POISSON APPROXIMATION FOR SUMS OF INDEPENDENT BIVARIATE BERNOULLI VECTORS

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1. Introduction. It is a well known fact as Poisson's theorem that for a given sequence of \{p_n, n \geq 1\} such that \( p_n \to 0 \) \((n \to \infty)\) we have

\[
P_n(m) = \frac{\lambda_n^m}{m!} e^{-\lambda_n} \to 0 \quad \text{as } n \to \infty
\]

for all non-negative integer \( m \) where

\[
\lambda_n = n p_n, \quad P_n(m) = \binom{n}{m} p_n^m (1 - p_n)^{n-m}.
\]

Furthermore, if \( np_n \to \lambda \) \((n \to \infty)\) then we have

\[
P_n(m) \to \frac{\lambda^m}{m!} e^{-\lambda} \quad \text{as } n \to \infty.
\]

R. von Mises in the paper [3] has showed that if \( \{X_k, j=1, 2, \ldots, n_k\} \) is a sequence of independent random variables such that

\[
\lambda_n = \sum_{j=1}^{n_k} \rho_{k_j}, \quad \sum_{j=1}^{n_k} \rho_{k_j} \to \lambda > 0 \quad (k \to \infty),
\]

then

\[
P\left[ \sum_{j=1}^{n_k} X_{k_j} = m \right] \to \frac{\lambda^m}{m!} e^{-\lambda} \quad (k \to \infty).
\]

In (1977) J. Mačys (see. [2]) has proved that the conditions (1.1) are necessary as well.

Let \( \{(X_k, Y_k), k \geq 1\} \) be a sequence of random vectors bivariate Bernoulli law, i.e.

\[
P[X_k=0, Y_k=0] = \rho_{00}, \quad P[X_k=1, Y_k=0] = \rho_{10},
\]

\[
P[X_k=0, Y_k=1] = \rho_{01}, \quad P[X_k=1, Y_k=1] = \rho_{11},
\]

where \( \rho_{00} + \rho_{10} + \rho_{01} + \rho_{11} = 1 \).
K. Kawamura in [1] has proved that if \( \{ (X_k, Y_k), k \geq 1 \} \) are mutually independent and identically distributed random vectors having bivariate Bernoulli probability, then

\[
P \left[ \sum_{k=1}^{n} (X_k, Y_k) = (n, m) \right] \rightarrow \min \left( \sum_{s=0}^{n,m} \frac{\lambda_{10}^{n-s} \lambda_{01}^{m-s} \lambda_{11}^s}{(n-s)!(m-s)!s!} e^{-\lambda_{10} + \lambda_{01} + \lambda_{11}} \right)
\]

as \( n \to \infty \), where \( np_{11} = \lambda_{11}, np_{10} = \lambda_{10} \) and \( np_{01} = \lambda_{01} \) are fixed values.

The main aim of this paper is to generalize Kawamura’s results [1] to nonidentically distributed random vectors \( \{ (X_k, Y_k), k \geq 1 \} \). The results presented in Section 2 extend those of Kawamura [1], and Mačys [2].

2. The result.

Let \( \{ (X_{kj}, Y_{kj}), j = 1, 2, \ldots, n_k, k \geq 1 \} \) be a sequence of independent bivariate Bernoulli vectors with

\[
\sum_{j=1}^{n_k} (X_{kj}, Y_{kj}) = (n_k, m_k)
\]

where \( p_{kj}(0, 0) + p_{kj}(0, 1) + p_{kj}(1, 0) + p_{kj}(1, 1) = 1 \).

Let

\[
S_k = \sum_{j=1}^{n_k} (X_{kj}, Y_{kj}), \quad k \geq 1.
\]

**Theorem.** In order that

\[
\lim_{k \to \infty} P[S_k = (n, m)] = \min \left( \sum_{s=0}^{n,m} \frac{\lambda_{10}^{n-s} \lambda_{01}^{m-s} \lambda_{11}^s}{(n-s)!(m-s)!s!} e^{-\lambda_{10} + \lambda_{01} + \lambda_{11}} \right)
\]

may hold for all \( n, m \geq 0 \) it is necessary and sufficient that for \( k \to \infty \)

\[
(2.1) \quad \sum_{j=1}^{n_k} p_{kj}(1, 0) \to \lambda_{10},
\]

\[
(2.2) \quad \sum_{j=1}^{n_k} p_{kj}(0, 1) \to \lambda_{01},
\]

\[
(2.3) \quad \sum_{j=1}^{n_k} p_{kj}(1, 1) \to \lambda_{11}
\]

and

\[
(2.4) \quad \min_{1 \leq k \leq n_k} p_{kj}(0, 0) \to 1.
\]

**Proof.** For the sake of simplicity the index \( k \) will be omitted in the proof of Theorem, i.e. instead of \( p_{kj} \) we write \( p_j \).

