side of (11.8).

LEMMA 12.2. *When $n \geqq 10$, we have*

$$(12.2) \qquad \frac{x^2\sqrt{n-x}\,F_2(x)}{(1-x)^5}+\frac{X^2\sqrt{n-X}\,F_2(X)}{(X-1)^5}$$

$$> \frac{1}{X-1}\left\{\frac{32}{3}\cdot\frac{n(n-1)^{n+2}(n-4)^2(n^2-n+1)\sqrt{2n^2-n+8}}{\sqrt{2n+1}\,(8n^3+18n+1)^2}\right.$$

$$+\frac{1}{96}\cdot\frac{n(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^{5/2}(2n+1)^n}\cdot(8n^3-24n^2-18n+61)$$

$$\left.\times(8n^4-12n^3-28n^2+27n+5)\right\}\quad \text{for}\quad \alpha_1(n)\leqq x<1.$$

Proof. By means of (7.6) and the inequality continued to it in the proof of Lemma 7.1, we obtain

$$\frac{x^2\sqrt{n-x}\,F_2(x)}{(1-x)^5}>\frac{1}{t^5}\cdot\frac{4(n-4)^2\sqrt{2n^2-n+8}}{(2n+1)^{5/2}}$$

(12.3)
$$\times\frac{8Bn(n-1)^3(n^2-n+1)(2n+1)^2t^4}{3(8n^3+18n+1)^2}$$

$$=\frac{32}{3}\cdot\frac{n(n-1)^{n+2}(n-4)^2(n^2-n+1)\sqrt{2n^2-n+8}}{\sqrt{2n+1}\,(8n^3+18n+1)^2}\cdot\frac{1}{t}\quad\text{for}\quad\alpha_1(n)\leqq x<1,$$

where $t=1-x$, when $n\geqq 10$.

Since $x^2\sqrt{n-x}$ is increasing for $0<x<\frac{4}{5}n$ and $\frac{2n+10}{2n+1}<\frac{4}{5}n$ for $n\geqq 10$, we obtain analogously from (7.8) appeared in the proof of Lemma 7.2

$$\frac{X^2\sqrt{n-X}\,F_2(X)}{(X-1)^5}>\frac{1}{s^5}\cdot\sqrt{n-1}\cdot\frac{n(n+2)^{n-2}(2n-5)^{n-2}}{96(n-1)^3(2n+1)^n}$$

(12.4)
$$\times(8n^3-24n^2-18n+61)(8n^4-12n^3-28n^2+27n+5)s^4$$

$$=\frac{1}{96}\cdot\frac{n(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^{5/2}(2n+1)^n}\cdot(8n^3-24n^2-18n+61)$$

$$\times(8n^4-12n^3-28n^2+27n+5)\cdot\frac{1}{s}\quad\text{for}\quad 1<X<\frac{2n+10}{2n+1},$$

where $s=X-1$, when $n\geqq 10$.

By Lemma 6.2, we have

$$\frac{1}{t}=\frac{1}{1-x}>\frac{1}{X(x)-1}=\frac{1}{s}.$$

Therefore, we obtain immediately (12.2) from the above inequalities (12.3) and (12.4).   Q. E. D.

Finally, from Lemma 12.1 and Lemma 12.2 we obtain the following

LEMMA 12.3. *When* $n\geqq 12$, *we have the inequality*

(12.5) $$\left\{\frac{x^2\sqrt{n-x}\,F_2(x)}{(1-x)^5}+\frac{X^2\sqrt{n-X}\,F_2(X)}{(X-1)^5}\right\}\cdot\{\tilde{\lambda}(X)-\tilde{\lambda}(\beta)\}>n\sqrt{n-1}\,B$$

$$\times\left[\frac{16}{3}\cdot\frac{(n-1)^{5/2}(n-4)^2(n^2-n+1)\sqrt{2n^2-n+8}}{(2n+1)^{3/2}(8n^3+18n+1)^2}+\frac{1}{192}\cdot\frac{(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^{n+2}(2n+1)^{n+1}}\cdot\right.$$

$$\times(8n^3-24n^2-18n+61)(8n^4-12n^3-28n^2+27n+5)\bigg]$$

$$\times\left[\frac{(2n^4+2n^3-38n^2+83n-40)\beta-n(2n^3+4n^2+43n-40)}{9\{n(n^2-n+2)-2(n^2-n+1)\beta\}}\right]$$

$$for \quad \alpha_1(n)\leqq x<1.$$

§13. Some evaluations related with Lemma 12.3.

LEMMA 13.1. *We have the following inequalities*

$$\frac{(2n^4+2n^3-38n^2+83n-40)\beta-n(2n^3+4n^2+43n-40)}{(n-1)\{n(n^2-n+2)-2(n^2-n+1)\beta\}}$$

(13.1) $$\geqq\frac{2(10103\beta-13524)}{11(804-133\beta)} \quad for \quad n\geqq12,$$

(13.2) $$\frac{4(16353\beta-19728)}{15(1936-241\beta)} \quad for \quad n\geqq16,$$

and

(13.3) $$\frac{105379\beta-123246}{17(2772-307\beta)} \quad for \quad n\geqq18.$$

Proof. Setting $u=\dfrac{1}{n}$, the left hand side of the above inequalities can be expressed as

(13.4) $$\Phi(u):=\frac{(2+2u-38u^2+83u^3-40u^4)\beta-(2+4u+43u^2-40u^3)}{(1-u)\{1-u+2u^2-2u(1-u+u^2)\beta\}}.$$

For the moment, we consider in this expression that β is a constant such that $\sqrt{\dfrac{24}{13}}<\beta<\sqrt{2}$ and compute its logarithmic derivative with respect to u, we obtain

$$\frac{(2-76u+249u^2-160u^3)\beta-4-86u+120u^2}{(2+2u-38u^2+83u^3-40u^4)\beta-(2+4u+43u^2-40u^3)}+\frac{1}{1-u}$$

$$+\frac{1-4u+(2-4u+6u^2)\beta}{1-u+2u^2-2u(1-u+u^2)\beta}$$

$$=\frac{(2-76u+249u^2-160u^3)\beta-4-86u+120u^2}{(2+2u-38u^2+83u^3-40u^4)\beta-(2+4u+43u^2-40u^3)}$$

$$+\frac{2-6u+6u^2+(2-8u+12u^2-8u^3)\beta}{(1-u)\{1-u+2u^2-2u(1-u+u^2)\beta\}},$$

for which, making the common denominator of the right hand side, we obtain its numarator as follows:

$$- \{4+86u-120u^2-(2-76u+249u^2-160u^3)\beta\}(1-u)\{1-u+2u^2-2u(1-u+u^2)\beta\}$$

$$+ \{(2-2u-38u^2+83u^3-40u^4)\beta-(2+4u+43u^2-40u^3)\}$$

$$\times \{2-6u+6u^2+(2-8u+12u^2-8u^3)\beta\} = -Q_0(u)+Q_1(u)\beta+Q_2(u)\beta^2,$$

where we set

$$(13.5) \qquad \begin{cases} Q_0(u):=8+74u-206u^2+176u^3-34u^4, \\ Q_1(u):=2-72u+409u^2-660u^3+425u^4-68u^5, \\ Q_2(u):=4-16u+92u^2-332u^3+408u^4-160u^5-6u^6. \end{cases}$$

We can easily show that $Q_0(u)$ is increasing and $Q_1(u)$ and $Q_2(u)$ are decreasing for $0 \le u \le \frac{1}{12}$, because

$$Q_0'(u)=74-412u+528u^2-136u^3>74-\frac{412}{12}-\frac{136}{12^3}>0,$$

$$Q_1'(u)=-72+818u-1980u^2+1700u^3-340u^4$$

$$<-72+\frac{818}{12}-\frac{1980}{12^2}+\frac{1700}{12^3}<-72+68.17-13.75+0.99<0,$$

since $\frac{818}{1980 \times 2}=\frac{409}{1980}>\frac{1}{12}$, and

$$Q_2'(u)=-16+184u-996u^2+1632u^3-800u^4-36u^5$$

$$<-16+\frac{184}{12}-\frac{996}{12^2}+\frac{1632}{12^3}<-16+15.34-6.91+0.95<0,$$

since $\frac{184}{996 \times 2}=\frac{23}{249}>\frac{1}{12}$. Hence we see that $-Q_0(u)+Q_1(u)\beta+Q_2(u)\beta^2$ is decreasing with respect to u for $0 \le u \le \frac{1}{12}$, regarding as β is a constant. On the other hand we have

$$-Q_0(0)+Q_1(0)\beta+Q_2(0)\beta^2=-8+2\beta+4\beta^2>0$$

since $\sqrt{\frac{24}{13}}<\beta<\sqrt{2}$, and

$$-Q_0\Big(\frac{1}{12}\Big)+Q_1\Big(\frac{1}{12}\Big)\beta+Q_2\Big(\frac{1}{12}\Big)\beta^2$$

$$=-\Big(8+\frac{74}{12}-\frac{206}{12^2}+\frac{176}{12^3}-\frac{34}{12^4}\Big)$$

$$+\left(2-\frac{72}{12}+\frac{409}{12^2}-\frac{660}{12^3}+\frac{425}{12^4}-\frac{68}{12^5}\right)\beta$$

$$+\left(4-\frac{16}{12}+\frac{92}{12^2}-\frac{332}{12^3}+\frac{408}{12^4}-\frac{160}{12^5}-\frac{6}{12^6}\right)\beta^2$$

$$<-\left(8+\frac{74}{12}-\frac{206}{12^2}+\frac{176}{12^3}-\frac{34}{12^4}\right)+\left(2-\frac{72}{12}+\frac{409}{12^2}-\frac{660}{12^3}+\frac{425}{12^4}\right)\beta$$

$$+\left(4-\frac{16}{12}+\frac{92}{12^2}-\frac{332}{12^3}+\frac{408}{12^4}\right)\beta^2=\frac{1}{12^4}\cdot(-266174-31543\beta+64968\beta^2)<0 .$$

