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ON THE GROWTH OF ALGEBROID FUNCTIONS OF $\mu_*<\infty$

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1. Introduction. Let $f_0, \cdots, f_N$ $(N\geqq 1)$ be entire functions with no common zeros and denote by $T(r, f)$ the characteristic function of the system $f=(f_0, \cdots, f_N)$. Further, if $f_j\not\equiv 0$ $(0\leqq j\leqq N)$, we define $m_2(r, f)$ as follows:

$$m_2(r. f)=\left(\frac{N}{4\pi}\int_0^{2\pi}\sum_{l,j=0}^{N}\left\{\log\left|\frac{f_l(re^{i\theta})}{f_j(re^{i\theta})}\right|\right\}^2 d\theta\right)^{1/2}. \tag{1}$$

By Drasin and Shea [2], Pólya peaks of order ρ exist iff $\rho\in[\mu_*, \lambda_*]$ and $\rho<\infty$, where

$$\begin{aligned}\mu_*=\mu_*(T)=\inf\left\{\rho:\ \varliminf_{r,A\to\infty}\frac{T(Ar, f)}{A^\rho T(r, f)}=0\right\},\\ \lambda_*=\lambda_*(T)=\sup\left\{\rho:\ \varlimsup_{r,A\to\infty}\frac{T(Ar, f)}{A^\rho T(r, f)}=\infty\right\}.\end{aligned} \tag{2}$$

In [5], [6], Miles and Shea have shown

THEOREM A. *Suppose that f is meromorphic (i.e., $N=1$, $f=f_1/f_0=(f_0, f_1)$) with $\mu_*<\infty$. Then*

$$k_2(f)=\varlimsup_{r\to\infty}\frac{N(r, 0, f)+N(r, \infty, f)}{m_2(r, f)}\geqq\sup_{\mu_*\leqq\rho\leqq\lambda_*}C_1(\rho), \tag{3}$$

where

$$C_1(\rho)=\frac{|\sin\pi\rho|}{\pi\rho}\left\{\frac{2}{1+(\sin 2\pi\rho)/(2\pi\rho)}\right\}^{1/2}. \tag{4}$$

In this note, we shall extend Theorem A to systems of $\mu_*<\infty$. Our extension is the following:

THEOREM. *Let $f=(f_0, \cdots, f_N)$ $(f_j\not\equiv 0)$ be a system with $\mu_*<\infty$. Then*

$$k_2(f)=\varlimsup_{r\to\infty}\frac{\sum_{j=0}^{N}N(r, 0, f_j)}{m_2(r, f)}\geqq\sup_{\mu_*\leqq\rho\leqq\lambda_*}C_N(\rho), \tag{5}$$

where

$$C_N(\rho)=\frac{1}{N}\frac{|\sin\pi\rho|}{\pi\rho}\left\{\frac{2}{1+(\sin 2\pi\rho)/(2\pi\rho)}\right\}^{1/2}. \tag{6}$$

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Equality holds in (5) for $f=(1, \cdots, 1, f_N)$, where f_N is a Lindelöf function, i.e., an entire function having all zeros on a ray through 0 and $N(r, 0, f_N) \sim r^{\mu_*}$ $(r\to\infty)$.

The corresponding problem with $m_2(r, f)$ replaced by $T(r, f)$ in (5) has received much attention. Making use of the techniques developed by Edrei and Fuchs [3], Toda [8] obtained

THEOREM B. *Let $f=(f_0, \cdots, f_N)$ $(N\geqq 1)$ be a system and let λ, μ be the order and lower order of f, respectively. If $\mu<\infty$, then*

$$k_1(f)=\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N} N(r, 0, f_j)}{T(r, f)} \geqq \sup_{\mu\leqq\rho\leqq\lambda}\frac{N+1}{N}\frac{|\sin\pi\rho|}{4.4e(\rho+1)+|\sin\pi\rho|}. \tag{7}$$

Using (5), *we are able to sharpen his estimate* (7).

