S. SASAO
KODAI MATH. J.
5 (1982), 446—453

FIBRE HOMOTOPY SELF-EQUIVALENCES

BY SEIYA SASAO

Introduction.

Let ξ be a fibre space $p: E \to E$ which means that the projection p has the COVERING HOMOTOPY PROPERTY for CW-complexes. Then a map $f: E \to E$ is a fibre preserving map if $p \circ f = p$ and a map f_0 is fibre homotopic to a map f_1 if there exists a homotopy $f_t: E \to E$ such that $p \circ f_t = p$ for all t. Now we call a fibre preserving map $f: E \to E$ a fibre homotopy self-equivalence if there is a fibre preserving map $g: E \to E$ such that $g \circ f$ and $f \circ g$ are both fibre homotopic to the identity 1_E. Then it is clear that the set of fibre homotopy classes of fibre homotopy self-equivalences forms a group under the multiplication defined by the composite of maps, which we denote by $\mathcal{L}(\xi)$. This group $\mathcal{L}(\xi)$ has been studied by several authors ([4], [5]) and also the purpose of this note is to investigate $\mathcal{L}(\xi)$ for ξ, a sphere bundle over a sphere. By using Gottlieb's theorem, K. Tsukiyama showed in a preprint that there exists a split extension:

$$0 \longrightarrow \pi_{n+q}(S^q) \longrightarrow \mathcal{L}(\xi) \longrightarrow Z_2 \longrightarrow 0$$

for ξ, a S^q-bundle over S^n $(n+2 \leqq q)$. As a generalization of this result we prove

THEOREM A. *Let $\xi: S^q \to E \to S^n$ be a S^q-bundle over S^n $(n, q > 2)$ with a cross-section, so that there exists $\eta \in \pi_{n-1}(SO(q))$ with $i_*(\eta) = \xi$. If $J(\eta)$ is contained in $\Sigma^2(\pi_{n+q-3}(S^{q-2}))$ we have an exact sequence·*

$$0 \longrightarrow \pi_{n+q}(S^q)/[\pi_{n+1}(S^q), \iota_q] \longrightarrow \mathcal{L}(\xi) \longrightarrow Z_2 \sharp P_n^q \longrightarrow 0,$$

where P_n^q denotes the kernel of the homomorphism defined by Whitehead product $[\ , \iota_q]: \pi_n(S^q) \to \pi_{n+q-1}(S^q)$ and $\sharp$ denotes the semi-direct product with a relation $\tau \cdot b \cdot \tau = (-\iota_q)_* b$ for $\tau \neq 1 \in Z_2$.

Moreover, as a bi-product of the proof of Theorem A, we obtain

THEOREM B. *If $J(\eta) \circ \Sigma^{q-1} \pi_{n+1}(S^n) \subset [\pi_{n+1}(S^q), \iota_q]$, then fibre homotopy self-equivalences $f_0, f_1: E \to E$ are fibre homotopic if and only if they are homotopic.*

This research was partially supported by the Grant-in-Aid, No. 56540012 for science research from the Ministry of Education.
Received November 2, 1981

COROLLARY. *In the case* $n \leqq 2q-2$, *if* $J(\xi)\circ\Sigma^q(\pi_{n+1}(S^n))=0$ *we have the same conclusion as Teeorem B.*

For example, Let $V_{n,2}$ be the Stiefel manifold $O(n)/O(n-2)$ and let ξ_n be the fibring $S^{n-2}\to V_{n,2}\to S^{n-1}$. Then the conditions of Theorem A are fulfiled if $n\equiv 0 \bmod 4$, $n\geqq 8$. Thus, from well-known results $[h_{n-1}, \iota_{n-2}]\neq 0$ and $[h_{n-1}h_n, \iota_{n-2}]=0$ for $n\equiv 0 \bmod 4$, $h_n\neq 0\in\pi_n(S^{n-1})$ we obtain an exact sequence ($n\equiv 0 \bmod 4$):

$$0 \longrightarrow \pi_{2n-3}(S^{n-2}) \longrightarrow \mathcal{L}(\xi_n) \longrightarrow Z_2 \longrightarrow 0\,.$$

Analogously, in the cases of the complex and quaternion, there exist following exact sequences:

$$0 \longrightarrow \pi_{4n-4}(S^{2n-3})/[\pi_{2n}(S^{2n-3}), \iota_{2n-3}] \longrightarrow \mathcal{L}(\mu_n) \longrightarrow Z_2\# P^{2n-3}_{2n-1} \longrightarrow 0 \qquad (n\equiv 0 \bmod 4)\,,$$

$$0 \longrightarrow \pi_{8n-6}(S^{4n-5}) \longrightarrow \mathcal{L}(v_n) \longrightarrow Z_2 \longrightarrow 0 \qquad (n\equiv 0 \bmod 24)\,,$$

where μ_n is the fibring: $S^{2n-3}\to W_{n,2}\to S^{2n-1}$ and v_n is the fibring: $S^{4n-5}\to X_{n,2}\to S^{4n-1}$.