Therefore, we can say that the original function $\Phi(u)$ is firstly increasing and then change to become decreasing at a point of the interval $0<u<\frac{1}{12}$.

Finally we shall compute the values of $\Phi(u)$ at $u=\frac{1}{12}, \frac{1}{16}$ and $\frac{1}{18}$, and compare them with the one at $u=0$. Since we have

$$2\cdot12^4+2\cdot12^3-38\cdot12^2+83\cdot12-40=40412=4\cdot10103 ,$$

$$2\cdot16^4+2\cdot16^3-38\cdot16^2+83\cdot16-40=130824=8\cdot16353 ,$$

$$2\cdot18^4+2\cdot18^3-38\cdot18^2+83\cdot18-40=210758=2\cdot105379$$

and

$$2\cdot12^3+4\cdot12^2+43\cdot12-40=4508 ,$$

$$2\cdot16^3+4\cdot16^2+43\cdot16-40=9864 ,$$

$$2\cdot18^3+4\cdot18^2+43\cdot18-40=13694 ,$$

we obtain

$$\Phi\left(\frac{1}{12}\right)=\frac{4\cdot10103\beta-12\cdot4508}{11(12\cdot134-2\cdot133\beta)}=\frac{2(10103\beta-13524)}{11(804-133\beta)} ,$$

$$\Phi\left(\frac{1}{16}\right)=\frac{8\cdot16353\beta-16\cdot9864}{15(16\cdot242-2\cdot241\beta)}=\frac{4(16353\beta-19728)}{15(1936-241\beta)} ,$$

$$\Phi\left(\frac{1}{18}\right)=\frac{2\cdot105379\beta-18\cdot13694}{17(18\cdot308-2\cdot307\beta)}=\frac{105379\beta-123246}{17(2772-307\beta)} .$$

We see easily $\Phi\left(\frac{1}{18}\right)<\Phi(0)=2(\beta-1)$, because

$$105379\beta-123246<34(\beta-1)(2772-307\beta) ,$$

which is equivalent to

$$10438\beta^2+693\beta-28998<0 .$$

In fact we have

$$10438\beta^2+693\beta-28998<10438\cdot2+693\cdot\sqrt{2}-28998=-8122+693\cdot\sqrt{2}<0$$

for $\sqrt{\frac{24}{13}}<\beta<\sqrt{2}$. Thus, we see that

$$\Phi(0)>\Phi\left(\frac{1}{18}\right)>\Phi\left(\frac{1}{16}\right)>\Phi\left(\frac{1}{12}\right).$$

From these things we obtain easily (13.1), (13.2) and (13.3).   Q. E. D.

LEMMA 13.2. *We have the following inequalities*

$$\frac{16}{3}\cdot\frac{(n-1)^{7/2}(n-4)^2(n^2-n+1)\sqrt{2n^2-n+8}}{n(2n+1)^{3/2}(8n^3+18n+1)^2}\geqq$$

(13.6) $$\frac{2^9\cdot(11)^{7/2}\cdot(71)^{1/2}\cdot133}{3^2\cdot5^3\cdot(14041)^2}\qquad for\quad n\geqq12\,,$$

(13.7) $$\frac{2^{11/2}\cdot3^2\cdot5^{7/2}\cdot7^{1/2}\cdot241}{(11)^{3/2}\cdot(11019)^2}\qquad for\quad n\geqq16\,,$$

and

(13.8) $$\frac{2^{11/2}\cdot7^2\cdot(17)^{7/2}\cdot307\cdot(319)^{1/2}}{3^3\cdot(37)^{3/2}\cdot(46981)^2}\qquad for\quad n\geqq18\,.$$

Proof. Setting $u=\frac{1}{n}$, the left hand side of the above inequalities can be expressed as

$$\frac{16}{3}\cdot\frac{(1-u)^{7/2}(1-4u)^2(1-u+u^2)\sqrt{2-u+8u^2}}{(2+u)^{3/2}(8+18u^2+u^3)^2}.$$

First, we show that $(1-4u)^4(2-u+8u^2)$ is decreasing for $0\leqq u\leqq\frac{1}{12}$, because

$$\{(1-4u)^4(2-u+8u^2)\}'=-(1-4u)^3(33-36u+192u^2)<0\,.$$

Hence, putting $n=12, 16$ and 18 in the expression above of n, we obtain these inequalities.   Q. E. D.

LEMMA 13.3. *Setting* $u=\frac{1}{n}$, *we have*

(13.9) $$\frac{1}{192}\cdot\frac{(n+2)^{n-2}(2n-5)^{n-2}}{n(n-1)^{n+1}(2n+1)^{n+1}}\cdot(8n^3-24n^2-18n+61)(8n^4-12n^3-28n^2+27n+5)$$

$$=\frac{1}{2^{17/2}\cdot3}\cdot\frac{\sqrt{2-5u}}{(1-u)^2(1+2u)^2(2+u)}\cdot\frac{e(n-1)(e(n/2))^2}{\sqrt{e(2n)}(e((2n-5)/5))^{5/2}}$$

$$\times(8-24u-18u^2+61u^3)(8-12u-28u^2+27u^3+5u^4)\,.$$

Proof. The left hand side of (13.9) can be written as

$$\frac{1}{2^6 \cdot 3} \cdot \frac{(1+2u)^{n-2}(2-5u)^{n-2}}{(1-u)^{n+1}(2+u)^{n+1}} \cdot (8-24u-18u^2+61u^3)(8-12u-28u^2+27u^3+5u^4) .$$

Using the notation $e(n)=e_n=\left(1+\dfrac{1}{n}\right)^n$ introduced in §10, we have

$$(1-u)^{n+1}=(1-u)^2/e(n-1) , \quad (2+u)^{n+1}=2^n \cdot (2+u) \cdot \sqrt{e(2n)} ,$$

$$(1+2u)^{n-2}=\frac{1}{(1+2u)^2} \cdot (e(n/2))^2 , \quad (2-5u)^{n-2}=2^{n-5/2} \cdot \sqrt{2-5u} \Big/ \left(e\Big(\frac{2n-5}{5}\Big)\right)^{5/2} .$$

Putting these into the above expression, we can obtain easily the right hand side of (13.9).   Q. E. D.

LEMMA 13.4. *The function* $e(n-1)(e(n/2))^2/\sqrt{e(2n)} \ (e((2n-5)/5))^{5/2}$ *of* n *is decreasing for* $n>5/2$.

