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ON A MINIMAX FORMULA OF LEJA'S

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1. Introduction. Let D be a domain in the compact complex plane containing ∞. Leja [5] presented two discrete formulas for the Green's function of D (if it exists) with the logarithmic singularity at ∞ in terms of the points of ∂D. Pommerenke [9] later proved a hyperbolic version of one of these formulas—the one which involves the Fekete-Leja extremal points. We will prove here a hyperbolic version of the other formula, which is a minimax formula. We will also point out an extremal property of the points involved in this type of minimax formula. The author would like to thank Professor James A. Jenkins for introducing Leja's work to him and for many suggestions.

2. Notation and Known Results. The unit disc $\{|z|<1\}$ will be denoted by Δ. Given two complex numbers a and b such that $1-\bar{b}a\neq 0$, we let

$$[a, b]=(a-b)/(1-\bar{b}a). \tag{1}$$

Then $d(a, b)=|[a, b]|$ defines a metric in Δ. Let E in the following denote a given compact set in Δ. Then the capacity of E is defined as follows ([10]; see also [8, 4]). Let

$$V(x_0, x_1, \cdots, x_n)=\textstyle\prod_{0\leq i<j\leq n}|[x_i, x_j]|, \qquad \text{for} \quad x=\{x_0, \cdots, x_n\}\subset E;$$

$$V_n=V_n(E)=\max_x V(x_0, \cdots, x_n)(x=\{x_0, \cdots, x_n\}\subset E) \quad \text{and}$$

$$v_n=v_n(E)=V_n^{2/[n(n+1)]}.$$

Then $\lim_{n\to\infty} v_n(E)$ exists and is called the *capacity* of E, denoted by ρ or $\rho(E)$.

Given $x=\{x_0, \cdots, x_n\}\subset E$ and $z\in\Delta$, we let

$$\delta_n^{(j)}(z; x)=\textstyle\prod_{i\neq j}|[z, x_i]|, \qquad \text{for} \quad j=0, 1, \cdots, n, \quad \text{and} \tag{2}$$

$$\delta_n=\max_x\min_{0\leq j\leq n}\delta_n^{(j)}(x_j; x). \tag{3}$$

Then we have

THEOREM 1. *$\delta_n^{1/n}\to\rho$ as $n\to\infty$* ([8]; see also [4]).

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Following Leja, if $\eta=\{\eta_0, \eta_1, \cdots, \eta_n\}\subset E$ and $V_n(E)=V(\eta)$ we call η an *n-extremal system* of E. In the following, for each $n>0$, $\eta=\{\eta_0, \cdots, \eta_n\}$ always stands for an n-extremal system of E. Furthermore we assume that $\eta_0, \cdots, \eta_n$ have been so arranged that

$$(4)\qquad \delta_n^{(0)}(\eta_0; \eta)\leqq\delta_n^{(j)}(\eta_j; \eta) \qquad \text{for} \quad n\geqq j\geqq 0.$$

Then we have

THEOREM 2. $[\delta_n^{(0)}(\eta_0; \eta)]^{1/n}\to\rho$ *as* $n\to\infty$ ([9]; see also [5, 2]).

Let D denote the component of $\Delta-E$ with $\partial\Delta$ as one of its boundary components.

Now for each $n>0$ and an n-extremal system η, let

$$(5)\qquad \begin{aligned} &L_n(z; \eta)=\prod_{1\leq i\leq n}[z, \eta_i]\cdot\prod_{1\leq i\leq n}\{(1-\bar{\eta}_i)/(1-\eta_i)\}, \quad \text{and} \\ &l_n(z)=L_n^{1/n}(z; \eta). \end{aligned}$$

Here $l_n(z)$ is first defined in a neighborhood of 1 with $l_n(1)=1$ and then extended continuously to $G=D\cup\partial\Delta\cup D^*$, where D^* is the domain of reflection of D with respect to $\partial\Delta$.

In what follows, a continuum is always meant to be a non-degenerate one. Then we have

THEOREM 3. ([9]; see also [5, 7]) *Assume* $\rho>0$. *Then* $l(z)=\lim_{n\to\infty}l_n(z)$ *exists locally uniformly in* G. $l(z)$ *is locally analytic and is of single-valued modulus in* G. *If* b *is a boundary point of* D *that lies on a continuum contained in* E, *then* $|l(z)|\to\rho$ *as* $z\to b$, $z\in D$. *Furthermore*, $|l(z)|=1$ *for* $|z|=1$. *If* E *is the union of a finite number of continua then* $\log|1/l(z)|/\log|1/\rho|$ *is the harmonic measure of* E *with respect to* D. *If* E *is a continuum then* $w=l(z)$ *maps* D *conformally onto* $\{w: \rho<|w|<1\}$ *with* $l(1)=1$.