COROLLARY 1. *Let $f=(f_0, \cdots, f_N)$ $(N\geqq 1)$ be a system with $\mu_*<\infty$. Then*

$$k_1(f)=\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N} N(r, 0, f_j)}{T(r, f)} \geqq \sup_{\mu_*\leqq\rho\leqq\lambda_*}\frac{N+1}{N}\frac{|\sin\pi\rho|}{\pi\rho/\sqrt{2}+1/4\sqrt{2}+|\sin\pi\rho|/N}.$$

COROLLARY 2. *Let $y(z)$ be an N-valued algebroid function with $\mu_*<\infty$. Then*

$$k_1(y\,;a_0, \cdots, a_N)=\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N} N(r, a_j, y)}{T(r, y)}$$

$$\geqq \sup_{\mu_*\leqq\rho\leqq\lambda_*}\frac{N+1}{N}\frac{|\sin\pi\rho|}{\pi\rho/\sqrt{2}+1/4\sqrt{2}+|\sin\pi\rho|/N}.$$

Remark. For $\mu\leqq 1$, Ozawa [7] obtained the correct value of

$$\inf_{\text{lower ord } y=\mu} K_1(y\,;a_0, \cdots, a_N).$$

2. Lemmas

LEMMA 1. ([1]) *Let $f=(f_0, \cdots, f_N)$ $(N\geqq 1)$ be a system and let $a_0, \cdots, a_N$ be complex numbers such that $F=a_0f_0+\cdots+a_Nf_N\not\equiv 0$. Further, define $\|F\|$ and $m(r, F)$ as follows*:

$$\|F\|=\frac{|F|}{\sqrt{|f_0|^2+\cdots+|f_N|^2}\sqrt{|a_0|^2+\cdots+|a_N|^2}},\quad m(r, F)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{\|F\|}d\theta.$$

Then

$$T(r, f)=m(r, F)+N(r, 0, F)+O(1).$$

LEMMA 2. ([8]) *Let $f=(f_0, \cdots, f_N)$ $(N\geqq 1)$ be a system. Then*

$$T(r, f_j/f_i)-O(1)<T(r, f)<\sum_{k\neq j}T(r, f_k/f_j)+O(1).$$

LEMMA 3. ([1]) *Let* $A=(a_{ij})_{j=0,\cdots,N}^{i=0,\cdots,N}$ *be a regular matrix and let*

$$(F_0, \cdots, F_N)^t=A(f_0, \cdots, f_N)^t.$$

Then

$$T(r, f)-O(1)<T(r, F)<T(r, f)+O(1),$$

where $F=(F_0, \cdots, F_N)$.

LEMMA 4. ([9]) *let* $y(z)$ *be an N-valued algebroid function and let* $F(z, y)=A_0(z)y^N+\cdots+A_N(z)=0$ *be the defining equation of* y. *Further, let* A *be the system* $(A_0, \cdots A_N)$. *Then*

$$NT(r, y)=T(r, A)+O(1).$$

LEMMA 5. *Let* $a_0, \cdots, a_N$ $(N\geqq 1)$ *be positive numbers. Then*

$$N\Big(\sum_{j=0}^{N} a_j\Big)^2 \geqq \sum_{j=0}^{N}\sum_{l>j}(a_l+a_j)^2.$$

The proof is clear.

LEMMA 6. *Let* $f=(f_0, \cdots, f_N)$ *be a system* $(f_j\not\equiv 0)$. *Then*

$$m_2(r, f)\geqq(N+1)T(r, f)-\sum_{j=0}^{N} N(r, 0, f_j)+O(\log r).$$

Proof.

$$m_2(r, f)=\Big(\frac{N}{2\pi}\int_0^{2\pi}\frac{1}{2}\sum_{j=0}^{N}\sum_{l=0}^{N}\Big\{\log|f_l/f_j|\Big\}^2 d\theta\Big)^{1/2}$$

$$\geqq\Big(\frac{N}{2\pi}\int_0^{2\pi}\sum_{j=0}^{N}\Big\{\log\max_l|f_l/f_j|\Big\}^2 d\theta\Big)^{1/2}$$

$$\geqq\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=0}^{N}\log\max_l|f_l/f_j|\, d\theta$$

