H. UEDA
KODAI MATH. J.
5 (1982), 360—369

ON ENTIRE FUNCTIONS EXTREMAL FOR THE $\cos\pi\rho$ THEOREM HAVING PRESCRIBED ASYMPTOTIC GROWTH

BY HIDEHARU UEDA

Introduction. If $f(z)$ is a nonconstant entire function, then Hadamard's three-circles theorem asserts that $\log M(r, f)$ is a convex, increasing function of $\log r$, where

$$M(r, f)=\max_{|z|=r}|f(z)|.$$

Hence, by well-known properties of logarithmically convex functions,

$$\log M(r, f)=\log M(r_0, f)+\int_{r_0}^{r}\frac{\Psi(t)}{t}dt \qquad (r\geqq r_0>0),$$

where $\Psi(t)$ is a nonnegative, nondecreasing function of t.

Valiron [6, p 130] showed the following result.

THEOREM A. *Let $\Lambda(r)$ be given by*

$$\Lambda(r)=\text{constant}+\int_{\alpha}^{r}\frac{\Psi(t)}{t}dt \qquad (r\geqq\alpha>0), \tag{1}$$

where $\Psi(t)$ is nonnegative, nondecreasing, and unbounded. Assume further that

$$\Lambda(r)<r^K, \tag{2}$$

for some $K>0$ and all sufficiently large r. Then there exists an entire function $f(z)$ such that

$$\log M(r, f)\sim\Lambda(r) \qquad (r\to\infty).$$

(In Theorem A, the hypothesis (2) can be ommitted. The proof is due to Clunie [1].)

If $f(z)$ is an entire function of order ρ (<1) and put

$$m^*(r, f)=\min_{|z|=r}|f(z)|,$$

then the classical $\cos\pi\rho$ theorem of Valiron and Wiman asserts that

Received April 15, 1981

$$\limsup_{r\to\infty} \frac{\log m^*(r, f)}{\log M(r, f)} \geqq \cos \pi\rho . \tag{3}$$

Now, suppose that $f(z)$, of order $\rho<1$, is extremal for (3) in the sense

$$\log m^*(r, f) \leqq \{\cos \pi\rho + o(1)\} \log M(r, f) \qquad (r \to \infty) . \tag{4}$$

In [3], Drasin and Shea characterized the $f(z)$ extremal for (3).

THEOREM B. *If $f(z)$ has order $\rho<1$ and satisfies (4), then*

$$\log M(r, f) = r^{\rho} L(r) , \tag{5}$$

where $L(r)$ varies slowly in a very long set G, i.e.

$$\lim_{\substack{r\to\infty \\ r\in G}} \frac{L(\sigma r)}{L(r)} = 1 \qquad (0<\sigma<\infty) \tag{6}$$

holds (uniformly for σ in any interval $A^{-1}\leqq\sigma\leqq A$, $A>1$), with

$$G = \bigcup_{n=1}^{\infty} [a_n, b_n] \qquad (a_n \to \infty, b_n/a_n \to \infty) \tag{7}$$

satisfying

$$\lim_{r\to\infty} \frac{1}{\log r} \int_{G\cap[1,r]} \frac{dt}{t} = 1 . \tag{8}$$

The exceptional set $E\equiv(0, \infty)-G$ on which (6) may fail can actually occur. This is shown by examples of Hayman [4] for $\rho=1/2$, and Drasin [2] for general $\rho<1$.

Combining Theorem B with Theorem A, the following problem is naturally raised.

Problem. Let ρ (<1) and G be given, where G is a very long set. Further, let $L(r)$ be a slowly varying function in G such that $r^{\rho}L(r) \neq O(\log r)$ is a convex, increasing function of $\log r$. Then is it always possible to find an entire function $f(z)$, of order ρ, such that

$$\log M(r, f) \sim r^{\rho} L(r) \qquad (r \to \infty) , \tag{9}$$

$$\log m^*(r, f) \leqq \{\cos \pi\rho + o(1)\} \log M(r, f) \qquad (r \to \infty)? \tag{10}$$

In this note, we consider the above problem for the special case $G=(0, \infty)$.