Proof. From the definition of $e(n)$, we obtain

$$\frac{d}{dn}\Big\{(n-1) \log \Big(1+\frac{1}{n-1}\Big)+n \log \Big(1+\frac{2}{n}\Big)-n \log \Big(1+\frac{1}{2n}\Big)$$

$$-\Big(n-\frac{5}{2}\Big) \log \Big(1+\frac{5}{2n-5}\Big)\Big\}=\log \Big(1+\frac{1}{n-1}\Big)+\log \Big(1+\frac{2}{n}\Big)-\log \Big(1+\frac{1}{2n}\Big)$$

$$-\log \Big(1+\frac{5}{2n-5}\Big)-(n-1)\frac{1/(n-1)^2}{1+(1/(n-1))}-n\frac{2/n^2}{1+(2/n)}+n\frac{1/2n^2}{1+(1/2n)}$$

$$+\Big(n-\frac{5}{2}\Big)\frac{10/(2n-5)^2}{1+(5/(2n-5))}=\log \frac{(n+2)(2n-5)}{(n-1)(2n+1)}-\frac{1}{n}-\frac{2}{n+2}+\frac{1}{2n+1}+\frac{5}{2n}$$

$$=\log \frac{2n^2-n-10}{2n^2-n-1}+\frac{3(5n+2)}{2n(n+2)(2n+1)} .$$

Setting $u=\dfrac{1}{n}$, the above expression can be written as

$$\frac{3}{2} \cdot \frac{u^2(5+2u)}{(1+2u)(2+u)}+\log \frac{2-u-10u^2}{2-u-u^2}$$

which becomes 0 at $u=0$. Its derivative with respect to u is

$$\frac{3u(20+37u+20u^2+4u^3)}{2(1+2u)^2(2+u)^2}-\frac{1+20u}{2-u-10u^2}+\frac{1+2u}{2-u-u^2}$$

$$=\frac{3u(20+37u+20u^2+4u^3)}{2(1+2u)^2(2+u)^2}-\frac{9u(4-u)}{(2-u-10u^2)(2-u-u^2)}$$

$$=-\frac{3u(16+388u+1160u^2+903u^3-195u^4-530u^5-244u^6-40u^7)}{2(1+2u)^2(2+u)^2(2-u-10u^2)(2-u-u^2)}<0$$

for $0<u<2/5$ and hence we see that

$$\log \frac{2n^2-n-10}{2n^2-n-1} + \frac{3(5n+2)}{2n(n+2)(2n+1)} < 0 \quad \text{for} \quad n > \frac{5}{2},$$

which implies the statement of this lemma. 　　　　　　　　　　Q. E. D.

LEMMA 13.5. *We have the following inequalities*

$$\frac{1}{192} \cdot \frac{(n+2)^{n-2}(2n-5)^{n-2}}{n(n-1)^{n+1}(2n+1)^{n+1}} \cdot (8n^3-24n^2-18n+61)(8n^4-12n^3-28n^2+27n+5) \geqq$$

(13.10) 　　　　$\dfrac{\sqrt{19}\cdot 167\cdot 1459}{2^{31/2}\cdot 3^{7/2}\cdot 5^2} \cdot \dfrac{e_{15}\cdot (e_8)^2}{\sqrt{e_{32}}\cdot (e_{5.4})^{5/2}}$ 　　　*for* $12 \leqq n < 16$,

(13.11) 　　　　$\dfrac{2933\cdot 10409}{2^{41/2}\cdot 3^{5/2}\cdot 5\cdot 11} \cdot \dfrac{e_{17}\cdot (e_9)^2}{\sqrt{e_{36}}\cdot (e_{6.2})^{5/2}}$ 　　　*for* $16 \leqq n < 18$,

and

(13.12) 　　　　$\dfrac{\sqrt{31}\cdot 38617\cdot 761243}{2^{15}\cdot 3^6\cdot 5^2\cdot 17^2\cdot 37}$ 　　　*for* $n \geqq 18$.

Proof. Regarding the right hand side of (13.9), we see that the function $\dfrac{\sqrt{2-5u}}{(1-u)^2(1+2u)^2(2+u)}$ is decreasing for $0 < u \leqq \dfrac{1}{12}$, because

$$\frac{d}{du}\log\frac{\sqrt{2-5u}}{(1-u)^2(1+2u)^2(2+u)} = -\frac{5}{2(2-5u)} + \frac{2}{1-u} - \frac{4}{1+2u} - \frac{1}{2+u}$$

$$= -\frac{5}{2(2-5u)} - \frac{1}{2+u} - \frac{2(1-4u)}{(1-u)(1+2u)} < 0,$$

and both of $8-24u-18u^2+61u^3$ and $8-12u-28u^2+27u^3+5u^4$ are clearly decreasing for $0 < u \leqq \dfrac{1}{12}$.

Now, putting $u = \dfrac{1}{12}, \dfrac{1}{16}, \dfrac{1}{18}$ in the expression

$$\frac{1}{2^{17/2}\cdot 3} \cdot \frac{\sqrt{2-5u}}{(1-u)^2(1+2u)^2(2+u)} \cdot (8-24u-18u^2+61u^3)(8-12u-28u^2+27u^3+5u^4),$$

we obtain respectively:

$$\frac{1}{2^{17/2}\cdot 3} \cdot \frac{(12)^{9/2}\cdot\sqrt{19}}{(11)^2\cdot(14)^2\cdot 25} \cdot \frac{8\cdot(12)^3-24\cdot(12)^2-18\cdot 12+61}{(12)^3}$$

$$\times \frac{8\cdot(12)^4-12(12)^3-28\cdot(12)^2+27\cdot 12+5}{(12)^4}$$

$$= \frac{\sqrt{19}\cdot 10213\cdot 141449}{2^{17/2}\cdot 3\cdot(11)^2\cdot(14)^2\cdot 25\cdot(12)^{5/2}} = \frac{\sqrt{19}\cdot 167\cdot 1459}{2^{31/2}\cdot 3^{7/2}\cdot 5^2},$$

$$\frac{1}{2^{17/2}\cdot 3}\cdot\frac{(16)^{9/2}\cdot\sqrt{27}}{(15)^2\cdot(18)^2\cdot 33}\cdot\frac{8\cdot(16)^3-24\cdot(16)^2-18\cdot 16+61}{(16)^3}$$

$$\times\frac{8\cdot(16)^4-12\cdot(16)^3-28\cdot(16)^2+27\cdot 16+5}{(16)^4}$$

$$=\frac{3^{3/2}\cdot 26397\cdot 468405}{2^{17/2}\cdot 3\cdot(15)^2\cdot(18)^2\cdot 33\cdot(16)^{5/2}}=\frac{2933\cdot 10409}{2^{41/2}\cdot 3^{5/2}\cdot 5\cdot 11}$$

and

$$\frac{1}{2^{17/2}\cdot 3}\cdot\frac{(18)^{9/2}\cdot\sqrt{31}}{(17)^2\cdot(20)^2\cdot 37}\cdot\frac{8\cdot(18)^3-24\cdot(18)^2-18\cdot 18+61}{(18)^3}$$

$$\times\frac{8\cdot(18)^4-12\cdot(18)^3-28\cdot(18)^2+27\cdot 18+5}{(18)^4}$$

$$=\frac{\sqrt{31}\cdot 38617\cdot 761243}{2^{17/2}\cdot 3\cdot(17)^2\cdot(20)^2\cdot 37\cdot(18)^{5/2}}=\frac{\sqrt{31}\cdot 38617\cdot 761243}{2^{15}\cdot 3^6\cdot 5^2\cdot(17)^2\cdot 37}.$$

From these values and Lemma 13.4, we obtain easily (13.10), (13.11) and (13.12).

Q. E. D.

§ 14. Fine evaluations of $V(x, X(x))$.

For the present purpose, we shall do more fine evaluations of the function $V(x, X(x))$ for $\alpha_1(n)\leqq x<1$ than the ones in § 9 and § 10.

LEMMA 14.1. *We have the following inequalities*

$$\frac{112}{27}\cdot\frac{(n-1)^{5/2}(n-4)^2(n^2-n+1)(2n+1)^{3/2}(2n^2-n+8)^{1/2}}{(8n^3+18n+1)^2}$$

$$\times\left[\frac{1}{(2n^2-n+8)^2}+\frac{4n^2+9n-10}{4(n-1)(n+5)^2(2n^2-n-10)}\right]>$$

(14.1) $\qquad\dfrac{2^8\cdot(11)^4\cdot(71)^{1/2}\cdot 133}{3^7\cdot 7\cdot(14041)^2}+\dfrac{2^2\cdot 3^3\cdot 7\cdot(11)^3\cdot(71)^{1/2}\cdot 193}{(17)^2\cdot(14041)^2}\quad for\quad 12\leqq n<16$,

(14.2) $\qquad\dfrac{2^3\cdot 3^{7/2}\cdot 5^{5/2}\cdot 7^{3/2}\cdot(37)^{3/2}\cdot 241}{(319)^2\cdot(11019)^2}+\dfrac{2^{20}\cdot 5^{3/2}\cdot(37)^{3/2}\cdot 181\cdot 241}{3^{29/2}\cdot 7^{1/2}\cdot(11019)^2}$

$$for\quad 16\leqq n<18$$

and

(14.3) $\qquad\dfrac{2^{19/2}\cdot 3^4\cdot 7^3\cdot(17)^{3/2}\cdot 307\cdot(319)^{1/2}}{5\cdot(23)^2\cdot 31\cdot(46981)^2}\quad for\quad n\geqq 18$.