$$=\sum_{j=0}^{N}\{T(r, f)-N(r, 0, f_j)+O(\log r)\}$$

$$=(N+1)T(r, f)-\sum_{j=0}^{N}N(r, 0, f_j)+O(\log r).$$

LEMMA 7. ([3]) *Let* f *be meromorphic and let* $\{a_j\}$, $\{b_j\}$ *be the sequences of its zeros and poles. Further let* s, R *be positive numbers such that* $2s<R/2$. *Then*

$$\log|f(z)|=\log\Big|\prod_{s<|a_j|<R}E\Big(\frac{z}{a_j}, q\Big)\Big|-\log\Big|\prod_{s<|b_j|<R}E\Big(\frac{z}{b_j}, q\Big)\Big|+W(z)+O(\log|z|),$$

where if $2s\leqq|z|=r\leqq R/2$,

$$|W(z)|\leqq V_q(s, r, R)=\begin{cases} A\Big\{\Big(\frac{r}{s}\Big)^q T(2s, f)+\Big(\frac{r}{R}\Big)^{q+1}T(2R, f)\Big\} & (q\geqq 1)\\ A\Big\{T(2s, f)\log\Big(\frac{r}{s}\Big)+\Big(\frac{s}{R}\Big)T(2R, f)\Big\} & (q=0),\end{cases}$$

A an absolute constant >0.

LEMMA 8. (cf. [3], [4, Theorem 1.11]) *Let f be meromorphic and let s, R be positive numbers such that* $2s<R/2$. *Then if* $2s\leqq|z|=r\leqq R/2$.

$$T(r,\ f)\leqq K_q r^{q+1}\int_s^R \frac{N(t,\ 0,\ f)+N(t,\ \infty,\ f)}{t^{q+1}(t+r)}dt+BV_q(s,\ r,\ R)$$

for suitable constants $K_q(>0)$, $B(>0)$.

The following lemma, which is an extension to systems of a result due to Miles and Shea [6], plays an important role for the proof of Theorem.

LEMMA 9. *Let* $f=(f_0,\ \cdots,\ f_N)$ $(f_j\not\equiv 0)$ *be a system satisfying* $\mu_*<\lambda_*$. *If* $\mu_*<\rho<\lambda_*$, $\rho\neq 1,\ 2,\ \cdots$, *there exist positive sequences* s_n, r_n, R_n *tending to* ∞ *and* $\xi_n\to 0$ *such that*

$$s_n=o(r_n),\quad r_n=o(R_n)\qquad (n\to\infty), \tag{8}$$

$$N(t)\leqq N(r_n)\left(\frac{t}{r_n}\right)^{\rho}\quad (s_n\leqq t\leqq R_n)\quad \left(N(t)\equiv\sum_{j=0}^{N}N(r,\ 0,\ f_j)\right), \tag{9}$$

$$\begin{aligned} &T(2R_n,\ f)<\xi_n N(r_n)\left(\frac{R_n}{r_n}\right)^{\rho},\\ &T(2s_n,\ f)<\xi_n N(r_n)\left(\frac{s_n}{r_n}\right)^{\rho}. \end{aligned} \tag{10}$$

Proof. By the fact that $T(r,\ f)$ has Pólya peaks of orders $\rho\pm\varepsilon$ for small $\varepsilon>0$ and the continuity of $T(r,\ f)$, there exist sequences s_n, t_n, R_n, $A_n\to\infty$ and $\delta_n\to 0$ such that

$$t_n/s_n\longrightarrow\infty,\quad R_n/t_n\longrightarrow\infty\qquad (n\to\infty),$$

$$T(t,\ f)\leqq T(t_n,\ f)(t/t_n)^{\rho}\qquad (s_n\leqq t\leqq 2R_n), \tag{11}$$

$$T(t,\ f)<\delta_n T(t_n,\ f)(t/t_n)^{\rho}\qquad (s_n\leqq t\leqq A_n s_n,\ A_n^{-1}R_n\leqq t\leqq 2R_n). \tag{12}$$

(See [6, p 177].) Choose $r_n\in[s_n,\ 2R_n]$ such that

$$N(r_n)r_n^{-\rho}\geqq N(t)t^{-\rho}\qquad (s_n\leqq t\leqq 2R_n). \tag{13}$$

Applying Lemma 8 to f_l/f_j $(l\neq j\,;\ l,\ j=0,\ \cdots,\ N)$, we have

$$T(t_n,\ f_l/f_j)\leqq K_q t_n^{q+1}\int_{s_n}^{R_n}\frac{N(t,\ 0,\ f_l)+N(t,\ 0,\ f_j)}{t^{q+1}(t+t_n)}dt+BV_q(s_n,\ t_n,\ R_n\,;\ f_l/f_j)$$