THEOREM. *Let ρ (<1) and $L(r)$ be given, where $L(r)$ is a slowly varying function (in $(0, \infty)$) such that $r^{\rho}L(r) \neq O(\log r)$ is a convex, increasing function of* $\log r$. *Then it is always possible to find an entire function $f(z)$, of order ρ, such that* (9) *and* (10) *hold.*

Observe that for $\rho=0$, the inequality (10) is not a restriction, so that our theorem is proved by Valiron (Theorem A) for $\rho=0$.

1. Preliminaries.

LEMMA 1. *Let $\Lambda(r)$ be given by* (1). *Then there exists a function $\phi(t)$, satisfying the conditions*

(i) *$\phi(t)$ is a continuous function which is continuously differentiable off a discrete set D,*

(ii) *$\phi(t)$ is strictly increasing and unbounded,*

(iii) *$\phi(1)=0$,*

and such that

$$\Lambda_1(r)\equiv\int_1^r\frac{\phi(t)}{t}dt=\Lambda(r)+O(\log r)\qquad(r\to\infty)\,. \tag{1.1}$$

Proof. Taking the term $O(\log r)$ in (1.1) into consideration, we may assume that $\Lambda(r)$ is given beforehand by

$$\Lambda(r)=\int_1^r\frac{\Psi(t)}{t}dt\qquad(\Psi(1)\equiv\Psi(1+0)=0)\,, \tag{1.2}$$

where $\Psi(t)$ is nondecreasing, unbounded, and continuous on the right. Put

$$X(t)=[\Psi(t)]\,. \tag{1.3}$$

By the properties of $\Psi(t)$, $X(t)$ takes the values $0, n_1, n_2, \cdots, n_k, \cdots$, say, where $\{n_k\}_1^\infty$ is a strictly increasing sequence of positive integers. Define the sequence $\{t_k\}_0^\infty$ by $t_0=1$ and

$$X(t)=n_k\qquad(t_k\leqq t<t_{k+1}\,;\ k=1, 2, \cdots)\,. \tag{1.4}$$

Further, take a sequence $\{m_k\}_0^\infty$ of positive numbers satisfying

$$m_k\geqq\max\left\{\left(\frac{t_{k+1}}{t_k}-1\right)\left(\log\frac{t_{k+1}}{t_k}\right)^{-1}(n_{k+1}-n_k)-1,\ 1\right\}. \tag{1.5}$$

Now, consider the following function $\phi(t)$ $(t\geqq t_0=1)$:

$$\phi(t)=n_{k+1}-(n_{k+1}-n_k)\left\{1-\left(\frac{t-t_k}{t_{k+1}-t_k}\right)^{m_k}\right\}^{1/m_k}(t_k\leqq t\leqq t_{k+1})\,. \tag{1.6}$$

As is easily seen, $\phi(t)$ satisfies the conditions (i), (ii), (iii) with $D=\{t_k\}_1^\infty$. By (1.2) and (1.3)

$$\tilde{\Lambda}(r)\equiv\int_1^r\frac{X(t)}{t}dt\leqq\Lambda(r)\leqq\tilde{\Lambda}(r)+\log r\qquad(r\geqq1)\,. \tag{1.7}$$

By (1.6) and (1.4), $\phi(t)\geqq X(t)$ $(t\geqq1)$, so that

$$\Lambda_1(r) \equiv \int_1^r \frac{\phi(t)}{t}\, dt \geqq \tilde{\Lambda}(r). \tag{1.8}$$