Proof. Setting $u=\dfrac{1}{n}$, the left hand side can be written as in Lemma 10.1:

$$\frac{2^4\cdot 7}{3^3}\cdot\frac{(1-u)^{5/2}(1-4u)^2(1-u+u^2)(2+u)^{3/2}(2-u+8u^2)^{1/2}}{(8+18u^2+u^3)^2}$$

$$\times \left[\frac{u}{(2-u+8u^2)^2} + \frac{4+9u-10u^2}{4(1-u)(1+5u)^2(2-u-10u^2)} \right].$$

We see that $(1-4u)^2(2-u+8u^2)^{1/2}$ is decreasing for $0<u<\dfrac{1}{4}$, because its logarithmic derivative is

$$-\frac{8}{1-4u} - \frac{1-16u}{2(2-u+8u^2)} = -\frac{33-36u+192u^2}{2(1-4u)(2-u+8u^2)} < 0$$

and $\dfrac{u}{(2-u+8u^2)^2}$ is increasing for $0<u<\dfrac{1}{4}$, because its logarithmic derivative is

$$\frac{1}{u} + \frac{2(1-16u)}{2-u+8u^2} = \frac{2+u-24u^2}{u(2-u+8u^2)} > 0.$$

$4+9u-10u^2$ is also increasing for $0<u<\dfrac{1}{4}$, We showed in the proof of Lemma 10.1 that $(1-u)(1+5u)^2(2-u-10u^2)$ is increasing for $0<u\leqq\dfrac{1}{10}$. Making use of these facts, we can do the following evaluations.

When $12\leqq n<16$ $\left(\dfrac{1}{16}<u\leqq\dfrac{1}{12}\right)$, we have

(the left hand side of (10.1))$>$

$$\frac{2^4\cdot7}{3^3} \cdot \frac{(11/12)^{5/2}\cdot(2/3)^2\cdot133/(12)^2\cdot(33/16)^{3/2}\cdot(284)^{1/2}/12\cdot(12)^6}{(14041)^2}$$

$$\times \left[\frac{1/16\cdot(16)^4}{(504)^2} + \frac{1158/(16)^2\cdot(12)^5}{4\cdot11\cdot(17)^2\cdot266} \right]$$

$$= \frac{2^2\cdot7\cdot(11)^4\cdot(71)^{1/2}\cdot133}{3^3\cdot(14041)^2} \cdot \left[\frac{2^6}{3^4\cdot7^2} + \frac{3^6\cdot193}{11\cdot(17)^2\cdot133} \right]$$

$$= \frac{2^8\cdot(11)^4\cdot(71)^{1/2}\cdot133}{3^7\cdot7\cdot(14041)^2} + \frac{2^2\cdot3^3\cdot7\cdot(11)^3\cdot(71)^{1/2}\cdot193}{(17)^2\cdot(14041)^2}.$$

When $16\leqq n<18$ $\left(\dfrac{1}{18}<u\leqq\dfrac{1}{16}\right)$, we have

(the left hand side of (10.1))$>$

$$\frac{2^4\cdot7}{3^3} \cdot \frac{(15/16)^{5/2}\cdot(3/4)^2\cdot241/(16)^2\cdot(37/18)^{3/2}\cdot(504)^{1/2}/16\cdot(16)^6}{(33057)^2}$$

$$\times \left[\frac{1/18\cdot(18)^4}{(638)^2} + \frac{1448/(18)^2\cdot(16)^5}{4\cdot15\cdot(21)^2\cdot486} \right]$$

$$= \frac{2^2\cdot5^{5/2}\cdot7^{3/2}\cdot(37)^{3/2}\cdot241}{3^{5/2}\cdot(11019)^2} \cdot \left[\frac{2\cdot3^6}{(319)^2} + \frac{2^{18}\cdot181}{3^{12}\cdot5\cdot7^2} \right]$$

$$= \frac{2^3 \cdot 3^{7/2} \cdot 5^{5/2} \cdot 7^{3/2} \cdot (37)^{3/2} \cdot 241}{(319)^2 \cdot (11019)^2} + \frac{2^{20} \cdot 5^{3/2} \cdot (37)^{3/2} \cdot 181 \cdot 241}{3^{29/2} \cdot 7^{1/2} \cdot (11019)^2}.$$

When $18 \leqq n$ $\left(0 < u \leqq \frac{1}{18}\right)$, we have

(the left hand side of (10.1))$>$

$$\frac{2^4 \cdot 7}{3^3} \cdot \frac{(17/18)^{5/2} \cdot (7/9)^2 \cdot 307/(18)^2 \cdot 2^{3/2} \cdot (638)^{1/2}/18 \cdot (18)^6}{(46981)^2}$$

$$\times \left[0 + \frac{4 \cdot (18)^5}{4 \cdot 17 \cdot (23)^2 \cdot 620}\right] = \frac{2^{19/2} \cdot 3^4 \cdot 7^3 \cdot (17)^{3/2} \cdot 307 \cdot (319)^{1/2}}{5 \cdot (23)^2 \cdot 31 \cdot (46981)^2}. \qquad \text{Q. E. D.}$$

LEMMA 14.2. *We have the following inequalities*

$$\frac{1}{96} \cdot \frac{n(n+2)^{n-2}(2n-5)^{n-2}}{(n-1)^{n+4}(2n+1)^n} \cdot (8n^3 - 24n^2 - 18n + 61)(8n^4 - 12n^3 - 28n^2 + 27n + 5) >$$

(14.4) $$\frac{(19)^{1/2} \cdot 10213 \cdot 141449}{2^{21/2} \cdot 3^{3/2} \cdot 7^2 \cdot (11)^5} \cdot \frac{e_{15} \cdot (e_8)^2}{\sqrt{e_{32} \cdot (e_{5.4})^{5/2}}} \qquad for \quad 12 \leqq n < 16,$$

(14.5) $$\frac{2933 \cdot 10409}{2^{23/2} \cdot 3^{9/2} \cdot 5^4} \cdot \frac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36} \cdot (e_{6.2})^{5/2}}} \qquad for \quad 16 \leqq n < 18$$

and

(14.6) $$\frac{7 \cdot 38617 \cdot 761243}{2^{29/2} \cdot 3^{15} \cdot 5} \qquad for \quad n \geqq 18.$$

Proof. Setting $u = \frac{1}{n}$, the left hand side can be written as (10.3):

$$\frac{1}{2^{15/2} \cdot 3} \cdot \frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2} \cdot \frac{e(n-1)(e(n/2))^2}{\sqrt{e(2n)(e((2n-5)/5))^{5/2}}}$$

$$\times (8 - 24u - 18u^2 + 61u^3)(8 - 12u - 28u^2 + 27u^3 + 5u^4).$$

We showed that $\frac{\sqrt{2-5u}}{(1-u)^5(1+2u)^2}$ is decreasing for $0 < u < \frac{41 - \sqrt{1161}}{2 \cdot 130} = \frac{1}{37.53\cdots}$

and increasing for $\frac{41 - \sqrt{1161}}{2 \cdot 130} < u < \frac{41 + \sqrt{1160}}{2 \cdot 130} = \frac{1}{3.46\cdots}$ and its value at $u =$

$\frac{41 - \sqrt{1161}}{2 \cdot 130}$ is $1.4095\cdots (> 7/5)$, and that the both functions $8 - 24u - 18u^2 + 61u^3$

and $8 - 12u - 28u^2 + 27u^3 + 5u^4$ are decreasing for $0 < u \leqq (1/10)$. By means of these facts and Lemma 13.4 we can do the following evaluations.

When $12 \leqq n < 16$ $\left(\frac{1}{16} < u \leqq \frac{1}{12}\right)$, we have

(the left hand side of (10.2))$>$

$$\frac{1}{2^{15/2} \cdot 3} \cdot \frac{\sqrt{19} \cdot (12)^{13/2}}{(11)^5 \cdot (14)^2} \cdot \frac{e_{15} \cdot (e_8)^2}{\sqrt{e_{32} \cdot (e_{5.4})^{5/2}}} \cdot \frac{1}{(12)^7} \cdot 10213 \cdot 141449$$

$$= \frac{(19)^{1/2} \cdot 10213 \cdot 141449}{2^{21/2} \cdot 3^{3/2} \cdot 7^2 \cdot (11)^5} \cdot \frac{e_{15} \cdot (e_8)^2}{\sqrt{e_{32} \cdot (e_{5.4})^{5/2}}} \cdot$$

When $16 \leqq n < 18 \left(\frac{1}{18} < u \leqq \frac{1}{16} \right)$, we have

(the left hand side of (10.2))$>$

$$\frac{1}{2^{15/2} \cdot 3} \cdot \frac{\sqrt{27} \cdot (16)^{13/2}}{(15)^5 \cdot (18)^2} \cdot \frac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36} \cdot (e_{6.2})^{5/2}}}$$

$$\times \frac{1}{(16)^7} \cdot 26397 \cdot 468405 = \frac{2933 \cdot 10409}{2^{23/2} \cdot 3^{9/2} \cdot 5^4} \cdot \frac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36} \cdot (e_{6.2})^{5/2}}} \cdot$$