Hence

$$\sum_{\substack{l,j=0\\(l\neq j)}}^{N}T(t_n,\ f_l/f_j)\leqq K_q t_n^{q+1}\int_{s_n}^{R_n}\frac{2N\cdot N(t)}{t^{q+1}(t+t_n)}dt+\sum_{\substack{l,j=0\\(l\neq j)}}^{N}BV_q(s_n,\ t_n,\ R_n\,;\ f_l/f_j). \tag{14}$$

Here we choose $q=[\rho]$. Then by (11) and Lemma 2, we have

(15) $$V_q(s_n, t_n, R_n; f_l/f_j)=o(T(t_n, f)).$$

Thus, (14), (15) and Lemma 2 imply

$$(N+1)T(t_n, f)\leqq 2NK_q t_n^{q+1}\int_{s_n}^{R_n}\frac{N(t)}{t^{q+1}(t+t_n)}dt+o(T(t_n, f)).$$

Further, using (13), we have

$$(N+1)T(t_n, f)\leqq 2NK_q t_n^{q+1}N(r_n)\int_{s_n}^{R_n}\Big(\frac{t}{r_n}\Big)^{\rho}\frac{dt}{t^{q+1}(t+t_n)}+o(T(t_n, f))$$

$$<2NK_qN(r_n)\Big(\frac{t_n}{r_n}\Big)^{\rho}\int_0^{\infty}\frac{du}{u^{q+1-\rho}(u+1)}+o(T(t_n, f)).$$

Since $q<\rho<q+1$, the integral in the right hand side converges. Hence

(16) $$T(t_n, f)<\Big\{\frac{2N}{N+1}+o(1)\Big\}\tilde{K}_qN(r_n)\Big(\frac{t_n}{r_n}\Big)^{\rho} \qquad (n\to\infty).$$

Now, from (12) and (16), we have

$$T(2R_n, f)<\delta_nT(t_n, f)\Big(\frac{2R_n}{t_n}\Big)^{\rho}<\delta_n\Big(\frac{2N}{N+1}+o(1)\Big)\tilde{K}_qN(r_n)\Big(\frac{2R_n}{r_n}\Big)^{\rho}.$$

Putting $\xi_n=2^{\rho}\delta_n\{2N/(N+1)+o(1)\}\tilde{K}_q$ $(\to 0)$, we obtain the first inequality of (10). In the same way, we have the second. It remains to prove (8). To do this, it suffices to show $r_n\in(A_ns_n, A_n^{-1}R_n)$. If $r_n\notin(A_ns_n, A_n^{-1}R_n)$, we have (12) with $t=r_n$. It follows from this and (16) that

(17) $$T(r_n, f)<\delta_n\Big(\frac{2N}{N+1}+o(1)\Big)\tilde{K}_qN(r_n).$$

On the other hand, we have from Lemma 1

(18) $$N(r_n)-O(1)<(N+1)T(r_n, f).$$

(17) and (18) yield $1\leqq 2\delta_nN\tilde{K}_q$ $(\to 0$ as $n\to\infty)$, a contradiction. This completes the proof of Lemma 9.

3. Proof of Theorem.

Case 1) Assume first that $\mu_*=\lambda_*$. Let λ be the order of f. In this case $\lambda=\mu_*=\lambda_*$. We may assume that $\lambda\neq 1, 2, \cdots$. Choose $q=[\lambda]$. By Lemma 2, the order of f_l/f_j $(l\neq j)$ does not exceed λ. Let $\{z_k^{(l,j)}\}$, $\{w_k^{(l,j)}\}$ be the sequences of the zeros and poles of f_l/f_j $(z_k^{(l,j)}\neq 0,\ w_k^{(l,j)}\neq 0)$. Then we can write