From (1.4), (1.5) and (1.6) it follows that

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \frac{\phi(t)-X(t)}{t}\, dt \\
&= (n_{k+1}-n_k)\int_{t_k}^{t_{k+1}} \left[1-\left\{1-\left(\frac{t-t_k}{t_{k+1}-t_k}\right)^{m_k}\right\}^{1/m_k}\right] t^{-1} dt \\
&< \frac{n_{k+1}-n_k}{t_k}\int_0^1 \{1-(1-Y^{m_k})^{1/m_k}\}(t_{k+1}-t_k)\, dY \qquad (1.9)\\
&\leqq \frac{n_{k+1}-n_k}{t_k}(t_{k+1}-t_k)\int_0^1 \{1-(1-Y)^{1/m_k}\}\, dY \\
&= (n_{k+1}-n_k)\left(\frac{t_{k+1}}{t_k}-1\right)\frac{1}{m_k+1} \leqq \log \frac{t_{k+1}}{t_k} \qquad (k=0, 1, 2, \cdots).
\end{aligned}$$

Assume that $t_k \leqq r \leqq t_{k+1}$ and put

$$F(r) = \log \frac{r}{t_k} - \int_{t_k}^r \frac{\phi(t)-X(t)}{t}\, dt. \tag{1.10}$$

Then for $t_k \leqq r \leqq t_{k+1}$

$$rF'(r) = 1-(n_{k+1}-n_k)\left[1-\left\{1-\left(\frac{r-t_k}{t_{k+1}-t_k}\right)^{m_k}\right\}^{1/m_k}\right].$$

From this, we see that $rF'(r)$ is strictly decreasing for $t_k \leqq r \leqq t_{k+1}$, and $t_k F'(t_k) = 1$, $t_{k+1}F'(t_{k+1}) = 1-(n_{k+1}-n_k) \leqq 0$. Thus, there exists a $t'_k \in (t_k, t_{k+1}]$ such that $F'(r) > 0$ $(t_k \leqq r < t'_k)$, $F'(r) \leqq 0$ $(t'_k \leqq r \leqq t_{k+1})$. Hence by (1.9) and (1.10)

$$F(r) \geqq \min\{F(t_k), F(t_{k+1})\} = 0 \qquad (t_k \leqq r \leqq t_{k+1}). \tag{1.11}$$

Combining (1.9), (1.10), and (1.11), we have for $t_k \leqq r \leqq t_{k+1}$

$$\begin{aligned}
\Lambda_1(r) - \tilde{\Lambda}(r) &= \int_1^r \frac{\phi(t)-X(t)}{t}\, dt \qquad (1.12)\\
&\leqq \sum_{l=0}^{k-l} \log \frac{t_{l+1}}{t_l} + \log \frac{r}{t_k} = \log r.
\end{aligned}$$

Therefore, (1.1) follows from (1.2), (1.7), (1.8) and (1.12).

This completes the proof of Lemma 1.

LEMMA 2. *Let ρ (<1) and $L(r)$ be given, where $L(r)$ is a slowly varying function (in $(0, \infty)$) such that $\Lambda(r) \equiv r^\rho L(r) \neq O(\log r)$ is a convex, increasing function of* $\log r$. *Corresponding to $\Lambda(r)$, define $\phi(t)$ and $\Lambda_1(r)$ as in Lemma* 1. *Then*

(1.13) $$\lambda(r)\equiv\frac{d\log(\Lambda_1(r)+1)}{d\log r}=\frac{\phi(r)}{\Lambda_1(r)+1}\longrightarrow\rho\qquad(r\to\infty).$$

Proof. Put

(1.14) $$\Lambda_1(r)=r^\rho L_1(r).$$

Then $L_1(r)$ is a slowly varying function in $(0, \infty)$ such that $\Lambda_1(r)\neq O(\log r)$ is a convex, increasing function of $\log r$. Define $h(r)$ by

(1.15) $$\lambda(r)=\rho+h(r).$$

By the definition of $\lambda(r)$ and the properties of $\phi(r)$, $\lambda(r)$ is a positive, continuous function for $r>1$, which is continuously differentiable off a discrete set $D=\{t_k\}$. By (1.13), (1.14) and (1.15)

(1.16) $$\Lambda_1(r)+1=r^\rho L_1(r)+1=\exp\left(\int_1^r\frac{\lambda(t)}{t}dt\right)=r^\rho\exp\left(\int_1^r\frac{h(t)}{t}dt\right).$$