Here, we list up some notations which are used throughout this note.

X^Y: the space of continuous maps $Y\to X$ endowed with $C-O$ topology.

X^Y_0: the sub-space of X^Y consisting of a base point preserving maps.

$\pi_0(X, x_0)$: the set of path-connected components of X with the distinguished point x_0.

J: the J-homomorphism: $\pi_r(SO(n))\to\pi_{n+r}(S^n)$.

Σ^n: the n-fold suspension functor.

$[\alpha, \beta]$: Whitehead product.

1. Preliminaries.

Let $\xi: S^q\to E \xrightarrow[p]{} S^n$ be a S^q-bundle over S^n with the characteristic class $\xi\in\pi_{n-1}(SO(q+1))$ and we consider the fibring

$$p^E: E^E \longrightarrow S^{n^E}$$

which is obtained by $p^E(f)=p\circ f$. Clearly the fibre over p is the space of fibre preserving maps: $E\to E$, which we denote by ξ^ξ. Then we have a part of the exact sequence associated with the fibring:

$$(1.1) \quad \pi_1(E^E, 1_E) \xrightarrow[p^E_*]{} \pi_1(S^{n^E}, p) \longrightarrow \pi_0(\xi^\xi, 1_E) \xrightarrow[i_*]{} \pi_0(E^E, 1_E) \longrightarrow \pi_0(S^{n^E}, p)\,.$$

Here we note that ξ^ξ, E^E are Hopf spaces with the multiplication defined by the composite of maps and $\pi_0(\xi^\xi, 1_E)$, $\pi(E^E, 1_E)$ are semi-groups and appropriate arrows are homomorphic.

Since $\mathcal{L}(\xi)$ is the group consisting of invertible elements of $\pi_0(\xi^\xi, 1_E)$ the sequence (1.1) is transformed into the exact sequence,

$$(1.2)\qquad \pi_1(E^E, 1_E) \xrightarrow[p_*^E]{} \pi_1(S^{n^E}, p) \xrightarrow[\partial]{} \mathcal{L}(\xi) \xrightarrow[i_*]{} \mathcal{E}(E) \xrightarrow[p_*^E]{} \pi_0(S^{n^E}, p),$$

where $\mathcal{E}(E)$ denotes the group of homotopy classes of self-homotopy equivalences of E. For, it is sufficient for exactness to show $(p_*^E)^{-1}(p)=i_*(\mathcal{L}(\xi))$. Let f be an element of $\mathcal{E}(E)$ such that $p\circ f=p$ and g be the inverse of f, i.e. $g\circ f\sim 1_E\sim f\circ g$. Since it holds that $p\sim p\circ(f\circ g)\sim(p\circ f)\circ g\sim p\circ g$ we may consider that f and g are both contained in $\mathcal{L}(\xi)$. Then $f\circ g$ is contained in ∂-image, i.e. $f\circ g=\partial(\sigma)$ for some $\sigma\in\pi_1(S^{n^E}, p)$ and this shows that f has a right inveres $g\circ\partial(\sigma^{-1})$. Analogousely f has a left inverse, hence f is an invertible element of $\pi_0(\xi^\xi, 1_E)$, i.e. $[f]\in i_*(\mathcal{L}(\xi))$.

Thus our purpose is to clarify the image of the homomorphism

$$p_*^E\colon \pi_1(E^E, 1_E) \longrightarrow \pi_1(S^{n^E}, p)$$

and the kernel of the morphism

$$p_*^E\colon \mathcal{E}(E) \longrightarrow \pi_0(S^{n^E}, p).$$

However, for computing these things, it is seemed to need some additional conditions. We assume that ξ has a cross-section ι_n and $n, q\geqq 3$, so that E has a CW-decomposition ([3]):

$$E=K\bigcup_\alpha e^{n+q}, \qquad K=S^q\vee S^n,$$

where $\alpha=J(\eta)+[\iota_n, \iota_q]$ for $\eta\in\pi_{n-1}(SO(q))$ with $i_*(\eta)=\xi$.

