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ON THE ALGEBRAIC STRUCTURES OF GRADED LIE ALGEBRAS OF SECOND ORDER

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§ 0. Introduction.

In 1973, by using of generalized Jordan triple systems of second order (=Kantor systems), I. L. Kantor [4] has given the models of graded Lie algebras of second order with involutive automorphism τ. In this note, we shall prove the converse, that is, if τ is an automorphism of a Lie triple system in a graded Lie algebra of second order such that $\tau^2=1$ (resp. -1), it characterizes the Kantor (resp. Freudenthal) system. We also give a simple connection between the two kinds of triple systems.

§ 1. A characterization of Kantor and Freudenthal systems.

We consider a graded Lie algebra of second order

$$\mathfrak{G}=\mathfrak{G}_{-2}\oplus\mathfrak{G}_{-1}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_1\oplus\mathfrak{G}_2 \qquad \text{(direct sum)}$$
$$[\mathfrak{G}_i, \mathfrak{G}_j]\subset\mathfrak{G}_{i+j} \tag{1.1}$$

over a field k of characteristic zero. Then the vector space $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ becomes a Lie triple system (L. t. s.) with a triple product $[[X, Y], Z]$ where $[\ ,\ ]$ is the Lie product of $\mathfrak{G}$ and elements X, Y, Z are in $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ (cf. [7]). Let τ be an automorphism of the L. t. s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ with respect to the triple product. Then τ is called an *ε-structure* on $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ ($\varepsilon=\pm1$) if $\tau^2=\varepsilon id$ and $\tau(\mathfrak{G}_{\pm1})=\mathfrak{G}_{\mp1}$.

Let V be a finite dimensional vector space over the field k. Then V is called a *Kantor* (resp. *Freudenthal*) *system* (cf. [4], [2], [8]) if V has a trilinear operation $\phi: V\times V\times V\to V$ such that

1) $[L(a, b), L(c, d)]=L(L(a, b)c, d)-\varepsilon L(c, L(b, a)d)$,
2) $K(K(a, b)c, d)=L(d, c)K(a, b)+\varepsilon K(a, b)L(c, d)$

for $a, b, c, d\in V$, where $L(a, b)c=\phi(a, b, c)$, $K(a, b)c=L(a, c)b-L(b, c)a$ and $\varepsilon=1$ (resp. -1).

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Now let $\mathfrak{G}$ be a graded Lie algebra of second order which form is of (1.1) and let τ be an ε-structure on the L. t. s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$. We denote by $\tau_{\pm1}$ the ε-structure τ restricted to $\mathfrak{G}_{\pm1}$, but, for simplicity, we sometimes use the same notation τ instead of $\tau_{\pm1}$ unless the confusion does not occur. When we write an element $a+\tau(x)$ in $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ as the column vector, the Lie triple product $[[a+\tau(x), b+\tau(y)], c+\tau(z)]$ in $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ can be denoted by

$$\left[\begin{pmatrix} a \\ \tau(x)\end{pmatrix}\begin{pmatrix} b \\ \tau(y)\end{pmatrix}\begin{pmatrix} c \\ \tau(z)\end{pmatrix}\right]=\begin{pmatrix} K(a, b)z+L(a, y)c-L(b, x)c \\ \varepsilon\tau(K(x, y)c+L(x, b)z-L(y, a)z)\end{pmatrix} \tag{1.2}$$

for $a, b, c, x, y, z\in\mathfrak{G}_{-1}$ where $L(a, b)c=[[a, \tau(b)], c]$ and $K(a, b)c=[[a, b], \tau(c)]$. Moreover, by using of 2×2 matrix forms and column vectors, the right side of (1.2) can be rewritten as the following form:

$$\begin{pmatrix} L(a, y)-L(b, x) & K(a, b)\tau^{-1} \\ \varepsilon\tau K(x, y) & -\varepsilon\tau(L(y, a)-L(x, b))\tau^{-1}\end{pmatrix}\begin{pmatrix} c \\ \tau(z)\end{pmatrix} \tag{1.3}$$

for $a, b, c, x, y, z\in\mathfrak{G}_{-1}$.

If the Lie algebra $\mathfrak{G}$ is semi-simple, it is isomorphic to the standard imbedding (Lie algebra) of the L. t. s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ (see [6], [7]).

Now, let V be any Kantor (resp. Freudenthal) system. Then the direct sum $\mathfrak{T}(V)=V\oplus V$ becomes a L. t. s. with an ε-structure τ by the trilinear multiplication of (1.2) where $\tau_{-1}=1$ and $\tau_1=\varepsilon$ with $\varepsilon=1$ (resp. -1). And, the standard imbedding (Lie algebra) $\mathfrak{L}(V)$ of the L. t. s. $\mathfrak{T}(V)$ has a structure of graded Lie algebra of second order (see [4], [8]). Then, we have the following.

