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REMARKS ON NON EXPLOSION THEOREM FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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§ 1. Introduction.

In this paper, we give sufficient conditions in order that the solutions of the stochastic differential equations cannot explode. The results are improvement of author's previous one [6], since the concavity condition is not imposed on the function which appears in the restriction on the growth of the drift and the diffusion coefficients.

In § 2, using the method of Liapunov functions, we obtain a key lemma. And, by applying the lemma, we prove a generalization of Hasminskii's theorem [3] and an analogue of Wintner's theorem [1], [2] which gives continuability of the solutions of ordinary differential equations. In § 3, we give another direct proof of the analogue of Wintner's theorem, so that the smoothness condition on the function which appears in the restriction on the growth of the drift coefficient is weaker than that given by the method of Liapunov functions.

First of all we introduce notations and definitions. Let R^d denote Euclidean d-space. For $x\in R^d$ and $y\in R^d$, let $\langle x, y\rangle$ be the inner product of x and y and let $|x|$ be the Euclidean norm of x. For a $d\times d$-matrix $M=(m_{ij})$, define $|M|=\left(\sum_{i,j=1}^{d} m_{ij}^2\right)^{1/2}$. We shall denote by $C^{1,2}([0, T]\times R^d)$ the family of scalar functions defined on $[0, T]\times R^d$ which are twice continuously differentiable with respect to $x\in R^d$ and once with respect to $t\in[0, T]$. Let $(\Omega, \boldsymbol{F}, P)$ be a probability space with an increasing family $\{\boldsymbol{F}_t; t\geqq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$ and let $w(t)=(w_i(t))$, $i=1, \cdots, d$, be a d-dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. Consider the stochastic differential equation

$$dX(t)=b(t, X(t))dt+\sigma(t, X(t))dw(t)\,, \tag{1.1}$$

where $b(t, x)=(b_i(t, x))$, $i=1, \cdots, d$, is a d-vector function and $\sigma(t, x)=(\sigma_{ij}(t, x))$, $i, j=1, \cdots, d$, is a $d\times d$-matrix function, which are defined on $[0, \infty)\times R^d$ and Borel measurable with respect to the complete set of the variables. Equation (1.1) is equivalent to the system of d equations

$$dX_i(t)=b_i(t, X(t))dt+\sum_{i,j=1}^{d}\sigma_{ij}(t, X(t))dw_j(t)\,, \qquad i=1, \cdots, d\,. \tag{1.1$'$}$$

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Throughout this paper, we assume the following conditions:

(1.2) $b(t, x)$ and $\sigma(t, x)$ are continuous in (t, x) for any $T>0$, $b(t, x)$ and $\sigma(t, x)$ satisfy the local Lipschitz condition with respect to $x\in R^d$ if $t\leqq T$.

As is well known, the stochastic differential equation (1.1) with the initial condition $X(t_0)=x_0\in R^d$ $(t_0\geqq 0)$ has a pathwise unique solution for $t<e(t_0, x_0)$, where $e(t_0, x_0)=\lim_{n\uparrow\infty} e_n(t_0, x_0)$ and $e_n(t_0, x_0)=\inf\{t\geqq t_0; |X(t)|\geqq n\}\wedge n$ (cf. [6]). The random time $e(t_0, x_0)$ is called *the explosion time* of the solution of (1.1) with the initial condition $X(t_0)=x_0$. The following remark enables us to understand the meaning of the explosion time $e(t_0, x_0)$ (see [4; § 2, Chap. IV] and [6, § 3]).

Remark. If $b(t, x)$ and $\sigma(t, x)$ satisfy (1.2), then
$\lim_{t\uparrow e(t_0, x_0)} |X(t)|=\infty$ for $e(t_0, x_0)<\infty$, almost surely.

We are interested in the question whether the explosion occurs or does not. We shall use the differential generator

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{d} b_i(t, x)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j}$$

associated with the stochastic differential equation (1.1), where $a(t, x)=(a_{ij}(t, x))$ is a $d\times d$-matrix defined by $a(t, x)=\sigma(t, x)\sigma(t, x)^*$ (* means the transpose).

§ 2. Liapunov functions.

In the explosion problem, the Liapunov function approach provides an effective method. To begin with, we prove a key lemma which will yield the main theorem to us.