3. Main Results.

Assume now E has infinitely many points. For a set $x=\{x_0, \cdots, x_n\}$ of $n+1$ distinct points of E, we consider the *polynomials of Lagrange belonging to these points*:

$$(6)\qquad L_n^{(j)}(z; x)=\prod_{i\neq j}([z, x_i]/[x_j, x_i]), \qquad j=0, 1, \cdots, n$$

and let

$$(7)\qquad L_n(z)=\min_x\max_{0\leq j\leq n}|L_n^{(j)}(z; x)|, \qquad z\in G.$$

We have the following hyperbolic version of Leja's minimax formula [5].

THEOREM 4. *If* $\rho>0$, *then* $\lim_{n\to\infty}L_n^{1/n}(z)=|l(z)|/\rho$ *for* z *in* D.

For a proof of this theorem we need the following lemmas. We assume $\rho(E)>0$.

I. $\lim_{n\to\infty} L_n^{1/n}(z)=L(z)$ *exists for all* $z\in G$.

This is essentially proven in [5, §§ 3-4].

II. $L(z)=1/\rho$ *for* $|z|=1$.

If $|z|=1$ and $b\in E$, then $|[z, b]|=1$. Therefore by (7) and (3),

$$L_n(z)=\min_x \max_{0\leqq j\leqq n} \prod_{i\neq j}(1/|[x_j,\ x_i]|)$$

$$=1/\max_x \min_{0\leqq j\leqq n} \prod_{i\neq j}|[x_j,\ x_i]|=1/\delta_n\,.$$

II then follows from THEOREM 1.

III. $1\leqq L(z)\leqq|l(z)|/\rho$ *for* $z\in G$.

For a proof of the first inequality, use Lagrange interpolation formula as in [9, p. 943]; for the second, see [5, p. 67].

Assume now a point a in D is given and fixed. We will assume, as we may, that for each $n>0$,

$$L_n(a)=L_n^{(0)}(a\,;\zeta)=\delta_n^{(0)}(a\,;\zeta)/\delta_n^{(0)}(\zeta_0\,;\zeta) \tag{8}$$

$$\geqq L_n^{(j)}(a\,;\zeta)=\delta_n^{(j)}(a\,;\zeta)/\delta_n^{(j)}(\zeta_j\,;\zeta)$$

where $\zeta=\{\zeta_0,\ \zeta_1,\ \cdots,\ \zeta_n\}\subset E$ and $j=0,\ 1,\ \cdots,\ n$. Let $A_n=[\delta_n^{(0)}(\zeta_0\,;\zeta)]^{1/n}$. Then we have

IV. $0<\liminf_{n\to\infty} A_n\leqq\limsup_{n\to\infty} A_n\leqq\rho$.

By (8), for $j=0,\ 1,\ \cdots,\ n$,

$$|[a,\ \zeta_1]\cdots[a,\ \zeta_n]|/\delta_n^{(0)}(\zeta_0\,;\zeta)$$

$$\geqq|[a,\ \zeta_0]\cdots[a,\ \zeta_{j-1}][a,\ \zeta_{j+1}]\cdots[a,\ \zeta_n]/\delta_n^{(j)}(\zeta_j\,;\zeta)\,.$$

Thus

$$\delta_n^{(j)}(\zeta_j\,;\zeta)\geqq\delta_n^{(0)}(\zeta_0\,;\zeta)\cdot|[a,\ \zeta_0]/[a\,;\zeta_j]|,\qquad j=0,\ 1,\ \cdots,\ n. \tag{9}$$

Whence

$$V^2(\zeta_0,\ \cdots,\ \zeta_n)=\delta_n^{(n)}(\zeta_n\,;\zeta)\cdot\delta_n^{(n-1)}(\zeta_{n-1}\,;\zeta)\cdots\delta_n^{(0)}(\zeta_0\,;\zeta)$$

$$\geqq[\delta_n^{(0)}(\zeta_0\,;\zeta)]^{n+1}\cdot|[a,\ \zeta_0]|^n/|[a,\ \zeta_1]\cdots[a,\ \zeta_n]|.$$

Let $m=\min_{b\in E}|[a,\ b]|$ and $M=\max_{b\in E}|[a,\ b]|$. Then $m>0$ and

$$v_n(E)\geqq[V(\zeta_0,\ \cdots,\ \zeta_n)]^{2/[n(n+1)]}\geqq[\delta_n^{(0)}(\zeta_0\,;\zeta)]^{1/n}\cdot(m/M)^{1/(n+1)}\,.$$

Thus $\limsup_{n\to\infty} A_n\leqq\rho$.