The part if. It is easy to see that
(2.5) \[ P[S_2 = (0, 0)] = \prod_{g=1}^{n_k} p_g(0, 0), \]

(2.6) \[ P[S_2 = (1, 0)] = \sum_{t_1 < t_2} \left\{ \prod_{g=1}^{n_k} p_g(0, 0) \prod_{g \neq t_1, t_2} p_g(1, 0) \right\} \]

\[ = \prod_{g=1}^{n_k} p_g(0, 0) \sum_{t_1 < t_2} p_{t_1}(1, 0) / p_{t_2}(0, 0), \]

(2.7) \[ P[S_2 = (2, 0)] = \sum_{t_1 < t_2} \left\{ \prod_{g=1}^{n_k} p_g(1, 0) \prod_{g \neq t_1, t_2} p_g(0, 0) \right\} \]

\[ = \prod_{g=1}^{n_k} p_g(0, 0) \sum_{t_1 < t_2} p_{t_1}(1, 0) / p_{t_2}(0, 0). \]

In the same way, for every \( n > 2 \), we obtain

\[ P[S_2 = (n, 0)] = \sum_{t_1 < \cdots < t_n} \left\{ \prod_{g=1}^{n_k} p_g(1, 0) \prod_{g \neq t_1, t_2} p_g(0, 0) \right\} \]

\[ = \prod_{g=1}^{n_k} p_g(0, 0) \sum_{t_1 < \cdots < t_n} \left( \prod_{g=1}^{n_k} p_g(1, 0) / p_{t_i}(0, 0) \right). \]

Let \( n, m \geq 1 \) be given and assume that \( n = m \). If we put \( \delta = \max\{2n - n_k, 0\} \), then

\[ P[S_2 = (n, n)] = \sum_{t_1 < \cdots < t_n} \sum_{j_1 < \cdots < j_n} P\left[ \bigcap_{g=1}^{n_k} [X_{j_g} = 1], \bigcap_{i=1}^{n_k} [X_i = 0], \right. \]

\[ \cdot \bigcap_{r=1}^{n_k} [Y_{r_g} = 1], \bigcap_{s=1}^{n_k} [Y_s = 0] = \sum_{t_1 < \cdots < t_n} \left\{ \prod_{g=1}^{n_k} p_{t_g}(1, 0) \prod_{g \neq t_1, t_2} p_g(0, 0) \right\} \]

\[ + \sum_{t_1 < t_2} \sum_{t_1 < t_2 < t_3} \sum_{g \neq t_1, t_2, t_3} \left\{ \prod_{i=1}^{n_k} p_{t_i}(1, 0) \prod_{g \neq t_1, t_2, t_3} p_g(0, 0) \right\} \]

\[ + \cdots \sum_{t_1 < \cdots < t_n} \sum_{g \neq t_1, t_2, \ldots, t_n} \left\{ \prod_{i=1}^{n_k} p_{t_i}(1, 0) \prod_{g \neq t_1, t_2, \ldots, t_n} p_g(0, 0) \right\}. \]

Taking out the product \( \prod_{g=1}^{n_k} p_g(0, 0) \) before the sign of the first sum, we may write out

(2.8) \[ P[S_2 = (n, n)] = \sum_{g = \max\{2n - n_k, 0\}}^{n_k} \prod_{g=1}^{n_k} p_g(0, 0) \left\{ \prod_{t_1 < \cdots < t_n} \left( \prod_{g=1}^{n_k} p_{t_g}(1, 0) / p_{t_g}(0, 0) \right) \right\}. \]
Let now $n, m \geq 1$ be given and suppose that $n < m$. If we put $\delta = \max \{n + m - n^*, 0\}$, then

\[
P[S_k = (n, m)] = \sum_{\ell_1 < \ldots < \ell_k} \sum_{j_1 < \ldots < j_m} P \left[ \bigcap_{\ell = 1}^{n-k} [X_{t_{\ell}} = 1], \bigcap_{r = 1}^{m} [Y_{r} = 1], \bigcap_{i = 1}^{\delta} \left\{ \sum_{j_1 < \ldots < j_m-n^*} \left( \prod_{r = 1}^{n} p_{t_1}(1, 1) \prod_{r = 1}^{m-n} p_{t_r}(0, 1) \right) \right\} \right].
\]