When $18 \leqq n \left(0 < u \leqq \frac{1}{18} \right)$, we have

(the left hand side of (10.2))$>$

$$\frac{1}{2^{15/2} \cdot 3} \cdot \frac{7}{5} \cdot 1 \cdot \frac{1}{(18)^7} \cdot 38617 \cdot 761243 = \frac{7 \cdot 38617 \cdot 761243}{2^{29/2} \cdot 3^{15} \cdot 5}$$

LEMMA 14.3. *The function $e((n-1)/3)/e((2n-5)/6)$ of n is decreasing for* $n > 5/2$

Proof. From the definition of $e(n)$, we have

$$e\left(\frac{n-1}{3} \right) = \left(\frac{n+2}{n-1} \right)^{(n-1)/3} \quad \text{and} \quad e\left(\frac{2n-5}{6} \right) = \left(\frac{2n+1}{2n-5} \right)^{(2n-5)/6}$$

and hence

$$\frac{d}{dn} \left[\frac{n-1}{3} \log \frac{n+2}{n-1} - \frac{2n-5}{6} \log \frac{2n+1}{2n-5} \right]$$

$$= \frac{1}{3} \left(\log \frac{n+2}{n-1} - \log \frac{2n+1}{2n-5} \right) + \frac{n-1}{3} \cdot \frac{-3}{(n+2)(n-1)}$$

$$- \frac{2n-5}{6} \cdot \frac{-12}{(2n+1)(2n-5)}$$

$$= \frac{1}{3} \log \frac{2n^2-n-10}{2n^2-n-1} + \frac{3}{(n+2)(2n+1)} \cdot$$

Setting $u = 1/n$, the above expression can be written as

$$\frac{3u^2}{(1+2u)(2+u)} + \frac{1}{3} \log \frac{2-u-10u^2}{2-u-u^2} ,$$

which becomes 0 at $u = 0$. Its derivative with respect to u is

$$\frac{3u(4+5u)}{(1+2u)^2(2+u)^2} - \frac{3u(4-u)}{(2-u-10u^2)(2-u-u^2)}$$

$$= -\frac{27u(8+24u+12u^2-11u^3-6u^4)}{(1+2u)(2+u)^2(2-u-10u^2)(2-u-u^2)} < 0 \quad \text{for} \quad 0<u<\frac{2}{5}$$

and hence we see that

$$\frac{1}{3}\log\frac{2n^2-n-10}{2n^2-n-1} + \frac{3}{(n+2)(2n+1)} < 0 \quad \text{for} \quad n>\frac{5}{2},$$

which implies the statement of this lemma. Q. E. D.

LEMMA 14.4. *We have the following inequalities*

$$\frac{n}{12(n-1)^{n-3/2}} \cdot \frac{1}{(2n+1)^{n-3/2}(n+2)^{3/2}(2n-5)^{3/2}(4n^2-2n+7)^2}$$

$$\times[4n(n-1)^{n-4}(n+5)(2n+1)^{n-3}(4n^2-2n+7)^2(48n^4-52n^3+342n^2$$

$$-198n+616)-3(n-4)(n+2)^{n-2}(2n-5)^{n-2}(256n^6-992n^5+80n^4$$

$$+4120n^3-3592n^2-5474n+3010]<$$

(14.7) $\quad \dfrac{2^{19/2}\cdot3\cdot17\cdot1489}{5^2\cdot7^{3/2}\cdot(11)^{5/2}\cdot(19)^{3/2}} - \dfrac{2^{19/2}\cdot(11)^{1/2}\cdot13836023}{3^7\cdot5^2\cdot7^{5/2}\cdot(111)^2}\cdot\left(\dfrac{e_5}{e_{4.5}}\right)^3 \quad for \quad 12\leqq n<16,$

(14.8) $\quad \dfrac{2^{19/2}\cdot7\cdot4657}{3^{15/2}\cdot5^{5/2}\cdot(11)^{3/2}} - \dfrac{3^{15/2}\cdot5^{1/2}\cdot748947}{2^{19/2}\cdot7^2\cdot11\cdot(181)^2}\cdot\left(\dfrac{e_{17/3}}{e_{31/6}}\right)^3 \quad for \quad 16\leqq n<18$

and

(14.9) $\quad \dfrac{2\cdot3^3\cdot23\cdot1210861}{5^{3/2}\cdot(17)^{5/2}\cdot(31)^{3/2}\cdot(37)^{3/2}} - \dfrac{7\cdot(17)^{1/2}\cdot3431922239}{2^{12}\cdot3^6\cdot5^{5/2}\cdot31\cdot37} \quad for \quad n\geqq18.$

Proof. Setting $u=1/n$, the left hand side can be written as (10.7):

$$\frac{(1+5u)(48-52u+342u^2-198u^3+616u^4)}{3(1+2u)^{3/2}(2-5u)^{3/2}(1-u)^{5/2}(2+u)^{3/2}}$$

$$-\frac{(1-4u)\sqrt{1-u}}{4(1+2u)^{5/2}(2-5u)(2+u)}\cdot\frac{1}{(4-2u+7u^2)^2}\cdot\left(e\left(\frac{n-1}{3}\right)\middle/e\left(\frac{2n-5}{6}\right)\right)^3$$

$$\times(256-992u+80u^2+4120u^3-3592u^4-5474u^5+3010u^6).$$

We showed in the proof of Lemma 10.3 that the functions of u:

$(1+5u)(48-52u+342u^2-198u^3+616u^4)$ increasing,

$(1+2u)^3(2-5u)^3(1-u)^5(2+u)^3$ decreasing,

$(1-4u)\sqrt{1-u}$ decreasing,

$(1+2u)^{5/2}(2-5u)(2+u)$ increasing,

$256-992u+80u^2+4120u^3-3592u^4-5474u^5+3010u^6$ decreasing

for $0<u\leqq(1/10)$. By means of these facts and Lemma 14.3 we can do the following evaluations.

When $12 \leqq n < 16$ $\left(\dfrac{1}{16} < u \leqq \dfrac{1}{12}\right)$, we have

(the left hand side of (10.6))$<$

$$\frac{17 \cdot 952960 \cdot (12)^7}{3 \cdot (12)^5 \cdot (14)^{3/2} \cdot (19)^{3/2} \cdot (11)^{5/2} \cdot (25)^{3/2}}$$

$$- \frac{8 \cdot \sqrt{11} \cdot (12)^{9/2}}{4 \cdot (12)^{3/2} \cdot (14)^{5/2} \cdot 19 \cdot 25} \cdot \frac{(16)^4}{(999)^2} \cdot (e_5/e_{4.5})^3 \cdot \frac{525768874}{(12)^6}$$

$$= \frac{2^{19/2} \cdot 3 \cdot 17 \cdot 1489}{5^2 \cdot 7^{3/2} \cdot (11)^{5/2} \cdot (19)^{3/2}} - \frac{2^{19/2} \cdot (11)^{1/2} \cdot 13836023}{3^7 \cdot 5^2 \cdot 7^{5/2} \cdot (111)^2} \cdot (e_5/e_{4.5})^3 .$$

When $16 \leqq n < 18$ $\left(\dfrac{1}{18} < u \leqq \dfrac{1}{16}\right)$, we have

(the left hand side of (10.6))$<$

$$\frac{21 \cdot 3017736 \cdot (16)^7}{3 \cdot (16)^5 \cdot (18)^{3/2} \cdot (27)^{3/2} \cdot (15)^{5/2} \cdot (33)^{3/2}}$$

$$- \frac{12 \cdot \sqrt{15} \cdot (16)^{9/2}}{4 \cdot (16)^{3/2} \cdot (18)^{5/2} \cdot 27 \cdot 33} \cdot \frac{(18)^4}{(1267)^2} \cdot (e_{17/3}/e_{31/6})^3 \cdot \frac{3275894178}{(16)^6}$$

$$= \frac{2^{19/2} \cdot 7 \cdot 4657}{3^{15/2} \cdot 5^{5/2} \cdot (11)^{3/2}} - \frac{3^{15/2} \cdot 5^{1/2} \cdot 748947}{2^{19/2} \cdot 7^2 \cdot 11 \cdot (181)^2} \cdot (e_{17/3}/e_{31/6})^3 .$$

When $n \geqq 18$ $\left(0 < u \leqq \dfrac{1}{18}\right)$, we have

(the left hand side of (10.6))$<$

$$\frac{23 \cdot 4843444 \cdot (18)^7}{3 \cdot (18)^5 \cdot (20)^{3/2} \cdot (31)^{3/2} \cdot (17)^{5/2} \cdot (37)^{3/2}}$$

$$- \frac{14 \sqrt{17} \cdot (18)^{9/2}}{4 \cdot (18)^{3/2} \cdot (20)^{5/2} \cdot 31 \cdot 37} \cdot \frac{1}{4^2} \cdot 1 \cdot \frac{6863844478}{(18)^6}$$

$$= \frac{2 \cdot 3^3 \cdot 23 \cdot 1210861}{5^{3/2} \cdot (17)^{5/2} \cdot (31)^{3/2} \cdot (37)^{3/2}} - \frac{7 \cdot (17)^{1/2} \cdot 3431922239}{2^{12} \cdot 3^6 \cdot 5^{5/2} \cdot 31 \cdot 37} \qquad \text{Q. E. D.}$$

§ 15. Proof of Theorem C for $n \geqq 18$.