On the other hand, let $\eta=\{\eta_0,\ \cdots,\ \eta_n\}$ be an n-extremal system of E and suppose

$$\max_j|L_n^{(j)}(a\,;\eta)|=|L_n^{(p)}(a\,;\eta)|=\delta_n^{(p)}(a\,;\eta)/\delta_n^{(p)}(\eta_p\,;\eta)$$

where $0 \leqq p \leqq n$. Then by (8), (7) and (4),

$$L_n(a)=|[a, \zeta_1] \cdots [a, \zeta_n]|/\delta_n^{(0)}(\zeta_0 ; \zeta)$$
$$\leqq |L_n^{(p)}(a ; \eta)| \leqq M^n/\delta_n^{(p)}(\eta_p ; \eta) \leqq M^n/\delta_n^{(0)}(\eta_0 ; \eta).$$

Thus $\delta_n^{(0)}(\zeta_0 ; \zeta) \geqq \delta_n^{(0)}(\eta_0 ; \eta)(m/M)^n$. The left hand inequality of IV then follows from this and THEOREM 2.

Let ζ be as in (8). By IV, we can choose a subsequence of $\{A_n\}=\{[\delta_n^{(0)}(\zeta_0 ; \zeta)]^{1/n}\}$ which tends to a limit ρ' with $0<\rho'<\rho$. By abuse of language, we will use the original indexing for this subsequence and the subsequent related subsequences. By I, the corresponding subsequence $\{[\delta_n^{(0)}(a ; \zeta)]^{1/n}\}$ converges to $L(a)\cdot\rho'$. Corresponding to this (sub)sequence, we now consider the sequence of functions $\{h_n(z)\}$, where

$$h_n(z)=\left([z, \zeta_1] \cdots [z, \zeta_n]\cdot\frac{1-\bar{\zeta}_1}{1-\zeta_1}\cdots\frac{1-\bar{\zeta}_n}{1-\zeta_n}\right)^{1/n}, \qquad z\in G, \tag{10}$$

is defined in a similar manner as $l_n(z)$ in (5). $h_n(z)$ is locally analytic and $|h_n(z)|$ is single-valued. Since $\{h_n(z)\}$ is evidently uniformly bounded on any compact set in G, there exists by Montel's lemma a subsequence $\{h_n(z)\}$ such that $h_n(z)$ converges uniformly in any compact set in G. Let $h(z)$ be the limit function. Then by (10),

$$|h(a)|=L(a)\rho' \quad \text{and} \quad |h(z)|=1 \quad \text{for} \quad |z|=1. \tag{11}$$

Assume now for a fixed $n>0$ and $z\in G$, $\max_j |L_n^{(j)}(z ; \zeta)|=|L_n^{(j_z)}(z ; \zeta)|$. Then by this, (7), (6), (9) and (10)

$$L_n(z)\leqq |L_n^{(j_z)}(z ; \zeta)|=\delta_n^{(j_z)}(z ; \zeta)/\delta_n^{(j_z)}(\zeta_{j_z} ; \zeta)$$
$$\leqq [\delta_n^{(j_z)}(z ; \zeta)/\delta_n^{(0)}(\zeta_0 ; \zeta)]\cdot |[a, \zeta_{j_z}]/[a, \zeta_0]|$$
$$=[\delta_n^{(0)}(z ; \zeta)/\delta_n^{(0)}(\zeta_0 ; \zeta)]\cdot |[z, \zeta_0]/[z, \zeta_{j_z}]|\cdot|[a, \zeta_{j_z}]/[a, \zeta_0]|$$
$$=|h_n^n(z)/\delta_n^{(0)}(\zeta_0 ; \zeta)|\cdot|[z, \zeta_0]/[z, \zeta_{j_z}]|\cdot|[a, \zeta_{j_z}]/[a, \zeta_0]|.$$

Since $[\delta_n^{(0)}(\zeta_0 ; \zeta)]^{1/n}\to\rho'$ as $n\to\infty$, we have from this

$$L(z)\cdot\rho' \leqq |h(z)|. \tag{12}$$

Therefore by III,

$$|h(z)| \geqq \rho' \qquad \text{for} \quad z\in G. \tag{13}$$

Consider now the harmonic function

$$u(z)=\log|\rho h(z)/\rho'|-\log|l(z)|, \qquad z\in D. \tag{14}$$

If E is the union of a finite number of continua in Δ, then by (13), (11) and THEOREM 3, $u(z)$ has non-negative boundary values on ∂D. Consequently by