Thus taking into account (2.8) and the above given equality, for all $n, m \geq 1$, we may write

\[
(2.9) \quad P[S_k = (n, m)] = \min_{i = 1}^{\min \{n, m\}} \sum_{\delta = \max \{n + m - n^*, 0\}}^{\delta} \left\{ \prod_{\ell = 1}^{n-k} p_{t_1}(1, 0) \prod_{r = 1}^{m-n} p_{t_r}(0, 1) \right\}.
\]

In order to prove that

\[
\prod_{\ell = 1}^{n} p_{t_1}(1, 0) \rightarrow e^{-(1+1+1)} \quad \text{as} \quad k \rightarrow \infty,
\]

we consider the inequality $1 + y \leq e^y$, $y \in [-1, \infty)$. Putting $y = -x$, $x \in [0, 1)$, we have

\[
e^{-x/(1-x)} \leq 1-x \leq e^{-x}, \quad x \in [0, 1).
\]

Now putting $A_k = p_{t_1}(1, 0) + p_{t_2}(0, 1) + p_{t_3}(1, 1)$ and using the last inequality, we obtain

\[
e^{\min_{i \geq 1} p_{t_i}(0, 0)} \sum_{k \geq 1} A_k \leq \prod_{i = 1}^{n} p_{t_i}(0, 0) \leq e^{-2A_k}.
\]
Taking into account (2.1)-(2.4) one can prove that
\[ \lim_{k \to \infty} \prod_{g=1}^{n_k} \rho_g(0, 0) = e^{-(\lambda_{10} + \lambda_{01} + \lambda_{11})}. \]

Now we are going to prove that
\[
\sum_{t_1 < t_2} \prod_{l=1}^{n_2} \rho_{t_1}(1, 0) \prod_{p=1}^{m_2} \rho_{t_2}(1, 1) \prod_{r=1}^{n_1} \rho_{t_2}(0, 1) \to \lambda_{10}^{n_2} / (n - s)! \cdot \lambda_{11}^{s} / s! \cdot \lambda_{01}^{m-s} / (m-s)! \quad \text{as} \quad k \to \infty.
\]

In the case \(m=0\) the proof is by induction with respect to \(n\). The case \(n=1\) is obvious. Let \(n=2\), then by (2.1) and (2.4)
\[
\sum_{t_1 < t_2} \rho_{t_1}(1, 0) \rho_{t_2}(1, 0) \to \lambda_{10}^2 / 2
\]
as
\[
2 \sum_{t_1 < t_2} \rho_{t_1}(1, 0) \rho_{t_2}(1, 0) = \left( \sum_{t=1}^{n_1} \rho_t(1, 0) \right)^2 - \sum_{t=1}^{n_1} \rho_t(1, 0)
\]
and
\[
0 \leq \sum_{t=1}^{n_1} \rho_t(1, 0) \leq (1 - \min \rho_t(0, 0)) \sum_{t=1}^{n_1} \rho_t(1, 0).
\]
Assume that
\[
\sum_{t_1 < \cdots < t_{n-1}} \prod_{l=1}^{n-1} \rho_{t_l}(1, 0) \to \lambda_{10}^{n-1} / (n-1) !.
\]
Multiplying the left hand side of the last relation by \(\sum_{t_{n-1} = 1}^{n_k} \rho_{t_{n-1}}(1, 0)\) we obtain
\[
\sum_{t_1 < \cdots < t_{n-1}} \sum_{t_{n-1} = 1}^{n_k} \prod_{l=1}^{n-1} \rho_{t_l}(1, 0) + \sum_{t_1 < \cdots < t_{n-1}} \prod_{l=1}^{n-1} \rho_{t_l}(1, 0) + \cdots
\]
\[
+ \sum_{t_1 < \cdots < t_{n-1}} \prod_{l=1}^{n-1} \rho_{t_l}(1, 0) + \cdots + \sum_{t_1 < \cdots < t_{n-1}} \prod_{l=1}^{n-1} \rho_{t_l}(1, 0).
\]
Because every sum from among the first \((n-1)\) sums we may estimate by
\[
(1 - \min \rho_t(0, 0)) \sum_{t_1 < \cdots < t_{n-1}} \prod_{l=1}^{n-1} \rho_{t_l}(1, 0)
\]
so the first \((n-1)\) sums tend to 0. We have then
\[
n \sum_{t_1 < \cdots < t_n} \prod_{l=1}^{n} \rho_{t_l}(1, 0) \to \lambda_{10}^n / (n-1) !.
\]
The last relation proves that
(2.11) \[ \sum_{t_1<...<t_n} \prod_{p=1}^{n} p_{t_p}(1, 0) \longrightarrow \lambda_{t_0}^n/n!. \]