In this section, we shall prove that Conjecture C is true for $n \geqq 18$. We suppose $n \geqq 18$ in this section. By means of (12.5), (13.3), (13.8) and (13.12), we obtain the following inequality

$$\left\{ \frac{x^2 \sqrt{n-x}\, F_2(x)}{(1-x)^5} + \frac{X^2 \sqrt{n-X}\, F_2(X)}{(X-1)^5} \right\} \cdot \{\tilde{\lambda}(X) - \tilde{\lambda}(\beta)\} > B n^2 \sqrt{n-1}$$

$$\times \left[\frac{2^{11/2} \cdot 7^2 \cdot (17)^{7/2} \cdot 307 \cdot (319)^{1/2}}{3^3 \cdot (37)^{3/2} \cdot (46981)^2} + \frac{\sqrt{31} \cdot 38617 \cdot 761243}{2^{15} \cdot 3^6 \cdot 5^2 \cdot (17)^2 \cdot 37} \right]$$

$$\times \frac{1}{3^2 \cdot 17} \cdot \frac{105379\beta - 123246}{2772 - 307\beta} \qquad \text{for} \quad \alpha_1(n) \leq x < 1 .$$

Now, we do some numerical computation as follows.

(i)
$$\frac{2^{11/2} \cdot 7^2 \cdot (17)^{5/2} \cdot 307 \cdot (319)^{1/2}}{3^5 \cdot (37)^{3/2} \cdot (46981)^2}$$

$$= \frac{2^5 \cdot 7^2 \cdot (17)^2 \cdot 307}{3^5 \cdot 37 \cdot (46981)^2} \times \sqrt{\frac{2 \cdot 17 \cdot 319}{37}} = \frac{32 \cdot 49 \cdot 289 \cdot 307}{243 \cdot 37 \cdot (46981)^2} \times \sqrt{\frac{2 \cdot 17 \cdot 319}{37}}$$

$$> \frac{7.0101896}{10^6} \times 17.1211896 > \frac{1.2002278}{10^4} .$$

(ii)
$$\frac{\sqrt{31} \cdot 38617 \cdot 761243}{2^{15} \cdot 3^8 \cdot 5^2 \cdot (17)^3 \cdot 37} = \frac{1}{10^2} \times \frac{38617 \cdot 761243}{8192 \cdot 6561 \cdot 4913 \cdot 37} \times \sqrt{31}$$

$$> \frac{1}{10^2} \times \frac{3.0088003}{10^3} \times 5.5677643 > \frac{1.67522908}{10^4}$$

(iii)
$$\beta(n) \geq \beta(18) = \frac{-1 + \sqrt{613}}{17} = 1.3975786 \cdots > 1.3975786 \qquad \text{(by (11.6))} .$$

(iv)
$$\frac{105379\beta - 123246}{2772 - 307\beta} > \frac{147275.4352894 - 123246}{2772 - 429.0566302}$$

$$= \frac{24029.4352894}{2342.9433698} = 10.25608881 \cdots > 10.2560888 .$$

By means of the above evaluations, we obtain

$$\left\{ \frac{x^2 \sqrt{n-x} \, F_2(x)}{(1-x)^5} + \frac{X^2 \sqrt{n-X} \, F_2(X)}{(X-1)^5} \right\} \cdot \{ \tilde{\lambda}(X) - \tilde{\lambda}(\beta) \}$$

$$> \frac{Bn^2 \sqrt{n-1}}{10^4} [1.2002278 + 1.67522908] \times 10.2560888$$

$$> \frac{Bn^2 \sqrt{n-1}}{10^4} \times 29.4909411 .$$

Thus, we have proved the following

LEMMA 15.1. *When* $n \geq 18$, *we have the inequality*

(15.1)
$$\left\{ \frac{x^2 \sqrt{n-x} \, F_2(x)}{(1-x)^5} + \frac{X^2 \sqrt{n-X} \, F_2(X)}{(X-1)^5} \right\} \{ \tilde{\lambda}(X) - \tilde{\lambda}(\beta) \}$$

$$> Bn^2 \sqrt{n-1} \times \frac{29.4909411}{10^4} \qquad for \quad \alpha_1(n) \leq x < 1 ,$$

where $X = X(x)$.

Next we shall do analogous evaluations for the function $V(x, X(x))$, which

is more fine than the ones done in § 10. By means of (9.5), (14.3), (14.6) and (14.9), we obtain the following inequality

$$\frac{V(x, X(x))}{sB\sqrt{n-1}} > \left[\frac{2^{19/2} \cdot 3^4 \cdot 7^3 \cdot (17)^{3/2} \cdot 307 \cdot (319)^{1/2}}{5 \cdot (23)^2 \cdot 31 \cdot (46981)^2} + \frac{7 \cdot 38617 \cdot 761243}{2^{29/2} \cdot 3^{15} \cdot 5} \right] n$$

$$- \frac{s+t}{s} \left[\frac{2 \cdot 3^3 \cdot 23 \cdot 1210861}{5^{3/2} \cdot (17)^{5/2} \cdot (31)^{3/2} \cdot (37)^{3/2}} - \frac{7 \cdot (17)^{1/2} \cdot 3431922239}{2^{12} \cdot 3^6 \cdot 5^{5/2} \cdot 31 \cdot 37} \right]$$

$$\text{for} \quad \alpha_1(n) \leqq x < 1$$

Now, we do some numerical computations as follows.

(i)
$$\frac{2^{19/2} \cdot 3^4 \cdot 7^3 \cdot (17)^{3/2} \cdot 307 \cdot (319)^{1/2}}{5 \cdot (23)^2 \cdot 31 \cdot (46981)^2}$$

$$= \frac{1}{10} \times \frac{1024 \cdot 81 \cdot 343 \cdot 17 \cdot 307}{529 \cdot 31 \cdot (46981)^2} \times \sqrt{2 \cdot 17 \cdot 319}$$

$$> \frac{1}{10} \times \frac{4.1020836}{10^3} \times 104.1441309 > \frac{427.2079314}{10^4}.$$

(ii)
$$\frac{7 \cdot 38617 \cdot 761243}{2^{29/2} \cdot 3^{15} \cdot 5} = \frac{1}{10} \times \frac{7 \cdot 38617 \cdot 761243}{16384 \cdot 14348907} \times \sqrt{2}$$

$$> \frac{1}{10} \times \frac{8.75308456}{10} \times 1.41421356 > \frac{1237.8730876}{10^4}.$$

(iii)
$$\frac{2 \cdot 3^3 \cdot 23 \cdot 1210861}{5^{3/2} \cdot (17)^{5/2} \cdot (31)^{3/2} \cdot (37)^{3/2}}$$

$$= \frac{1}{10^2} \times \frac{8 \cdot 27 \cdot 23 \cdot 1210861}{4913 \cdot 961 \cdot 1369} \times \sqrt{5 \cdot 17 \cdot 31 \cdot 37}$$

$$< \frac{1}{10^2} \times \frac{9.3068407}{10} \times 312.2418935 < 2.905985563.$$

(iv)
$$\frac{7 \cdot (17)^{1/2} \cdot 3431922239}{2^{12} \cdot 3^6 \cdot 5^{5/2} \cdot 31 \cdot 37} = \frac{1}{10^3} \times \frac{7 \cdot 3431922239}{512 \cdot 729 \cdot 31 \cdot 37} \times \sqrt{5 \cdot 17}$$

$$> \frac{1}{10^3} \times 56.11443206 \times 9.21954445 > 0.517349500.$$

Making use of the above evaluations, we obtain

$$\frac{V(x, X(x))}{sB\sqrt{n-1}} > \frac{1}{10^4} \times [427.2079314 + 1237.8730876]n$$

$$- \frac{s+t}{s} [2.905985563 - 0.517349500]$$

$$= \frac{1665.081019}{10^4} n - \frac{23886.36063}{10^4} \frac{s+t}{s}.$$

Thus, we have proved the following

LEMMA 15.2. *When* $n \geq 18$, *we have the inequality*

$$(15.2) \qquad V(x, X(x)) > \frac{sB\sqrt{n-1}}{10^4} \times \left(1665.081019n - 23886.36063 \frac{s+t}{s}\right)$$

$$for \quad \alpha_1(n) \leq x < 1.$$

Proof of Theorem C for $n \geq 18$. By means of (11.8), Lemma 15.1 and Lemma 15.2, we have the inequality