Minimum Principle $u(z)\geqq 0$ on D. Thus $|h(z)\cdot\rho/\rho'|\geqq|l(z)|$, or

$$|h(z)/\rho'|\geqq|l(z)/\rho|, \qquad z\in D.$$

In particular,

$$|h(a)/\rho'|\geqq|l(a)/\rho|.$$

Thus by (11)

$$L(a)\geqq|l(a)/\rho|.$$

Noting that a is an arbitrary point in D, it follows from this and III that we have

V. *If E is the union of a finite number of continua in Δ then $L(z)=|l(z)|/\rho$, $z\in D$.*

If E is not the union of a finite number of continua in Δ, we let $D_1\subset D_2\subset D_3\subset\cdots\subset D_m\to D$ be an exhaustion of D, where for each m, D_m is bounded by a finite number of Jordan curves with $\partial\Delta\subset\partial D_m$. Let $E_m=\partial D_m-\partial\Delta$ and $\rho^{(m)}=\rho(E_m)$. We can show easily (see for instance [3, THEOREM 16.2.2]) that

$$\lim_{m\to\infty}\rho^{(m)}=\rho\,. \tag{15}$$

Next, denote the l and L functions corresponding to E_m or D_m by $l^{(m)}$ and $L^{(m)}$ respectively. By considering the sequence of harmonic measures

$$\{\log(1/|l^{(m)}(z)|)/\log(1/\rho^{(m)})\},$$

we see easily that

$$\lim_{m\to\infty}|l^{(m)}(z)|\leqq|l(z)|, \qquad z\in D.$$

On the other hand, by [9, THEOREM 1], we have

$$\lim_{m\to\infty}|l^{(m)}(z)|\geqq|l(z)|, \qquad z\in D.$$

Thus

$$\lim_{m\to\infty}|l^{(m)}(z)|=|l(z)|, \qquad z\in D. \tag{16}$$

It follows from (15), (16) and V that

$$\lim_{m\to\infty}L^{(m)}(z)=|l(z)|/\rho\,, \qquad z\in D.$$

By definition and with an obvious justification, we have $L^{(m)}(z)\leqq L(z)$ for $z\in D_m$. Therefore

$$\lim_{m\to\infty}L^{(m)}(z)=|l(z)|/\rho\leqq L(z)\,, \qquad z\in D. \tag{17}$$

THEOREM 4 then follows from (17) and III. Q.E.D.

Return now to the harmonic function $u(z)$ in (14). Assume first that E is the union of a finite number of continua in Δ. We notice that $u(a)=0$ and $u(z)\geqq 0$ for z in D. Therefore $u(z)\equiv 0$ in D. Thus $|h(z)\cdot\rho/\rho'|=|l(z)|$ in G. Since $|h(1)|=|l(1)|=1$, $\rho=\rho'$, and consequently, by the definition of h and l,

$h(z)\equiv l(z)$ in G. For a general compact set E in Δ with $\rho=\rho(E)>0$, by (12), (11) and THEOREM 4, we again have $\rho=\rho'$ and $h(z)\equiv l(z)$ in G. We thus have shown

THEOREM 5. *Let E be a compact set in Δ with $\rho(E)>0$. Given a fixed point a in D and $n>0$, let $\zeta^{(n)}=\{\zeta_0^{(n)}, \cdots, \zeta_n^{(n)}\}\subset E$ be such that $L_n(a)=L_n^{(0)}(a\,;\zeta^{(n)})$. Let*

$$g_n(z)=\left([z, \zeta_1^{(n)}]\cdots[z, \zeta_n^{(n)}]\cdot\frac{1-\overline{\zeta_1^{(n)}}}{1-\zeta_1^{(n)}}\cdots\frac{1-\overline{\zeta_n^{(n)}}}{1-\zeta_n^{(n)}}\right)^{1/n}$$

where $g_n(z)$ is defined in G in a similar manner as $l_n(z)$ in (5). *Then*

(i) $[\delta_n^{(0)}(\zeta_0^{(n)}\,;\zeta^{(n)})]^{1/n}\to\rho(E)$ *as $n\to\infty$, and*

(ii) $g_n(z)\to l(z)$ *locally uniformly in G as $n\to\infty$.*

The same argument proves the parabolic version of this theorem.

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