In the same way one can prove that

(2.12) \[ \sum_{t_1<...<t_s} \prod_{p=1}^{s} p_{t_p}(1, 1) \longrightarrow \lambda_{t_1}^s/s! \]

and

(2.13) \[ \sum_{j_1<...<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) \longrightarrow \lambda_{j_0}^{m-s}/(m-s)!. \]

Let us put

\[ B_{nk}(n-s, s, m-s) = \sum_{t_1<...<t_{n-s}} \prod_{p=1}^{n-s} p_{t_p}(1, 0) \sum_{t_1<...<t_{s}} \prod_{p=1}^{s} p_{t_p}(1, 1) \]

\[ \cdot \sum_{j_1<...<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1). \]

Then, taking into account (2.11)-(2.13), we have

(2.14) \[ B_{nk}(n-s, s, m-s) \longrightarrow \lambda_{n-s}^{n-s}/(n-s)! \cdot \lambda_{t_1}^{s}/s! \cdot \lambda_{j_0}^{m-s}/(m-s)!. \]

as \( k \to \infty \).

Let us define

\[ A^{s}_{t_1,...,t_{n-s}}(1, 1) = \sum_{t_1<...<t_{n-s}} \prod_{p=1}^{n-s} p_{t_p}(1, 1) \]

\[ \sum_{\neq t_1,...,t_{n-s}} \prod_{p=1}^{s} p_{t_p}(1, 1) \]

and

\[ A^{s}_{t_1,...,t_{n-s}}(0, 1) = \sum_{j_1<...<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) \]

\[ \sum_{\neq j_1,...,j_{m-s}} \prod_{r=1}^{s} p_{j_r}(0, 1). \]

Taking into account the above considerations it may be proved that

\[ A^{s}_{t_1,...,t_{n-s}}(1, 1) \] and \( A^{s}_{t_1,...,t_{n-s}}(0, 1) \) tend to 0 as \( k \to \infty \).

It is easy to see that

\[ \sum_{t_1<...<t_{n-s}} \prod_{p=1}^{n-s} p_{t_p}(1, 0) \sum_{\neq t_1,...,t_{n-s}} \prod_{p=1}^{s} p_{t_p}(1, 1) \sum_{j_1<...<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) \]

\[ = B_{nk}(n-s, s, m-s) = (Z_1 + Z_2 + Z_3), \]

where

\[ Z_1 = \sum_{t_1<...<t_{n-s}} \prod_{p=1}^{n-s} p_{t_p}(1, 0) \sum_{\neq t_1,...,t_{n-s}} \prod_{p=1}^{s} p_{t_p}(1, 1), \]

\[ Z_2 = \sum_{t_1<...<t_{n-s}} \prod_{p=1}^{n-s} p_{t_p}(1, 0) \sum_{\neq t_1,...,t_{n-s}} \prod_{p=1}^{s} p_{t_p}(1, 0) \sum_{j_1<...<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1), \]

\[ Z_3 = \sum_{j_1<...<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1). \]
\[ Z_{3} = \sum_{t_{1} < \cdots < t_{n-2}} \prod_{i=1}^{n-2} p_{t_{i}}(1, 0) A^{s}_{t_{1}, \ldots, t_{n-2}} A^{s}_{(0, 1), t_{1}, \ldots, t_{n-2}} A^{s}_{t_{1}, \ldots, t_{n-2}} (0, 1) \]

and \( Z_{3} \to 0 \) as \( k \to \infty \), \( 1 \leq i \leq 3 \).