THEOREM 1. *Let $\mathfrak{G}$ be a graded Lie algebra of second order which form is of* (1.1). *If the L. t. s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ has a* 1*-structure* (*resp.* -1*-structure*) τ, *$\mathfrak{G}_{-1}$ is a Kantor* (*resp. Freudenthal*) *system with respect to the trilinear operation $\phi(a, b, c)=[[a, \tau(b)], c]$ for $a, b, c\in\mathfrak{G}_{-1}$. Moreover, if $\mathfrak{G}$ is semi-simple, it is isomorphic to the standard imbedding* (*Lie algebra*) *$\mathfrak{L}(\mathfrak{G}_{-1})$ of the L. t. s. $\mathfrak{T}(\mathfrak{G}_{-1})$.*

Proof. Assume that the L. t. s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ in $\mathfrak{G}$ has an ε-structure τ. Then we show that $\mathfrak{G}_{-1}$ is a Kantor or Freudenthal system with respect to the trilinear operation $\phi(a, b, c)$ $(=L(a, b)c)=[[a, \tau(b)], c]$ corresponding to $\varepsilon=1$ or -1 respectively. The adjoint representation ad of the Lie algebra $\mathfrak{G}$ is defined usually by $\mathrm{ad}(x)y=[x, y]$ for $x, y\in\mathfrak{G}$.

First, for $a, b, c, d, e\in\mathfrak{G}_{-1}$, we have

$$\begin{aligned}[L(a, b), L(c, d)]&=[\mathrm{ad}[a, \tau(b)], \mathrm{ad}[c, \tau(d)]]\\ &=\mathrm{ad}[[a, \tau(b)], [c, \tau(d)]]\\ &=\mathrm{ad}[[[a, \tau(b)], c], \tau(d)]+\mathrm{ad}[c, [[a, \tau(b)], \tau(d)]]\end{aligned}$$

by the Jacobi's identity. Since $[[a, \tau(b)], c]=L(a, b)c$ and $[[a, \tau(b)], \tau(d)]=$

$-\varepsilon\tau L(b, a)d$, the operation ϕ satisfies the axiom 1) of the triple system. Secondly, again by the Jacobi's identity, it holds that $[[a, b], \tau(c)]=L(a, c)b-L(b, c)a$, i.e., the definition of $K(a, b)$ in the Kantor or Freudenthal system coincides with the definition in the L.t.s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ of (1.2). Then we can prove the axiom 2):

$$\begin{aligned} K(K(a, b)c, d)e &= [[[[a, b], \tau(c)], d], \tau(e)] \\ &= -[[[\tau(c), d], [a, b]], \tau(e)] \\ &= -\mathrm{ad}[[\tau(c), d], [a, b]]\tau(e) \\ &= -[\mathrm{ad}[\tau(c), d], \mathrm{ad}[a, b]]\tau(e) \\ &= -\mathrm{ad}[\tau(c), d]\mathrm{ad}[a, b]\tau(e)+\mathrm{ad}[a, b]\mathrm{ad}[\tau(c), d]\tau(e) \\ &= L(d, c)K(a, b)e+\varepsilon K(a, b)L(c, d)e \end{aligned}$$

where the second equality is proved by the Jacobi's identity and the relation $[[a, b], d]\in\mathfrak{G}_{-3}=\{0\}$.

Now, let $\mathfrak{G}_{-1}$ be the Kantor (resp. Freudenthal) system with the product $L(a, b)c$ which is obtained from $\mathfrak{G}$ by the above method. Then the L.t.s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ in $\mathfrak{G}$ with the ε-structure τ is isomorphic to $\mathfrak{T}(\mathfrak{G}_{-1})$ by the mapping $a+\tau_{-1}(x)\to a+x$. Therefore, if we assume the semi-simplicity of $\mathfrak{G}$, the two Lie algebras $\mathfrak{G}$ and $\mathfrak{L}(\mathfrak{G}_{-1})$ are isomorphic.

EXAMPLES. For the graded simple Lie algebra of second order with a 1-structure in $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$, we know the models constructed by Tits-Koecher [5] and I. L. Kantor [4]. However, the Tits-Koecher's models are of $\mathfrak{G}_{\pm 2}=\{0\}$. For -1-structure, there are the models by H. Freudenthal [2] and B. N. Allison [1].