$$f_{l,j}(z)=\frac{f_l(z)}{f_j(z)}=z^{p_{l,j}}e^{P_{l,j}(z)}\frac{\prod E\Big(\frac{z}{z_k^{(l,j)}}, q\Big)}{\prod E\Big(\frac{z}{w_k^{(l,j)}}, q\Big)},$$

where $p_{l,j}$ is an integer and $P_{l,j}(z)=\alpha_q^{(l,j)}z^q+\cdots+\alpha_0^{(l,j)}$ is of degree $\leqq q$. Here we define $F_{l,j}(z)$ as follows:

$$F_{l,j}(z)=z^{p_{l,j}}e^{\hat{P}_{l,j}(z)}\prod E\Big(\frac{z}{|z_k^{(l,j)}|},q\Big)\prod E\Big(\frac{z}{|w_k^{(l,j)}|},q\Big),$$

where $\hat{P}_{l,j}(z)=|\alpha_q^{(l,j)}|z^q+\cdots+|\alpha_0^{(l,j)}|$. Let

$$c_m^{(l,j)}(r)=\frac{1}{2\pi}\int_0^{2\pi}(\log|f_{l,j}(re^{i\theta})|)e^{-im\theta}d\theta \qquad (m=0,\ \pm1,\ \cdots),$$

$$\gamma_m^{(l,j)}(r)=\frac{1}{2\pi}\int_0^{2\pi}(\log|F_{l,j}(re^{i\theta})|)e^{-im\theta}d\theta \qquad (m=0,\ \pm1,\ \cdots).$$

Then $|c_m^{(l,j)}(r)|\leqq|\gamma_m^{(l,j)}(r)|$ $(m=0,\ \pm1,\ \cdots)$ (See [5]), so that

(19) $$m_2(r,f)=\Big\{N\sum_{m=-\infty}^{+\infty}\sum_{j=0}^{N}\sum_{l>j}|c_m^{(l,j)}(r)|^2\Big\}^{1/2}$$
$$\leqq\Big\{N\sum_{m=-\infty}^{+\infty}\sum_{j=0}^{N}\sum_{l>j}|\gamma_m^{(l,j)}(r)|^2\Big\}^{1/2}.$$

It is clear that $c_m^{(l,j)}(r)=\overline{c_{-m}^{(l,j)}(r)}$ for $m\leqq-1$ and $c_0^{(l,j)}(r)=N(r,0,f_l/f_j)-N(r,\infty,f_l/f_j)$. By Edrei-Fuchs' computation [3],

$$|\gamma_m^{(l,j)}(r)|\begin{cases}=\dfrac{1}{2}|\alpha_m^{(l,j)}|r^m+\dfrac{1}{2m}\displaystyle\sum_{|z_k^{(l,j)}|\leqq r}\Big\{\Big(\frac{r}{|z_k^{(l,j)}|}\Big)^m-\Big(\frac{|z_k^{(l,j)}|}{r}\Big)^m\Big\}\\ \quad+\dfrac{1}{2m}\displaystyle\sum_{|w_k^{(l,j)}|\leqq r}\Big\{\Big(\frac{r}{|w_k^{(l,j)}|}\Big)^m-\Big(\frac{|w_k^{(l,j)}|}{r}\Big)^m\Big\} \qquad (1\leqq m\leqq q)\\ =\dfrac{1}{2m}\Big\{\displaystyle\sum_{|z_k^{(l,j)}|\leqq r}\Big(\frac{|z_k^{(l,j)}|}{r}\Big)^m+\sum_{|w_k^{(l,j)}|\leqq r}\Big(\frac{|w_k^{(l,j)}|}{r}\Big)^m\\ \quad+\displaystyle\sum_{|z_k^{(l,j)}|>r}\Big(\frac{r}{|z_k^{(l,j)}|}\Big)^m+\sum_{|w_k^{(l,j)}|>r}\Big(\frac{r}{|w_k^{(l,j)}|}\Big)^m\Big\} \quad (m\geqq q+1).\end{cases}$$

Now, we use Lemma 5. Let $\{z_k^{(l)}\}$ $(l=0,\cdots,N)$ be the zeros of f_l. If we put

$$a_l=\frac{1}{2m}\sum_{|z_k^{(l)}|\leqq r}\Big\{\Big(\frac{r}{|z_k^{(l)}|}\Big)^m-\Big(\frac{|z_k^{(l)}|}{r}\Big)^m\Big\}\equiv|\gamma_m^{(l)}(r)|,$$