The relation (2.10) finishes the proof of the part if, because

\[
\sum_{t_{1} < \cdots < t_{n-2}} \prod_{i=1}^{n-2} p_{t_{i}}(1, 0) \cdot \sum_{s.t._{1} < \cdots < t_{n-2}} \prod_{p=1}^{s} p_{t_{p}}(1, 1) \cdot \sum_{s.t._{1} < \cdots < t_{n-2}} \prod_{r=1}^{m} p_{t_{r}}(0, 1)
\]

\[
\leq \left( \frac{1}{\min p_{j}(0, 0)} \right)^{n+m-2} \cdot \sum_{t_{1} < \cdots < t_{n-2}} \prod_{i=1}^{n-2} p_{t_{i}}(1, 0)
\]

Proof of the part only. From (2.5)-(2.7) we have

\[
(2.15) \quad \prod_{a=1}^{n} p_{x}(0, 0) \to e^{-\lambda_{10} + \lambda_{01} + \lambda_{11}},
\]

\[
(2.16) \quad \sum_{t_{1}=1}^{n} p_{t_{1}}(1, 0)/p_{t_{1}}(0, 0) \to \lambda_{10},
\]

\[
(2.17) \quad \sum_{t_{2} \leq t_{5}} p_{t_{2}}(1, 0)/p_{t_{2}}(0, 0) \to \lambda_{01},
\]

and

\[
(2.18) \quad \sum_{j_{1}=1}^{\infty} p_{j_{1}}(1, 0)/p_{j_{1}}(0, 0) \to \lambda_{01}/2, \quad \sum_{j_{1}=1}^{\infty} p_{j_{1}}(0, 1)/p_{j_{2}}(0, 0) \to \lambda_{01}/2.
\]

From (2.15)-(2.17) we get

\[
\sum_{t_{1}=1}^{n} \left( p_{t_{1}}(1, 0)/p_{t_{1}}(0, 0) \right)^{2} \to 0, \quad \sum_{j_{1}=1}^{\infty} \left( p_{j_{1}}(0, 1)/p_{j_{1}}(0, 0) \right)^{2} \to 0,
\]

which implies that

\[
\max \left( p_{t_{1}}(1, 0)/p_{t_{1}}(0, 0) \right) \to 0, \quad \max \left( p_{j_{1}}(0, 1)/p_{j_{1}}(0, 0) \right) \to 0.
\]

Now we will prove that \( \max \left( p_{t_{1}}(1, 1)/p_{t_{1}}(0, 0) \right) \to 0 \). Indeed, from (2.8) for \( n=m=1 \) and (2.15) we have
\[
\sum_{t_1=1}^{n_1} p_{t_1}(1, 0)/p_{t_1}(0, 0) + \sum_{t_2=1}^{n_2} p_{t_2}(0, 1)/p_{t_2}(0, 0) \longrightarrow \lambda_{10} \cdot \lambda_{01} + \chi^{11}.
\]

On the other hand, taking into account the inequality
\[
\sum_{t_1=1}^{n_1} p_{t_1}(1, 0)p_{t_1}(0, 1)/p_{t_2}(0, 0) \leq \max (p_{f}(0, 1)/p_{c}(0, 0)) \sum_{t_1=1}^{n_1} p_{t_1}(1, 0)/p_{t_1}(0, 0),
\]
(2.15) and (2.16), we get
\[
\sum_{t_1=1}^{n_1} p_{t_1}(1, 1)/p_{t_1}(0, 0) \longrightarrow \lambda_{11}.
\]

In the same way, putting in (2.8) \( n=m=2 \), we obtain
\[
\sum_{t_1=1}^{n_1} p_{t_1}(1, 1)\rho_{t_2}(1, 1)/p_{t_2}(0, 0)p_{t_2}(0, 0) \longrightarrow \lambda_{11}^2/2.
\]
From (2.18) and (2.19) it follows that \( \max (p_{f}(1, 1)/p_{c}(0, 0)) \rightarrow 0 \), and therefore \( \min p_{f}(0, 0) \rightarrow 1 \) as \( k \rightarrow \infty \).

The last relation and (2.16) imply that
\[
\sum_{t=1}^{n} p_{t}(1, 0) \longrightarrow \chi_{10},
\]
because
\[
\min p_{f}(0, 0) \sum_{t_1=1}^{n_1} p_{t_1}(1, 0)/p_{t_1}(0, 0) \leq \sum_{t_1=1}^{n_1} p_{t_1}(1, 0) \leq \sum_{t_1=1}^{n_1} p_{t_1}(1, 0)/p_{t_1}(0, 0).
\]
In the same way one can prove that (2.2) and (2.3) are satisfied. Thus the proof of Theorem is completed.

REFERENCES


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