Since $\Lambda_1(r)$ is a convex, increasing function of $\log r$, we deduce from (1.15) and (1.16) that

(1.17) $$(\lambda(r))^2+rh'(r)\geqq 0\qquad(r\notin D).$$

First, we prove $\{h(r)\}^+\equiv\max\{h(r), 0\}\to 0$ $(r\to\infty)$. Suppose that there exists a sequence $\{r_n\}\uparrow\infty$ such that $h(r_n)=\delta$ for some $\delta>0$. Since $L_1(r)$ is a slowly varying function in $(0, \infty)$, (1.16) implies

(1.18) $$\int_r^{\sigma r}\frac{h(t)}{t}dt\longrightarrow 0\qquad(r\to\infty;\ \sigma(>1):\text{ fixed}).$$

Thus there is a $s_n\in(r_n, \sigma r_n)$ such that $h(s_n)=\delta/2$ for $n\geqq n_0(\sigma)$.

Now, to each r_n $(n\geqq n_0)$ we correspond r'_n by

$$r'_n=\inf\{s>r_n;\ h(s)=\delta/2\}.$$

By the continuity of $h(r)$, we easily see that $h(r'_n)=\delta/2$ and $h(r)>\delta/2$ $(r_n\leqq r<r'_n)$. It follows from this and (1.18) that

(1.19) $$r'_n/r_n\longrightarrow 1\qquad(n\to\infty).$$

Using the mean value theorem to $\lambda(r)$, we deduce from (1.17) and (1.15) that

(1.20) $$-\delta/2=\lambda(r'_n)-\lambda(r_n)=h(r'_n)-h(r_n)\geqq-\frac{[\lambda(r''_n)]^2}{r''_n}(r'_n-r_n)\qquad(r_n<r''_n<r'_n).$$

By (1.19) and (1.20), $\lambda(r''_n)\to\infty$ $(n\to\infty)$, which implies

(1.21) $$h(r''_n)>2\delta\qquad(n\geqq n_1(\delta)).$$

(1.21) and the fact that $h(r'_n)=\delta/2$ yield the existence of $u_n\in(r''_n, r'_n)$ satisfying

$h(u_n)=\delta$. Here, define r_n''' by

$$r_n'''=\sup\{u<r_n' ;\ h(u)=\delta\} .$$

Then it is easily to see that $h(r_n''')=\delta$ and

$$\delta/2<h(r)<\delta \qquad (r_n'''<r<r_n' ;\ n\geqq n_1(\delta)) . \tag{1.22}$$

On the other hand, as we stated above, the mean value theorem gives the existence of $r_n''''\in(r_n''', r_n')$ such that $h(r_n'''')>2\delta$ for $n\geqq n_1$. This is impossible. This and (1.18) show that

$$\{h(r)\}^+ \longrightarrow 0 \qquad (r\rightarrow\infty) . \tag{1.23}$$

Next, we prove

$$\{h(r)\}^-\equiv\max\{-h(r), 0\}\longrightarrow 0 \qquad (r\rightarrow\infty) . \tag{1.24}$$

Suppose that there exists a sequence $\{R_n\}\uparrow\infty$ such that $h(R_n)=-\delta'$ for some $\delta'>0$. Using (1.18), we see that $I_n\equiv\{s<R_n ;\ h(s)=-\delta'/2\}$ is not empty for $n\geqq n_2(\delta')$. Then if we put $R_n'=\sup I_n$, $h(R_n')=-\delta'/2$ and $R_n/R_n'\to 1$ $(n\to\infty)$. It follows from these and (1.17) that for some $R_n''\in(R_n', R_n)$

$$[\lambda(R_n'')]^2>(\delta'/2)(R_n/R_n'-1)^{-1}\longrightarrow\infty \qquad (n\rightarrow\infty) . \tag{1.25}$$

Since $\lambda(r)>0$ $(r>1)$, $\lambda(R_n'')=\rho+h(R_n'')\to\infty$ $(n\to\infty)$, by (1.25). However, the definition of R_n' implies that $h(r)<-\delta'/2$ for $R_n'<r\leqq R_n$. This is untenable. This and (1.18) give (1.24). Combining (1.23) and (1.24), we have the desired result.