$$(15.3) \qquad V(x, \beta) > \frac{sB\sqrt{n-1}}{10^4}$$

$$\times \left[29.4909411 \frac{n^2}{s} + 1665.081019n - 23886.36063 \frac{s+t}{s}\right]$$

$$for \quad \alpha_1(n) \leq x < 1.$$

Since we have the following inequality for $\alpha_1(n) \leq x < 1$

$$\frac{1}{s} = \frac{1}{X(x)-1} > \frac{1}{(2n+10)/(2n+1)-1} = \frac{2n+1}{9}, \quad \frac{t}{s} = \frac{1-x}{X(x)-1} < 1$$

by means of Lemma 6.2 and (6.1), we obtain

$$29.4909411 \frac{n^2}{s} + 1665.081019n - 23886.36063 \frac{s+t}{s}$$

$$> \frac{29.4909411}{9} n^2(2n+1) + 1665.081019n - 47772.72126$$

$$> 3.27677n^2(2n+1) + 1665.08101n - 47772.72126$$

$$\geq 3.27677 \times (18)^2 \times 37 + 1665.08101 \times 18 - 47772.72126$$

$$= 39281.91876 + 29971.45818 - 47772.72126 > 0.$$

Thus, when $n \geq 18$, we obtain the inequality (11.7)

$$V(x, \beta) > 0 \quad for \quad \alpha_1(n) \leq x < 1,$$

which implies Theorem C for $n \geq 18$ by the reason stated in § 11.

§ 16. Proof of Theorem C for $16 \leq n < 18$.

In this section, we shall prove that Conjecture C is true for $16 \leq n < 18$ by the same way as in § 15. We suppose $16 \leq n < 18$ through this section. By means of (12.5), (13.2), (13.7) and (13.11), we obtain the following inequality

$$\left\{\frac{x^2\sqrt{n-x}\,F_2(x)}{(1-x)^5} + \frac{X^2\sqrt{n-X}\,F_2(X)}{(X-1)^5}\right\}\{\tilde{\lambda}(X) - \tilde{\lambda}(\beta)\} > Bn^2\sqrt{n-1}$$

$$\times \left[\frac{2^{11/2} \cdot 3^2 \cdot 5^{7/2} \cdot 7^{1/2} \cdot 241}{(11)^{3/2} \cdot (11019)^2} + \frac{2933 \cdot 10409}{2^{41/2} \cdot 3^{5/2} \cdot 5 \cdot 11} \cdot \frac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36}} \cdot (e_{6.2})^{5/2}}\right]$$

$$\times \frac{2^2}{3^3 \cdot 5} \cdot \frac{16353\beta - 19728}{1936 - 241\beta} \qquad \text{for} \quad \alpha_1(n) \leq x < 1 .$$

Now, we do some numerical computations as follows.

(i) $\quad \dfrac{2^{15/2} \cdot 5^{5/2} \cdot 7^{1/2} \cdot 241}{3 \cdot (11)^{3/2} \cdot (11019)^2} = \dfrac{10^2 \cdot 32 \cdot 241}{3 \cdot 121 \cdot (11019)^2} \times \sqrt{2 \cdot 5 \cdot 7 \cdot 11}$

$$> 10^2 \times \frac{1.7497501}{10^7} \times 27.7488738 > \frac{4.8553594}{10^4} .$$

(ii) $\quad \dfrac{2933 \cdot 10409}{2^{37/2} \cdot 3^{11/2} \cdot 5^2 \cdot 11} = \dfrac{1}{10^2} \times \dfrac{2933 \cdot 10409}{131072 \cdot 729 \cdot 11} \times \sqrt{6}$

$$> \frac{1}{10^2} \times \frac{2.9046307}{10^2} \times 2.4494897 > \frac{7.1148629}{10^4} .$$

(iii) $\quad \dfrac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36}} \cdot (e_{6.2})^{5/2}} = \dfrac{(18/17)^{17} \cdot (10/9)^{18}}{(37/36)^{18} \cdot (72/62)^{15.5}} > \dfrac{2.6424143 \times 6.6624633}{1.6375178 \times 10.1527224}$

$$> \frac{17.6049882}{16.6252637} = 1.05892986 \cdots > 1.0589298 .$$

(iv) $\quad \dfrac{7.1148629}{10^4} \times 1.0589298 > \dfrac{7.5341403}{10^4} .$

(v) $\quad \beta(n) \geq \beta(16) = \dfrac{-1 + \sqrt{481}}{15} = 1.3954474 \cdots > 1.3954474 \quad \text{(by (11.6))} .$

(vi) $\quad \dfrac{16353\beta - 19728}{1936 - 241\beta} > \dfrac{22819.7513322 - 19728}{1936 - 336.3028234} = \dfrac{3091.7513322}{1599.6971766}$

$$= 1.932710376 \cdots > 1.93271037 .$$

By means of the above evaluations, we obtain

$$\left\{\frac{x^2 \sqrt{n-x} \, F_2(x)}{(1-x)^5} + \frac{X^2 \sqrt{n-X} \, F_2(X)}{(X-1)^5}\right\} \{\tilde{\lambda}(X) - \tilde{\lambda}(\beta)\}$$

$$> \frac{Bn^2 \sqrt{n-1}}{10^4} \times [4.8553594 + 7.5341403] \times 1.93271037$$

$$> \frac{Bn^2 \sqrt{n-1}}{10^4} \times 23.9453145 .$$

Thus, we have proved the following

LEMMA 16.1. *When* $16 \leq n < 18$, *we have the inequality*

(16.1)
$$\left\{ \frac{x^2\sqrt{n-x}\,F_2(x)}{(1-x)^5} + \frac{X^2\sqrt{n-X}\,F_2(X)}{(X-1)^5} \right\}\{\tilde{\lambda}(X) - \tilde{\lambda}(\beta)\}$$

$$> Bn^2\sqrt{n-1} \times \frac{23.9453145}{10^4} \quad for \quad \alpha_1(n) \leqq x < 1,$$

where $X = X(x)$.

Next, by means of (9.5), (14.2), (14.5) and (14.8) we obtain the following inequality

$$\frac{V(x, X(x))}{sB\sqrt{n-1}} > \left[\frac{2^3 \cdot 3^{7/2} \cdot 5^{5/2} \cdot 7^{3/2} \cdot (37)^{3/2} \cdot 241}{(319)^2 \cdot (11019)^2} + \frac{2^{20} \cdot 5^{3/2} \cdot (37)^{3/2} \cdot 181 \cdot 241}{3^{29/2} \cdot 7^{1/2} \cdot (11019)^2} \right.$$

$$\left. + \frac{2933 \cdot 10409}{2^{23/2} \cdot 3^{9/2} \cdot 5^4} \cdot \frac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36} \cdot (e_{6.2})^{5/2}}} \right] n$$

$$- \frac{s+t}{s}\left[\frac{2^{19/2} \cdot 7 \cdot 4657}{3^{15/2} \cdot 5^{5/2} \cdot (11)^{3/2}} - \frac{3^{15/2} \cdot 5^{1/2} \cdot 748947}{2^{19/2} \cdot 7^2 \cdot 11 \cdot (181)^2} \cdot \left(\frac{e_{17/3}}{e_{31/6}} \right)^3 \right]$$

$$for \quad \alpha_1(n) \leqq x < 1.$$

Now, we do some numerical computations as follows.

(i) $\dfrac{2^3 \cdot 3^{7/2} \cdot 5^{5/2} \cdot 7^{3/2} \cdot (37)^{3/2} \cdot 241}{(319)^2 \cdot (11019)^2} = 10^2 \times \dfrac{2 \cdot 27 \cdot 7 \cdot 37 \cdot 241}{(319)^2 \cdot (11019)^2} \times \sqrt{3 \cdot 5 \cdot 7 \cdot 37}$

$$> \frac{2.7280029}{10^5} \times 62.3297681 > \frac{17.0035788}{10^4}.$$

(ii) $\dfrac{2^{20} \cdot 5^{3/2} \cdot (37)^{3/2} \cdot 181 \cdot 241}{3^{29/2} \cdot 7^{1/2} \cdot (11019)^2} = 10 \times \dfrac{524288 \cdot 37 \cdot 181 \cdot 241}{14348907 \cdot 7 \cdot (11019)^2} \times \sqrt{3 \cdot 5 \cdot 7 \cdot 37}$

$$> \frac{6.9385070}{10^4} \times 62.3297681 > \frac{432.4755322}{10^4}.$$

(iii) $\dfrac{2933 \cdot 10409}{2^{23/2} \cdot 3^{9/2} \cdot 5^4} = \dfrac{1}{10^4} \times \dfrac{2933 \cdot 10409}{256 \cdot 243} \times \sqrt{2 \cdot 3}$

$$> \frac{1}{10^4} \times 490.7664127 \times 2.4494897 > \frac{12.0212727}{10^2}$$

and

$$\frac{2933 \cdot 10409}{2^{23/2} \cdot 3^{9/2} \cdot 5^4} \times \frac{e_{17} \cdot (e_9)^2}{\sqrt{e_{36} \cdot (e_{6.2})^{5/2}}} > \frac{12.0212727}{10^2} \times 1.0589298 > \frac{1272.9683895}{10^4}.$$

(iv) $\dfrac{2^{19/2} \cdot 7 \cdot 4657}{3^{15/2} \cdot 5^{5/2} \cdot (11)^{3/2}} = \dfrac{1}{10^3} \times \dfrac{4096 \cdot 7 \cdot 4657}{6561 \cdot 121} \times \sqrt{2 \cdot 3 \cdot 5 \cdot 11}$

$$< \frac{168.1933489}{10^3} \times 18.1659022 < 3.055383927.$$

(v) $\dfrac{3^{15/2} \cdot 5^{1/2} \cdot 748947}{2^{19/2} \cdot 7^2 \cdot 11 \cdot (181)^2} = \dfrac{2187 \cdot 748947}{1024 \cdot 49 \cdot 11 \cdot (181)^2} \times \sqrt{2 \cdot 3 \cdot 5}$

$> \dfrac{9.05845219}{10^2} \times 5.47722557 > 0.496151859$.