2. Barcus-Barratt operations.

We consider two fibrings

$$\begin{array}{ccccc} F_2 & \longrightarrow & E_0^E & \xrightarrow{r_2} & E_0^K \\ & & \downarrow p^E & & \downarrow p^K \\ F_1 & \longrightarrow & S_0^{n^E} & \xrightarrow[r_1]{} & S_0^{n^K}, \end{array}$$

where r_ι denotes the restriction map and $F_1=r_1^{-1}(p|K)$, $F_2=r_2^{-1}(\iota_K)$ i.e.

$$F_1=\{g\colon E \longrightarrow S^n\ g(_*)=*,\ g|K=p|K\},$$

$$F_2=\{f\colon E \longrightarrow E\ f(*)=*,\ f|K=i_K\}.$$

Then we have two boundary homomorphisms:

$$(2.1)\qquad \pi_1(E_0^K, i_K) \xrightarrow[\partial_2]{} \pi_0(F_2, 1_E)\cong\pi_{n+q}(E),$$

and

$$\pi_1(S_0^{n^K}, pK) \xrightarrow[\partial_1]{} \pi_0(F_1, p)\cong\pi_{n+q}(S^n).$$

By Barcus-Barratt (page 62 of [1]) we may regard ∂_1 and ∂_2 as Barcus-Barratt operations $p_*(\alpha)_{p|K}$ and α_{ι_K} respectively.

Now, using the identification $\pi_1(X_0^{S^n}, u)=\pi_{n+1}(X)$ given by Barcus-Barratt (page 59 of [1]) we obtain identifications:

$$\begin{array}{l}\pi_1(E_0^K, i_K)=\pi_1(E_0^{S^q}, \iota_q)\times\pi_1(E_0^{S^n}, \iota_n)=\pi_{q+1}(E)\times\pi_{n+1}(E)=\left\{\begin{bmatrix} x & y \\ u & v\end{bmatrix}\right\} \\ \big\downarrow \alpha_{\iota_K} \\ \pi_0(F_2, 1_E)=\pi_{n+q}(E)=\pi_{n+q}(S^q)\times\pi_{n+q}(S^n),\end{array}$$

where $x\in\pi_{q+1}(S^q)$, $y\in\pi_{n+1}(S^q)$, $u\in\pi_{q+1}(S^n)$ and $v\in\pi_{n+1}(S^n)$.

Then, from Theorem 4.1, 4.2, and 4.6 of [1], we obtain

LEMMA 2.2. $$\alpha_{\iota_K}\left(\begin{bmatrix} x & y \\ u & v\end{bmatrix}\right)=x\Sigma J(\eta)+J(\eta)\Sigma^{n-1}x+(-1)^{n+1}[y, \iota_q]$$
$$+J(\eta)\Sigma^{q-1}v+u\Sigma J(\eta)+(-1)^{n+q+1}[\iota_n, u].$$

If $J(\eta)$ is contained in Σ^2-image we have

$$x\Sigma J(\eta)=J(\eta)\Sigma^{n-1}x \qquad \text{for all} \quad x\in\pi_{q+1}(S^q)$$

from Hilton-Barratt formula, Hence Lemma 2.2 is restated as follows:

$$\partial_2\left(\begin{bmatrix} x & y \\ u & v\end{bmatrix}\right)=J(\eta)\Sigma^{q-1}v+(-1)^{n+1}[y, \iota_q]+u\Sigma J(\eta)+(-1)^{n+q-1}[\iota_n, u]. \tag{2.3}$$

Moreover, by using the identification:

$$\pi_1(S_0^{n^K}, p/K)=\pi_1(S_0^{n^{S^q}}, pt)\times\pi_1(S_0^{S^n}, 1_{S^n})=\pi_{q+1}(S^n)+\pi_{n+1}(S^n)$$

we have

LEMMA 2.4. $$\partial_1(u, v)=u\Sigma J(\eta)+(-1)^{n+q+1}[\iota_n, u]$$

3. The image of $p_*^E: \pi_1(E^E, 1_E)\to\pi_1(S^{n^E}, p)$.

First we note that the homomorphism:

$$p_*^E: \pi_1(E^E, 1_E)\longrightarrow\pi_1(S^{n^E}, p)$$

is equivalent to the homomorphism:

$$P_{0*}^E: \pi_1(E_0^E, 1_E)\longrightarrow\pi_1(S_0^{n^E}, p)$$

because of 2-connectedness of E and S^n.