LEMMA 3. *Let $\Lambda_1(r)=r^\rho L_1(r)$ be given as in Lemma 2, where $\rho\in(0,1)$. Put*

$$n(r)=\left[\frac{\sin\pi\rho}{\pi}(\Lambda_1(r)+1)\right], \tag{1.26}$$

and let $f(z)$ be the entire function with negative zeros with counting function $n(r)$. Then for a suitable branch of $\log f(z)$,

$$\log f(z)=\{e^{i\rho\theta}+o(1)\}\Lambda_1(r) \qquad (z=re^{i\theta},\ |\theta|<\pi,\ r\rightarrow\infty), \tag{1.27}$$

and the $o(1)$ tends to zero uniformly as $z\to\infty$ in any sector

$$-\pi+\eta\leqq\arg z\leqq\pi-\eta \qquad (\eta>0) .$$

Proof. Let m and M be given such that $0<m<\rho<M<1$. By (1.13), there exists a $r_0\equiv r_0(m, M)$ such that $r\geqq r_0$ implies

$$m\leqq\lambda(r)\leqq M . \tag{1.28}$$

It is clear that we may prove Lemma 3 with $n(r)$ replaced by

$$n_1(r)=\begin{cases}0 & (r\leqq r_0)\\ \left[\dfrac{\sin\pi\rho}{\pi}(\Lambda_1(r)-\Lambda_1(r_0))\right] & (r>r_0).\end{cases} \tag{1.29}$$

Since (1.29) implies that $f(z)$ is of genus zero, we have for a suitable branch of $\log f(z)$

$$\log f(z)=z\int_{r_1}^{\infty}\frac{n_1(t)}{t(t+z)}dt \qquad (|\arg z|<\pi), \tag{1.30}$$

where $r_1(>r_0)$ is the number satisfying $n_1(t)=0$ $(t<r_1)$ and $n_1(r_1)=1$. By (1.28) and (1.29)

$$\frac{n_1(s)}{n_1(r)+1}\leqq\frac{\Lambda_1(s)-\Lambda_1(r_0)}{\Lambda_1(r)-\Lambda_1(r_0)}\leqq\frac{\Lambda_1(r_0)+\pi}{\pi}\cdot\frac{\Lambda_1(s)}{\Lambda_1(r)} \tag{1.31}$$

$$\leqq\frac{\Lambda_1(r_0)+\pi}{\pi}\left(\frac{s}{r}\right)^M \qquad (s>r>r_1),$$

$$\frac{n_1(s)}{n_1(r)+1}\leqq\frac{\Lambda_1(r_0)+\pi}{\pi}\left(\frac{s}{r}\right)^m \qquad (r>s>r_1). \tag{1.32}$$

Noting that

$$e^{i\theta}\int_0^{\infty}\frac{u^{\alpha-1}}{u+e^{i\theta}}du=\frac{\pi}{\sin\pi\alpha}e^{i\alpha\theta} \qquad (0<\alpha<1,\ |\theta|<\pi),$$

we have, for given $\varepsilon>0$

$$\int_0^{K^{-1}}\left|\frac{u^{\alpha-1}}{u+e^{i\theta}}\right|du<\varepsilon/2,\ \int_K^{\infty}\left|\frac{u^{\alpha-1}}{u+e^{i\theta}}\right|du<\varepsilon/2 \qquad (K\equiv K(\varepsilon, m, M)>1) \tag{1.33}$$