Remark. Any automorphism τ of the L.t.s. $\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$ can be canonically extended to an automorphism τ of the Lie algebra $\mathfrak{G}$. In fact, under the notation of (1.3), we can define the automorphism $\tau:\mathfrak{G}\to\mathfrak{G}$ by $\tau(D+E)=\tau D\tau^{-1}+\tau(E)$ for $D+E\in(\mathfrak{G}_{-2}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_2)\oplus(\mathfrak{G}_{-1}\oplus\mathfrak{G}_1)$. Hence, any 1-structure τ can be extended to an involutive automorphism of $\mathfrak{G}$ and -1-structure τ becomes an automorphism with $\tau^4=1$ where $\tau^2=1$ in $\mathfrak{G}_{-2}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_2$.

§ 2. Simplicity of $\mathfrak{L}(V)$.

Let V be a Kantor (resp. Freudenthal) system with a triple product $L(a, b)c$ such that $L(V, V)V\neq\{0\}$ and let $\mathfrak{L}(V)$ be the standard imbedding (Lie algebra) of the L.t.s. $\mathfrak{T}(V)$. Usually, V is said to be simple if V has no subspaces $\{W\}$ except $\{0\}$ and V such that $L(W, V)V\subset W$, $L(V, W)V\subset W$ and $L(V, V)W\subset W$.

PROPOSITION 2. *If $\mathfrak{L}(V)$ is simple, V is simple. Conversely, if V is simple, $\mathfrak{L}(V)$ is semi-simple.*

Proof. Put $V=\mathfrak{G}_{-1}$. We assume that $\mathfrak{A}$ is any ideal of $\mathfrak{G}_{-1}$. Then $\mathfrak{A}\oplus\tau(\mathfrak{A})$ is an ideal of the L.t.s. $\mathfrak{T}(\mathfrak{G}_{-1})$ where $\tau_{-1}=1$ and $\tau_1=\varepsilon$ with $\varepsilon=1$ (resp. -1). Since $\mathfrak{T}(\mathfrak{G}_{-1})$ is simple by the simplicity of $\mathfrak{L}(\mathfrak{G}_{-1})$, we have $\mathfrak{A}=\{0\}$ or $\mathfrak{G}_{-1}$. This means that $\mathfrak{G}_{-1}$ is simple.

Conversely, we assume that $\mathfrak{G}_{-1}$ is simple. For the L.t.s. $\mathfrak{T}=\mathfrak{T}(\mathfrak{G}_{-1})$ of $\mathfrak{L}(\mathfrak{G}_{-1})$, it always holds $\mathfrak{R}(\mathfrak{L}(\mathfrak{G}_{-1}))=\mathfrak{R}(\mathfrak{T})\oplus[\mathfrak{R}(\mathfrak{T}), \mathfrak{T}]$ where $\mathfrak{R}(\mathfrak{L}(\mathfrak{G}_{-1}))$ and $\mathfrak{R}(\mathfrak{T})$ are the radicals of $\mathfrak{L}(\mathfrak{G}_{-1})$ and $\mathfrak{T}(\mathfrak{G}_{-1})$ respectively (cf. [6]). Under the notation of (1.3), since the mapping $\theta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ of $\mathfrak{T}$ makes the radical $\mathfrak{R}(\mathfrak{T})$ invariant and τ is an automorphism of $\mathfrak{T}$, there is an ideal $\mathfrak{A}$ of $\mathfrak{G}_{-1}$ such that $\mathfrak{R}(\mathfrak{T})=\mathfrak{A}\oplus\tau(\mathfrak{A})\subset\mathfrak{T}$. Because $\mathfrak{G}_{-1}$ is simple, if we assume $\mathfrak{A}=\mathfrak{G}_{-1}$, we have $\mathfrak{L}(\mathfrak{G}_{-1}, \mathfrak{G}_{-1})\mathfrak{G}_{-1}=\mathfrak{G}_{-1}$ and $\mathfrak{T}\subset\mathfrak{R}(\mathfrak{T})^{(n)}$ $(=[\mathfrak{T}, \mathfrak{R}(\mathfrak{T})^{(n-1)}, \mathfrak{R}(\mathfrak{T})^{(n-1)}]$ where $[\ ,\ ,\ ]$ is the triple product of the L.t.s. $\mathfrak{T})$ for any natural number n. But this contradicts $\mathfrak{G}_{-1}\neq\{0\}$. Therefore $\mathfrak{A}=\{0\}$, i.e., $\mathfrak{R}(\mathfrak{T})=\mathfrak{R}(\mathfrak{L}(\mathfrak{G}_{-1}))=\{0\}$.

§ 3. Isomorphisms of $\mathfrak{L}(V)$.