Now consider the diagram which is obtained from two fibrings as stated in § 2:

$$\begin{array}{ccc}
\pi_1(F_2, 1_E)=\pi_{n+q+1}(E) & \xrightarrow{p_*} & \pi_{n+q+1}(S^n)=\pi_1(F_1, p|K) \longrightarrow 0 \\
\downarrow & & \downarrow \\
\pi_1(E_0^E, 1_E) & \xrightarrow{p_{0*}^E} & \pi_1(S_0^{nE}, p) \\
\downarrow r_{2*} & & \downarrow r_{1*} \\
\pi_1(E_0^K, i_K) & \longrightarrow & \pi_1(S_0^{nK}, p|K) \\
\downarrow \partial_2 & & \downarrow \partial_1 \\
\pi_0(F_2, 1_E)=\pi_{n+q}(E) & \xrightarrow{p_*} & \pi_{n+q}(S^n)=\pi_0(F_1, p|K) \longrightarrow 0 .
\end{array}$$

Thus, using lemma 2.2 and 2.3, it is easy to obtain isomorphisms:

$$\begin{aligned}
&\pi_1(S_0^{nE}, p)/p_{0*}^E(\pi_1(E_0^E, 1_E)) \\
&\quad\cong \{\pi_1(S_0^{nE}, p)/\pi_1(F_1, p|K)\}/\{p_{0*}^E(\pi_1(E_0^E, 1_E)/\pi_1(F_2, 1_E))\} \\
&\quad\cong \partial_1^{-1}(p|K)/p_{0*}^K(\partial_2^{-1}(1_E)) \cong A/B ,
\end{aligned}$$

where $A=(u, v)|u\in\pi_{n+q}(S^n)$, $v\in\pi_{n+1}(S^n)$ with $u\Sigma J(\eta)+(-1)^{n+q+1}[\iota_n, u]=0\}$ and $B=A\cap\{(u, v)|J(\eta)\Sigma^{q-1}v+(-1)^{n+q+1}[y, \iota_q]=0$ for some $y\in\pi_{n+1}(S^q)\}$.

Then we have

LEMMA 3.1 *If $J(\eta)$ is contained in Σ^2-image we have*

$$\begin{aligned}
\pi_1(S_0^{nE}, p)/p_{0*}(\pi_1(E_0^E, 1_E)) &\cong \{0\} \quad \textit{if } J(\eta)\Sigma^{q-1}\pi_{n+1}(S^n)\subset[\pi_{n+1}(S^q), \iota_q] , \\
&\cong Z_2 \quad \textit{if otherwise.}
\end{aligned}$$

4. The kernel of $\mathcal{E}(E)\to\pi_0(S^{nE}, p)$=Image of $\mathcal{L}(\xi)\to\mathcal{E}(E)$.

First consider the following diagram (the continuation of the preceeding one):

$$\begin{array}{ccccc}
 & & \pi_1(E_0^K, i_K) & \longrightarrow & \pi_1(S_0^{nK}, p|K) \\
 & & \downarrow \partial_2 & & \downarrow \partial_1 \\
 & & \pi_{n+q}(S^q)\times\pi_{n+q}(S^n)=\pi_0(F_2, 1_E) & \longrightarrow & \pi_0(F_1, p)=\pi_{n+q}(S^n) \\
 & & \downarrow i_{2*} & & \downarrow i_{1*} \\
\pi_0(\xi^\xi, 1_E) & \xrightarrow{\dot{i}_*} & \pi_0(E_0^E, 1_E) & \longrightarrow & \pi_0(S_0^{nE}, p) \\
 & & \downarrow r_{2*} & & \downarrow r_{1*} \\
 & & \left\{\begin{bmatrix}\pi_q(S^q) & \pi_q(S^n) \\ \pi_n(S^q) & \pi_n(S^n)\end{bmatrix}\right\}=\pi_0(E_0^K, i_K) & \longrightarrow & \pi^0(S_0^{nK}, p|K)=\pi_q(S^n)\times\pi_n(S^n) .
\end{array} \tag{4.1}$$

Let $f: E\to E$ be a map such that $p\circ f=p$. From the commutativity of (4.1) we obtain

$$r_{2*}(f)=\begin{bmatrix} a & 0 \\ \beta & 1_{S^n} \end{bmatrix}, \ a\in\pi_q(S^q) \quad \text{and} \quad \beta\in\pi_n(S^n).$$