LEMMA. *Let $b(t, x)$ and $\sigma(t, x)$ satisfy* (1.2) *and suppose for each $T>0$, there exist positive numbers $c=c_T$ and $r=r_T$, and there exist a function $U=U_T \in C^{1,2}([0, T]\times R^d)$ such that*

$$LU(t, x)\leqq c \quad \text{for all} \quad t\leqq T \quad \text{and} \quad |x|\geqq r \tag{2.1}$$

and

$$\lim_{|x|\to\infty}\inf_{0\leqq t\leqq T} U(t, x)=\infty\,. \tag{2.2}$$

Then $P(e(t_0, x_0)=\infty)=1$ for all $t_0\geqq 0$ and $x_0\in R^d$.

Proof. Assume that there exists some (t_0, x_0) such that $P(e(t_0, x_0)\leqq T_0)>0$ for some T_0. In the following, we take a sample such that $e(t_0, x_0)\leqq T_0$. For simplicity of the notation we put $e=e(t_0, x_0)$. Let $T>T_0$ be arbitrary and be fixed. Then we can take positive numbers $c=c_T$, $r=r_T$ and a function $U=U_T$ in the hypothesis. By Remark in § 1, we notice that $|X(e-)|=\infty$ for such a sample and hence put

$$\rho=\sup\{t>t_0;\ |X(t)|=r\}\ .$$

Then Ito's formula concerning stochastic differentials implies that

$$U(t, X(t))-U(\rho, X(\rho))=\int_{\rho}^{t} LU(s, X(s))ds+M(t)-M(\rho)$$

for all $\rho \leqq t < e$, where

$$M(t)=\int_{t_0}^{t} \langle \operatorname{grad} U(s, X(s)), \sigma(s, X(s))dw(s)\rangle .$$

Notice that $M(t)=z(\phi(t))$, where $z(t)$ is a new Brownian motion process and $\phi(t)=\int_{t_0}^{t} |\sigma(s, X(s))^* \operatorname{grad} U(s, X(s))|^2 ds$ (see McKean [5; Problem 1, § 2.9]). Then we get by (2.1) that

$$U(t, X(t))-U(\rho, X(\rho)) \leqq c(t-\rho)+z(\phi(t))-z(\phi(\rho))$$

for all $\rho \leqq t < e$. Let t tend to e in the above equation. Then it follows from Remark in § 1 and (2.2) that

$$\begin{aligned}\infty &= \lim_{t \uparrow e} U(t, X(t))-U(\rho, X(\rho)) \\ &\leqq c(e-\rho)+\liminf_{t \uparrow e} z(\phi(t))-z(\phi(\rho)) \\ &< \infty ,\end{aligned}$$

which is a contradiction. Hence the proof is complete.

The following theorem is a generalization of Hasminskii's result [3, p. 113].

THEOREM 2.1. *Let $b(t, x)$ and $\sigma(t, x)$ satisfy* (1.2) *and suppose for each $T>0$, there exist positive numbers $c=c_T$ and $r=r_T$, and there exist a nonnegative function $V=V_T \in C^{1,2}([0, T] \times R^d)$ and a nondecreasing, differentiable function $\beta=\beta_T: [0, \infty) \to [0, \infty)$ such that*

$$LV(t, x) \leqq c\beta(V(t, x)) \quad \text{for all } t \leqq T \text{ and } |x| \geqq r . \tag{2.3}$$

$$\lim_{|x| \to \infty} \inf_{0 \leqq t \leqq T} V(t, x)=\infty \tag{2.4}$$

and

$$\int_0^{\infty} \frac{du}{1+\beta(u)}=\infty . \tag{2.5}$$

Then $P(e(t_0, x_0)=\infty)=1$ for all $t_0 \geqq 0$ and $x_0 \in R^d$.

Proof. Let $T>0$ be arbitrary and be fixed. Then, let $c=c_T$ and $r=r_T$ be positive numbers and let $V=V_T$ and $\beta=\beta_T$ be the functions in the hypothesis, respectively. We set

$$f(v)=\int_0^{v} \frac{du}{1+\beta(u)} \quad \text{and} \quad W(t, x)=f(V(t, x)) .$$

Then we see that $W \in C^{1,2}([0, T] \times R^d)$ and so

$$LW(t, x)=(LV(t, x))f'(V(t, x))$$
$$+\frac{1}{2}|\sigma(t, x)^*\text{grad } V(t, x)|^2 f''(V(t, x))$$
$$=\frac{LV(t, x)}{1+\beta(V(t, x))}$$
$$+\frac{1}{2}|\sigma(t, x)^*\text{grad } V(t, x)|^2\left[\frac{-\beta'(V(t, x))}{(1+\beta(V(t, x)))^2}\right]$$
$$\leqq c$$

for all $t\leqq T$ and $|x|\geqq r$, since (2.3) holds and since β is nondecreasing by the assumption. Further, for any $t\leqq T$,

$$W(t, x)\geqq f(\inf_{0\leqq t\leqq T} V(t, x)),$$

since f is nondecreasing by the definition. Then we obtain by (2.4) and (2.5) that

$$\inf_{0\leqq t\leqq T} W(t, x) \longrightarrow \infty \qquad \text{as} \quad |x|\to\infty .$$

Therefore, $W(t, x)$ satisfies (2.1) and (2.2) and Lemma applies if we take $U_T=W(t, x)$ for each $T>0$. Hence the proof is complete.