$$(\alpha=m,\ \rho,\ M;\ |\theta|\leqq\pi-\eta,\ \eta(>0)\text{: fixed}).$$

Hence, by (1.32) and (1.33), for $|z|=r>Kr_1$, $|\theta|\leqq\pi-\eta$

$$\left|z\int_{r_1}^{K^{-1}r}\frac{n_1(t)}{t(t+z)}dt\right|\leqq\frac{\Lambda_1(r_0)+\pi}{\pi}(n_1(r)+1)\int_{r_1}^{K^{-1}r}\left|\frac{1}{t+z}\right|\left(\frac{t}{r}\right)^{m-1}dt$$

$$\leqq 2\frac{\Lambda_1(r_0)+\pi}{\pi}n_1(r)\int_0^{K^{-1}}\left|\frac{u^{m-1}}{u+e^{i\theta}}\right|du \tag{1.34}$$

$$<\frac{\Lambda_1(r_0)+\pi}{\pi}\varepsilon n_1(r).$$

In the same way, by (1.31) and (1.33),

$$\left|z\int_{Kr}^{\infty}\frac{n_1(t)}{t(t+z)}dt\right|<\frac{\Lambda_1(r_0)+\pi}{\pi}\varepsilon n_1(r) \qquad (|z|=r>Kr_1,\ |\theta|\leqq\pi-\eta). \tag{1.35}$$

Finally with this choice of K, we choose σ positive but so small that

$$\sigma\int_{K^{-1}}^{K}\left|\frac{u^{\rho-1}}{u+e^{i\theta}}\right|du<\varepsilon \qquad (|\theta|\leqq\pi-\eta). \tag{1.36}$$

Since $L_1(r)$ is a slowly varying function, we have

$$(1-\sigma)\left(\frac{t}{r}\right)^{\rho}<\frac{n_1(t)}{n_1(r)}<(1+\sigma)\left(\frac{t}{r}\right)^{\rho} \qquad (Kr>t>K^{-1}r>r_2(\sigma)). \tag{1.37}$$

Then, if $|z|=r>Kr_2$, $|\theta|\leqq\pi-\eta$, (1.36) and (1.37) yield that

$$\left|z\int_{K^{-1}r}^{Kr}\frac{n_1(t)}{t(t+z)}dt-n_1(r)e^{i\theta}\int_{K^{-1}}^{K}\frac{u^{\rho-1}}{u+e^{i\theta}}du\right| \tag{1.38}$$

$$\leqq\sigma n_1(r)\int_{K^{-1}}^{K}\left|\frac{u^{\rho-1}}{u+e^{i\theta}}\right|du<\varepsilon n_1(r).$$

Combining (1.30), (1.33), (1.34), (1.35) and (1.38), we have for $|z|=r>\max(Kr_1, Kr_2)$, $|\theta|\leqq\pi-\eta$

$$\left|\log f(z)-\frac{\pi}{\sin\pi\rho}e^{i\rho\theta}n_1(r)\right|$$

$$\leqq\left|z\int_{r_1}^{K^{-1}r}\frac{n_1(t)}{t(t+z)}dt\right|+\left|z\int_{K^{-1}r}^{Kr}\frac{n_1(t)}{t(t+z)}dt-n_1(r)e^{i\theta}\int_{K^{-1}}^{K}\frac{u^{\rho-1}}{u+e^{i\theta}}du\right|$$

$$+\left|z\int_{Kr}^{\infty}\frac{n_1(t)}{t(t+z)}dt\right|+n_1(r)\int_{0}^{K^{-1}}\left|\frac{u^{\rho-1}}{u+e^{i\theta}}\right|du+n_1(r)\int_{K}^{\infty}\left|\frac{u^{\rho-1}}{u+e^{i\theta}}\right|du$$

$$<\left\{2\frac{\Lambda_1(r_0)+\pi}{\pi}+2\right\}\varepsilon n_1(r).$$

This proves (1.27).