Conversely we take an element of $\pi_0(E_0^K, i_K)$ with a form $\begin{bmatrix} a & 0 \\ \beta & 1_{S^n} \end{bmatrix}$ for $a\in\pi_q(S^q)$, $\beta\in\pi_n(S^q)$, i.e. a map $\hat{f}\colon K=S^q\vee S^n\to E$ defined by $\hat{f}|S^q=a\iota_q$ and $\hat{f}|S^n=\iota_n+\iota_q\circ\beta$. Since it holds

$$\hat{f}_*(\alpha)=a\alpha+a[\beta, \iota_q]$$

$\hat{f}$ is extendable over E if and only if $a[\beta, \iota_q]=0$. Let f be an extension of $\hat{f}$ in such a case. Since $p\circ f|K=p\circ\hat{f}=p|K$ the separation element $d(pf, p)$ $(\in\pi_{n+q}(S^n))$ is defined and we have

$$d(pf, p)=d(pf, pf')+d(pf', p)=p_*d(f, f')+d(pf', p)$$

for another extension of $\hat{f}$, f'. Thus we obtain

LEMMA 4.2. *There exists an exact sequence:*

$$i_*(\pi(\xi^\xi, 1_E)) \longrightarrow \left\{\begin{bmatrix} a & 0 \\ \beta & 1_{S^n} \end{bmatrix} a\in\pi_q(S^q),\ \beta\in\pi_n(S^q),\ [\beta, \iota_q]=0\right\} \longrightarrow 0.$$

Moreover, using the following identities obtained from the diagram (4.1):

$$\begin{aligned} & i_*(\pi_0(\xi^\xi, 1_E))\cap i_{2*}\{\pi_{n+q}(S^q)\times\pi_{n+1}(S^n)\} \\ &= i_{2*}(\pi_{n+q}(S^q)\times\partial_1\pi_1(S_0^{nE}, p|K)) \\ &= i_{2*}(\pi_{n+q}(S^q)\times(u\Sigma J(\eta)+(-1)^{n+q+1}[\iota_n, u])),\ u\in\pi_{q+1}(S^n), \end{aligned}$$

we can easily obtain

LEMMA 4.3. *There exists an exact sequence.*

$$0\longrightarrow C\longrightarrow i_*(\pi_0(\xi^\xi, 1_E))\longrightarrow D\longrightarrow 0,$$

where $C=i_{2*}\{(\pi_{n+q}(S^q)\times(\pi_{q+1}(S^n)\Sigma J(\eta)+(-1)^{n+q+1}[\iota_n, \pi_{n+1}(S^n)]))\}$ *and* D *denotes the middle term in lemma* 4.2.

5. The proof of Theorems.

We start from the diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \pi_1(S^{n^E}, p)/p^E_*(\pi_1(E^E, 1_E)) & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{L}(\xi) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & C\cap\mathcal{E}(E) & \longrightarrow & \mathcal{E}(E)\cap i_*(\pi_0(\xi^\xi, 1_E)) & \longrightarrow & D_1 & \longrightarrow & 0\,, \end{array}$$

where D_1 denotes the subgroup of D with $a=1$. On the other hand, the intersection $D_1\cap i_*(\pi_0(\xi^\xi, 1_E))$ is clearly contained in $\mathcal{E}(E)$. Then from lemma 2.2 we have an isomorphism:

$$D_1\cap\mathcal{E}(E)\cong\pi_{n+q}(S^q)/\{J(\eta)\Sigma^{q-1}\pi_{n+1}(S^n)+(-1)^{n+1}[\pi_{n+1}(S^q), \iota_q]\}\ .$$

Now Theorem B follows from lemma 3.1 by using the diagram (5.1) and the above isomorphism.

In the case $J(\eta)\Sigma^{q-1}\pi_{n+1}(S^n)\subset[\pi_{n+1}(S^q), \iota_q]$, Theorem A can be analogously obtained.

Finally we consider the case $J(\eta)\Sigma^{q-1}\pi_{n+1}(S^n)\not\subset[\pi_{n+1}(S^q), \iota_q]$.