Hasminskii's theorem is a special case of Theorem 2.1, where $V=V_T$ satisfies (2.3) for all $t\leqq T$ and $x\in R^d$ with the function $\beta=\beta_T(u)=u$ such that (2.5) holds obviously. In particular, if we take $V=V_T=|x|^2$ in Theorem 2.1, then we obtain the following restriction on the growth of the coefficients in order that $X(t)$ cannot explode.

COROLLARY. *Let $b(t, x)$ and $\sigma(t, x)$ satisfy* (1.2) *and suppose for each $T>0$, there exist positive numbers $c=c_T$ and $r=r_T$, and there exist a nondecreasing, differentiable function $\beta=\beta_T: [0, \infty)\to[0, \infty)$ such that*

$$2\langle x, b(t, x)\rangle+|\sigma(t, x)|^2\leqq c\beta(|x|^2) \tag{2.6}$$

for all $t\leqq T$ and $|x|\geqq r$, where β satisfies (2.5). *Then, $P(e(t_0, x_0)=\infty)=1$ for all $t_0\geqq 0$ and $x_0\in R^d$.*

EXAMPLE. Suppose that (cf. Yershov [7, Theorem 5.2])

$$|b(t, x)|^2+|\sigma(t, x)|^2\leqq C(1+|x|^2)\log(1+|x|) \tag{2.7}$$

with a constant $C>0$ for all $t\geqq 0$ and $x\in R^d$. Then we see that

$$2\langle x, b(t, x)\rangle+|\sigma(t, x)|^2\leqq|x|^2+|b(t, x)|^2+|\sigma(t, x)|^2$$
$$\leqq C'[|x|^2+(1+|x|^2)\log(1+|x|)]$$

with a constant $C'>0$ for all $t\geqq 0$ and $x\in R^d$. Therefore, if (2.7) holds, then (2.6) holds, where $\beta=\beta_T(u)=u+(1+u)\log(1+u^{1/2})$ satisfies (2.5), and hence

Corollary will apply.

In the above Theorem 2.1 and Corollary, the concavity condition is not imposed on the function $\beta=\beta_T(u)$, which improves author's previous results [6; Theorem 2.2, Corollary].

The following result corresponds to an analogue of Wintner's Theorem of continuability of solutions of the ordinary differential equation $dX(t)/dt=b(t, X(t))$.

THEOREM 2.2. *Let* $b(t, x)$ *and* $\sigma(t, x)$ *satisfy* (1.2) *and suppose for each* $T>0$, *there exist positive numbers* $c=c_T$, $c'=c'_T$ *and* $r=r_T$, *and there exist a nondecreasing, differentiable function* $\beta=\beta_T: [0, \infty)\to(0, \infty)$ *such that*

$$|b(t, x)|\leqq c\beta(|x|) \quad \text{for all} \quad t\leqq T \quad \text{and} \quad |x|\geqq r, \tag{2.8}$$

$$|\sigma(t, x)|^2\leqq c' \quad \text{for all} \quad t\leqq T \quad \text{and} \quad |x|\geqq r \tag{2.9}$$

and

$$\int_r^\infty \frac{du}{\beta(u)}=\infty. \tag{2.10}$$

Then, $P(e(t_0, x_0)=\infty)=1$ *for all* $t_0\geqq 0$ *and* $x_0\in R^d$.

Proof. For each $T>0$, let $c=c_T$, $c'=c'_T$ and $r=r_T$ be the positive numbers and let $\beta=\beta_T$ be the function in the hypothesis, respectively. We set

$$U(t, x)=\int_r^{|x|}\frac{du}{\beta(u)}-kt \qquad (k\equiv c+c'/2r\beta(r))$$

for all $t\leqq T$ and $|x|\geqq r$, and extend it smoothly to $|x|<r$. Then it is easy to see from (2.8) and (2.9) that $U\in C^{1,2}([0, T]\times R^d)$ satisfies

$$\begin{aligned} LU(t, x)=&-k+\frac{\langle x, b(t, x)\rangle}{|x|\beta(|x|)}+\frac{|\sigma(t, x)|^2}{2|x|\beta(|x|)} \\ &-\frac{1}{2}\left[\frac{\beta(|x|)+|x|\beta'(|x|)}{|x|^3\beta^2(|x|)}\right]|\sigma(t, x)^*x|^2 \\ \leqq& 0 \end{aligned}$$

for all $t\leqq T$ and $|x|\geqq r$. Therefore, we can get the conclusion by Lemma since (2.10) holds.