(vi) $\left(\dfrac{e_{17/3}}{e_{31/6}} \right)^3 = \left(\dfrac{20}{17} \right)^{17} \Big/ \left(\dfrac{37}{31} \right)^{31/2} > \dfrac{15.8444899}{15.5246018} = 1.020605237 \cdots$

> 1.020605237 .

(vii) $\dfrac{3^{15/2} \cdot 5^{1/2} \cdot 748947}{2^{19/2} \cdot 7^2 \cdot 11 \cdot (181)^2} \times \left(\dfrac{e_{17/3}}{e_{31/6}} \right)^3 > 0.496151895 \times 1.020605237 > 0.506375185$.

Making use of the above evaluations, we obtain

$$\dfrac{V(x, X(x))}{sB\sqrt{n-1}} > \dfrac{1}{10^4} \times [17.0035788 + 432.4755322 + 1272.9683895] n$$

$$- \dfrac{s+t}{s} [3.055383927 - 0.506375185] = \dfrac{1722.4475005}{10^4} n - \dfrac{25490.08742}{10^4} \dfrac{s+t}{s} .$$

Thus, we have proved the following

LEMMA 16.2.   *When* $16 \leqq n < 18$, *we have the inequality*

(16.2) $V(x, X(x)) > \dfrac{sB\sqrt{n-1}}{10^4} \times \left(1722.4475 n - 25490.08742 \dfrac{s+t}{s} \right)$

for $\alpha_1(n) \leqq x < 1$.

Proof of Theorem C for $16 \leqq n < 18$.   By means of (11.8), Lemma 16.1 and Lemma 16.2, we have the inequality

(16.3) $V(x, \beta)$

$$> \dfrac{sB\sqrt{n-1}}{10^4} \times \left[23.9453145 \dfrac{n^2}{s} + 1722.4475 n - 25490.08742 \dfrac{s+t}{s} \right]$$

for $\alpha_1(n) \leqq x < 1$.

By the same reason as in § 15, we have

$$23.9453145 \dfrac{n^2}{s} + 1722.4475 n - 25490.08742 \dfrac{s+t}{s}$$

$$> \dfrac{23.9453145}{9} n^2(2n+1) + 1722.4475 n - 25490.08742 \times 2$$

$$> 2.66059 n^2(2n+1) + 1722.4475 n - 50980.17484 .$$

Now, for the last cubic polynome of n we have

$$[2.66059 n^2(2n+1) + 1722.4475 n - 50980.17484]_{n=16}$$

$$=22476.66432+27559.16-50980.17484=-944.35052<0$$

and

$$[2.66059n^2(2n+1)+1722.4475n-50980.17484]_{n=16.1}$$

$$\doteqdot 22896.43093+27731.40475-50980.17484=-352.33916<0$$

and then

$$[2.66059n^2(2n+1)+1722.4475n-50980.17484]_{n=16.2}$$

$$>23321.39100+27903.6495-50980.17484=244.86566>0 .$$

Hence we see that when $n\geqq 16.2$ we obtain the inequality (11.7)

$$V(x,\ \beta)>0 \quad \text{for} \quad \alpha_1(n)\leqq x<1 ,$$

which implies Theorem C for $16.2\leqq n<18$.

Now, in order to prove (11.7) for $16\leqq n<16.2$ we put

$$(16.3) \qquad\qquad \alpha_2(n)=X_n^{-1}\Big(\frac{2n+9}{2n+1}\Big) .$$

For $\alpha_2(n)\leqq x<1$, we have

$$\frac{1}{s}\geqq\frac{2n+1}{8} , \qquad \frac{s+t}{s}<2$$

and so

$$23.9453145\frac{n^2}{s}+1722.4475n-25490.08742\frac{s+t}{s}$$

$$>\frac{23.9453145}{8}n^2(2n+1)+1722.4475n-25490.08742\times 2$$

$$>2.99316n^2(2n+1)+1722.4475n-50980.17484$$

$$\geqq 2.99316\times(16)^2\times 33+1722.4475\times 16-50980.17484$$

$$=25286.21586+27559.16-50980.17484=1865.20084>0 .$$

Then, for $\alpha_1(n)\leqq x<\alpha_2(n)$, we evaluate $\dfrac{s+t}{s}$. By Proposition 4 in (IV), §9, $\dfrac{X(x)-1}{1-x}$ is decreasing with respect to x for $0<x<1$, hence we have

$$1<\frac{s+t}{s}<\Big[\frac{s+t}{s}\Big]_{x=\alpha_2(n)} \qquad \text{for} \quad \alpha_1(n)\leqq x<\alpha_2(n)$$

and

$$\Big[\frac{s+t}{s}\Big]_{x=\alpha_2(n)}=1+\frac{1-\alpha_2(n)}{8/(2n+1)}=1+\frac{2n+1}{8}(1-\alpha_2(n)) ,$$

i.e.

(16.4) $$\frac{s+t}{s} < \frac{2n+9}{8} - \frac{2n+1}{8}\alpha_2(n).$$

By means of Proposition 1 in (IV), §2, we have

$$n-(n-1)(\alpha_2(n))^{1/(n-1)} < \frac{2n+9}{2n+1},$$

hence

(16.5) $$\left(\frac{2n^2-n-9}{2n^2-n-1}\right)^{n-1} < \alpha_2(n).$$

From (16.4) and (16.5), we obtain

(16.6) $$\frac{s+t}{s} < \frac{2n+9}{8} - \frac{2n+1}{8}\left(\frac{2n^2-n-9}{2n^2-n-1}\right)^{n-1}.$$

$\frac{2n^2-n-9}{2n^2-n-1}$ is monotone increasing with respect to n, and hence from (16.6) we obtain for $16 \leq n < 16.2$

$$\frac{s+t}{s} < \frac{2\times16.2+9}{8} - \frac{2\times16+1}{8}\cdot\left(\frac{2\times(16)^2-16-9}{2\times(16)^2-16-1}\right)^{16.2-1}$$

$$= \frac{41.4}{8} - \frac{33}{8}\cdot\left(\frac{487}{495}\right)^{15.2} < 5.175 - 4.125\times0.7806227$$

$$= 5.175 - 3.2200686375 = 1.9549313625 < 1.96.$$

Making use of this evaluation, for $\alpha_1(n) \leq x < \alpha_2(n)$ we have

$$23.9453145\frac{n^2}{s} + 1722.4475n - 25490.08742\frac{s+t}{s}$$

$$> \frac{23.9453145}{9}n^2(2n+1) + 1722.4475n - 25490.08742\times1.96$$

$$> 2.66059n^2(2n+1) + 1722.4475n - 49960.5713432$$

$$\geq 2.66059\times(16)^2\times33 + 1722.4475\times16 - 49960.5713432$$

$$= 22476.66432 + 27559.16 - 49960.5713432 = 75.2529768 > 0.$$

Thus, we have proved that when $16 \leq n < 16.2$ the inequality (11.7) is also true for $\alpha_1(n) \leq x < 1$. This fact implies that Conjecture C is ture for $16 \leq n < 16.2$.

From the observations in §15 and §16, we obtain the main theorem.

THEOREM. C. *The period function T as a function of τ and n is monotone decreasing with respect to $n \geq 16$ for any fixed τ $(0 < \tau < 1)$.*

Remark. In fact, Conjecture C is natural for small values of n $(2 < n < 30)$

from the numerical data obtained by M. Urabe (see Fig. 9 in [6]). From the arguments in §16, it seems to the author that in order to prove Theorem C for $2<n<16$ we have to develope another and more fine way than the ones in this paper.

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