2. Proof of Theorem. Let ρ $(0<\rho<1)$ and $\Lambda(r)=r^{\rho}L(r)\neq O(\log r)$ be given. By Lemma 1, we may replace $\Lambda(r)$ by $\Lambda_1(r)=r^{\rho}L_1(r)$. Further, by Lemma 3, there exists an entire function $f(z)$, of order ρ, such that

$$\log f(z)=\{e^{i\rho\theta}+o(1)\}\Lambda_1(r) \qquad (z=re^{i\theta},\ |\theta|<\pi,\ r\to\infty),$$

where $o(1)$ tends to zero uniformly as $z\to\infty$ in any sector $|\arg z|\leqq\pi-\eta$.

By the construction of $f(z)$, it is clear that

$$m^*(r, f)=|f(-r)|, \qquad M(r, f)=|f(r)|.$$

Hence

$$\log M(r, f)\sim\Lambda_1(r)\sim\Lambda(r) \qquad (r\to\infty),$$

$$\log m^*(r, f)<\log|f(re^{i(\pi-\eta)})|$$

$$\sim[\cos(\pi-\eta)\rho]\Lambda_1(r)$$

$$\leqq(\cos\pi\rho+o(1))\log M(r, f) \qquad (r\to\infty).$$

This completes the proof of our theorem.

Remark. From Lemma 3, we easily deduce

THEOREM C. *Let* ρ $(0<\rho<1)$ *and* $L(r)$ *be given, where* $L(r)$ *is a slowly varying function such that* $\Lambda(r)=r^{\rho}L(r)$ *is a convex, increasing function of* $\log r$. *Put*

$$n(r)=\left[\left(\frac{\sin\pi\rho}{\pi}+o(1)\right)\Lambda(r)\right],$$

and let $f(z)$ *be the entire function with negative zeros with counting function* $n(r)$. *Then*

$$\log|f(z)|=\{\cos\rho\theta+o(1)\}\,\Lambda(r)\qquad(z=re^{i\theta},\ |\theta|<\pi,\ r\to\infty),$$

and the $o(1)$ *tends to zero uniformly as* $z\to\infty$ *in any sector* $|\theta|\leqq\pi-\eta$ $(\eta>0)$.

In particular, we have the following

COROLLARY. *Let*

$$f(z)=\prod_{n=1}^{\infty}\left(1+\frac{z}{a_n}\right)\qquad(0<a_n\leqq a_{n+1})$$

be an entire function. Assume there are a constant ρ $(0<\rho<1)$ *and a slowly varying function* $L(r)$ *snch that* $\Lambda(r)=r^{\rho}L(r)$ *is a convex, increasing function of* $\log r$, *and such that*

$$n(r)\equiv n(r,0,f)=\left[\left(\frac{\sin\pi\rho}{\pi}+o(1)\right)\Lambda(r)\right].$$

Then

$$\log M(r,f)\sim\Lambda(r)\qquad(r\to\infty),$$

$$\log m^*(r,f)<(\cos\pi\rho+\varepsilon)\Lambda(r)\qquad(r>r_0(\varepsilon)).$$

For the special case $L(r)\equiv$constant, this was proved by Titchmarsh [5, Theorems I, III; p 185, p 191].

REFERENCES

[1] CLUNIE, J., On integral functions having prescribed asymptotic growth, Canad. J. Math. (1965), 396-404.

[2] DRASIN, D., A flexible proximate order, Bull. London Math. Soc. (1974), 129-135.

[3] DRASIN, D. AND SHEA, D. F., Convolution inequalities, regular variation and exceptional sets, J. Analyse Math. (1977), 232-293.

[4] HAYMAN, W. K., Some examples related to the $\cos\pi\rho$ theorem; Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press (1970), 149-170.

[5] TITCHMARSH, E. C., On integral functions with real negative zeros, Proc. London Math. Soc. (1927), 185-200.

[6] VALIRON, G., Sur les fonctions entières d'ordre fini et d'ordre nul, et en particulier les fonctions à correspondance régulière, Ann. Fac. Sci. Toulouse, (1913), 117-208.

DEPARTMENT OF MATHEMATICS
DAIDO INSTITUTE OF TECHNOLOGY
DAIDO-CHO, MINAMI-KU, NAGOYA, JAPAN