Such functions as $U(t, x)$ and $V(t, x)$ which appear in lemma and theorems of this section are said to be *Liapunov functions* of $X(t)$.

§ 3. Another proof of Theorem 2.2.

In this section, we give another direct proof of Theorem 2.2 without using the method of Liapunov functions. In the proof, the smoothness condition on the function $\beta=\beta_T$ of Theorem 2.2 can also be weakened.

THEOREM. *Assume all the conditions of Theorem 2.2 except that the differentiability condition on β is replaced by the condition such that β is continuous. Then, $P(e(t_0, x_0)=\infty)=1$ for all $t_0 \geqq 0$ and $x_0 \in R^d$.*

Proof. Assume that there exists some (t_0, x_0) such that $P(e(t_0, x_0) \leqq T_0)>0$ for some T_0. In the following, we take a sample such that $e(t_0, x_0) \leqq T_0$. For simplicity of the notation we put $e=e(t_0, x_0)$. Let $T>T_0$ be arbitrary and be fixed. Then we can take positive numbers $c=c_T$, $c'=c'_T$ and $r=r_T$, and a continuous function $\beta=\beta_T$ in the hypothesis. By Remark in § 1, we notice that $|X(e-)|=\infty$ for such a sample, and hence we put $\rho=\sup\{t ; |X(t)|=r\}$. Then we have by (1.1)′ that

$$X_i(t)=X_i(\rho)+\int_\rho^t b_i(s, X(s))ds+M_i(t)-M_i(\rho)$$

for all $\rho \leqq t<e$ $(i=1, \cdots, d)$, where

$$M_i(t)=\sum_{j=1}^d \int_{t_0}^t \sigma_{ij}(s, X(s))dw_j(s) .$$

By the time substitution rule (see [5; Problem 1, § 2.9]), we see that $M_i(t) =z_i(\phi_i(t))$ for a new Brownian motion process $z_i(t)$ and $\phi_i(t)=\sum_{j=1}^d \int_{t_0}^t \sigma_{ij}^2(s, X(s))ds$. Then (2.9) implies that $\phi_i(t) \leqq c'(T-t_0)$ for all $t_0 \leqq t<e$, and hence $|M_i(t)-M_i(\rho)| \leqq \sup_{t_0 \leqq t \leqq \phi_i(e)} |z_i(t)-z_i(\phi_i(\rho))| \equiv k_i<\infty$ $(i=1, \cdots, d)$. Thus, we have,

$$|X_i(t)| \leqq |X_i(\rho)|+\int_\rho^t |b_i(s, X(s))|ds+k_i$$

for all $\rho \leqq t<e$, which yields

$$|X(t)| \leqq dr+k+cd\int_\rho^t \beta(|X(s)|)ds \quad (k \equiv k_1+ \cdots +k_d)$$

for all $\rho \leqq t<e$, since (2.8) holds and since $|x| \leqq |x_1|+|x_2|+ \cdots +|x_d| \leqq d|x|$ for $x=(x_1, x_2, \cdots, x_d) \in R^d$. Set $u(t)=dr+k+cd\int_\rho^t \beta(|X(s)|)ds$. Then we see that $u(\rho)=dr+k$, $|X(t)| \leqq u(t)$ and $u'(t)=cd\beta(|X(t)|)$ for all $\rho \leqq t<e$, where $u'(t)$ is the sample derivative of $u(t)$ and is continuous. Since β is nondecreasing by the assumption, we get

$$u'(t) \leqq cd\beta(u(t)) , \qquad \rho \leqq t<e .$$

Divide the both sides of the above equation by $\beta(u(t))$, which is possible since β is positive by the assumption, and then integrate from ρ to $t(<e)$. Then, we obtain,

$$\int_{dr+k}^{u(t)} \frac{du}{\beta(u)} \leqq cd(t-\rho)$$

and therefore

$$\int_{dr+k}^{|X(t)|} \frac{du}{\beta(u)} \leqq cd(e-\rho)$$

for all $\rho \leqq t < e$. Let t tend to the time e in the above equation. Then, the left-hand side of the above equation becomes infinity since $|X(e-)|=\infty$ and since (2.10) holds, while the right-hand side is finite, which is a contradiction. Hence the proof is complete.

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