Two triple systems V_1, V_2, having triple products $L_1(a, b)c$, $L_2(x, y)z$ respectively, are weakly isomorphic if there are two one-to-one onto mappings $P, Q: V_1\to V_2$ such that $PL_1(a, b)c=L_2(Pa, Qb)Pc$ and $QL_1(a, b)c=L_2(Qa, Pb)Qc$ for $a, b, c\in V_1$ where we use the notation Pa instead of $P(a)$. Then we have $PK_1(a, b)c=K_2(Pa, Pb)Qc$ and $QK_1(a, b)c=K_2(Qa, Qb)Pc$ where $K_i(a, b)c=L_i(a, c)b-L_i(b, c)a$ $(i=1, 2)$.

PROPOSITION 3. *Two standard imbedding (Lie algebra) $\mathfrak{L}(\mathfrak{G}_{-1})$ and $\mathfrak{L}(\mathfrak{G}'_{-1})$ are isomorphic (by an isomorphism preserving the grading) if and only if two triple systems $\mathfrak{G}_{-1}$ and $\mathfrak{G}'_{-1}$ are weakly isomorphic. If σ is a grade-preserving isomorphism of $\mathfrak{L}(\mathfrak{G}_{-1})$ to $\mathfrak{L}(\mathfrak{G}'_{-1})$, we can have $P=\sigma|\mathfrak{G}_{-1}$ (the restriction of σ to $\mathfrak{G}_{-1}$) and $Q=\tau'^{-1}\sigma\tau|\mathfrak{G}_{-1}$ where τ and τ' are ε-structures in $\mathfrak{T}(\mathfrak{G}_{-1})$ and $\mathfrak{T}(\mathfrak{G}'_{-1})$ respectively.*

Proof. If $\mathfrak{G}_{-1}$ and $\mathfrak{G}'_{-1}$ are weakly isomorphic, there is an isomorphism $\sigma: \mathfrak{T}(\mathfrak{G}_{-1})\to\mathfrak{T}(\mathfrak{G}'_{-1})$ with respect to the triple product of the L.t.s. which is defined by $\sigma(a+\tau(b))=Pa+\tau'(Qb)$ for $a, b\in\mathfrak{G}_{-1}$. And, this σ can be canonically extended to an isomorphism of the standard imbedding (Lie algebra) by the same method as the Remark of Theorem 1.

Conversely, let σ be a grade-preserving isomorphism of $\mathfrak{L}(\mathfrak{G}_{-1})$ to $\mathfrak{L}(\mathfrak{G}'_{-1})$. When we put $P=\sigma|\mathfrak{G}_{-1}$ and $Q=\tau'^{-1}\sigma\tau|\mathfrak{G}_{-1}$, we have $PL_1(a, b)c=\sigma[[a, \tau(b)], c]$ $=[[\sigma(a), \sigma\tau(b)], \sigma(c)]=[[\sigma(a), \tau'\tau'^{-1}\sigma\tau(b)], \sigma(c)]=L_2(Pa, Qb)Pc$ for $a, b, c\in\mathfrak{G}_{-1}$. The other relation $QL_1(a, b)c=L_2(Qa, Pb)Qc$ can be proved similarly.

§ 4. A duality.

There is a simple connection between the Kantor systems and the Freudenthal systems.

THEOREM 4. *Let V be a Kantor (resp. Freudenthal) system with a triple product $L(a, b)c$. If there exist an automorphism Φ of V, i.e., $\Phi(L(a, b)c)=L(\Phi(a), \Phi(b))\Phi(c)$ for $a, b, c\in V$, such that $\Phi^2=-1$, V becomes a Freudenthal (resp. Kantor) system with respect to the new triple product $L(a, \Phi(b))c$ (resp. $-L(a, \Phi(b))c$). This mapping Φ is also an automorphism for the new product.*

EXAMPLE. Let V $(=\mathfrak{C})$ be the Cayley algebra over the complex numbers $\boldsymbol{C}$. Then V is a Kantor system by the triple product $L(a, b)c=a(\bar{b}c)+c(\bar{b}a)-b(\bar{a}c)$ for $a, b, c\in V$, where $-$ is the usual conjugation of V, and $\mathfrak{L}(V)$ is a simple Lie algebra of type $\boldsymbol{F}_4$. In this case, the right multiplication Φ is an automorphism for the triple product $a(\bar{b}c)$ in V where $\Phi(x)=xv$ for any $x\in V$ and some fixed $v\in V$ with $\mathrm{tr}(v)=0$ and $vv=-1$. Therefore Φ is also an automorphism with respect to the product $L(a, b)c$ and V becomes a Freudenthal system by the product $L(a, bv)c=a((\overline{bv})c)+c((\overline{bv})a)-(bv)(\bar{a}c)$ for $a, b, c\in V$ (cf. [3]).

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