We show that a homomorphism ϕ:

$$G=\pi_{n+q}(S^q)/[\pi_{n+1}(S^q), \iota_q] \longrightarrow \pi_0(\xi^\xi, 1_E)$$

can be defined, which makes the following diagram commute,

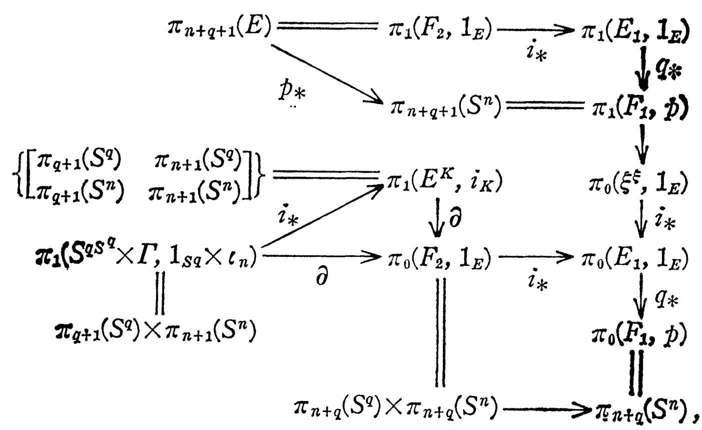
$$\begin{array}{ccc} \pi_0(\xi^\xi, 1_E) & \longrightarrow & \pi_0(E^E, 1_E) \\ \uparrow & & \uparrow \\ G & \longrightarrow & \pi_0(F_2, 1_E)\,, \end{array}$$

Then Theorem A follows from the diagram (5.1) and lemma 3.1.

Consider the following two fibrings which are obtained from the restriction of fibrings in § 2:

$$\begin{array}{ccccc} & & F_2 & & \\ & & \downarrow & & \\ \xi^\xi & \longrightarrow & E_1 & \xrightarrow{\ q\ } & F_1\ni p\,, \\ & & \downarrow & & \\ & & S^{q^{S^q}}\times\Gamma\ni(1_{S^q}\times\iota_n) & & \end{array}$$

where $E_1=(p^E)^{-1}(F_1)=(r_2)^{-1}((p^K)^{-1}(p\,|\,K))$ and Γ denotes the space of cross-sections: $S^n\to E$. Then we have the diagram consisting of a part of a the homotopy exact sequence of fibrings:



where i_* denotes the homomorphism induced by appropriate inclusions. Now we can easily deduce following results from the diagram (5.2):

(1) p_* is surjective $\Rightarrow q_*$ is surjective $\Rightarrow 0 \to \pi_0(\xi^\xi, 1_E) \to \pi_0(E_1, 1_E)$ is exact.

(2) $J(\eta)\Sigma^{q-1}h_{n+1} = \partial\begin{bmatrix} 0 & 0 \\ 0 & h_{n+1}\end{bmatrix} \notin \partial\pi_1(S^{q^{S^q}}\times\Gamma, 1_{S^q}\times\iota_n) \Rightarrow \iota_*(J(\eta)\Sigma^{q-1}h_{n+1}) \neq 0$,

where h_{n+1} is the generator of $\pi_{n+1}(S^n)$.

(3) $[\pi_{n+q}(S^q), \iota_q] = \partial\begin{bmatrix} * & \pi_{n+1}(S^q) \\ 0 & 0\end{bmatrix} \subset \partial\pi_1(S^{q^{S^q}}\times\Gamma, 1_{S^q}\times\iota_n) \Rightarrow i_*([\pi_{n+1}(S^q), \iota_q]) = 0$.

(4) $q_* i_*(\pi_{n+q}(S^q)) = 0 \Rightarrow \iota_*(\pi_{n+q}(S^q)) \subset \iota_*(\pi_0(\xi^\xi, 1_E))$.

Thus the homomorphism ϕ which we want is naturally defined by using (1)~(4).

REFERENCES

[1] W.D. BARCUS AND M.G. BARRATT, On the homotopy classification of the extensions of a fixed map, Trans. Amer. Math. Soc. **88** (1958), 59-74.

[2] M.G. BARRATT AND P.J. HILTON, On join operations of homotopy groups, Proc. Lond. Math. Soc. **3** (1953), 430-445.

[3] I.M. JAMES AND J.H.C. WHITEHEAD, The homotopy theory of sphere bundles over spheres, Proc. Lond. Math. Soc. (3) **4** (1954), 196-213.

[4] D.H. GOTTLIEB, On fibre spaces and the evaluation map, Ann. of Math. **87** (1968), 42-55.

[5] Y. NOMURA, The group of fibre homotopy equivalences, Bull. Nagoya Inst. Tech. **17** (1967), 66-71.

DEPARTMENT OF MATHEMATICS
OH-OKAYAMA MEGURO-KU
TOKYO JAPAN