Lemma 5 implies for $1\leqq m\leqq q$,

(20) $$N\sum_{j=0}^{N}\sum_{l>j}|\gamma_m^{(l,j)}(r)|^2\leqq N\sum_{j=0}^{N}\sum_{l>j}\{a_l+a_j+O(r^m)\}^2$$
$$\leqq N^2\Big\{\Big(\sum_{j=0}^{N}a_j\Big)+O(r^m)\Big\}^2=N^2\Big\{\sum_{j=0}^{N}|\gamma_m^{(j)}(r)|+O(r^m)\Big\}^2.$$

If we put

$$a_l=\frac{1}{2m}\Big\{\sum_{|z_k^{(l)}|\leqq r}\Big(\frac{|r_k^{(l)}|}{r}\Big)^m+\sum_{|z_k^{(l)}|>r}\Big(\frac{r}{|z_k^{(l)}|}\Big)^m\Big\}\equiv|\gamma_m^{(l)}(r)|,$$

Lemma 5 implies for $m\geqq q+1$,

$$(21)\qquad N\sum_{j=0}^{N}\sum_{l>j}|\gamma_m^{(l,j)}(r)|^2\leqq N\sum_{j=0}^{N}\sum_{l>j}(a_l+a_j)^2\leqq N^2\Big(\sum_{j=0}^{N}a_j\Big)^2=N^2\Big(\sum_{j=0}^{N}|\gamma_m^{(j)}(r)|\Big)^2.$$

Substituting (20), (21) into (19) we have

$$(22)\qquad m_2(r,\ f)\leqq N\Big(\sum_{m\neq 0}\Big\{\sum_{j=0}^{N}|\gamma_m^{(j)}(r)|\Big\}^2+N^2(r)+O(r^{2q})+O(r^q)\sum_{m=1}^{q}\sum_{j=0}^{N}|\gamma_m^{(j)}(r)|\Big)^{1/2}$$

$\Big(N(r)\equiv\sum_{j=0}^{N}N(r,\ 0,\ f_j).\Big)$ It is easy to see that for $m\geqq q+1$,

$$(23)\qquad \sum_{j=0}^{N}|\gamma_m^{(j)}(r)|=\frac{m}{2}\Big\{\int_0^r\Big(\frac{t}{r}\Big)^m\frac{N(t)}{t}dt+\int_r^\infty\Big(\frac{r}{t}\Big)^m\frac{N(t)}{t}dt\Big\}-N(r),$$

and for $1\leqq m\leqq q$,

$$(24)\qquad \sum_{j=0}^{N}|\gamma_m^{(j)}(r)|=\frac{m}{2}\int_0^r\Big\{\Big(\frac{r}{t}\Big)^m-\Big(\frac{t}{r}\Big)^m\Big\}\frac{N(t)}{t}dt+N(r).$$

Here we show that $N(r)$ has order λ. First, Lemma 1 gives

$$N(r)<(N+1)T(r,\ f)+O(1),$$

which implies that the order of $N(r)$ does not exceed λ. Next, we use the following estimate:

$$T(r,\ f_l/f_j)\leqq C_q\Big\{qr^q\int_0^r\frac{N(t,\ 0,\ f_l)+N(t,\ 0,\ f_j)}{t^{q+1}}dt$$
$$+(q+1)r^{q+1}\int_r^\infty\frac{N(t,\ 0,\ f_l)+N(t,\ 0,\ f_j)}{t^{q+2}}dt+O(r^q)+O(\log r)\Big\},$$

where $C_0=1$, $C_q=2(q+1)\{2+\log(q+1)\}$ if $q\geqq 1$. For the proof, see [4, p 102]. Hence

$$\sum_{\substack{l,j=0\\(l\neq j)}}^{N}T(r,\ f_l/f_j)\leqq 2NC_q\Big\{qr^q\int_0^r\frac{N(t)}{t^{q+1}}dt+(q+1)r^{q+1}\int_r^\infty\frac{N(t)}{t^{q+2}}dt\Big\}+O(r^q+\log r).$$

It follows from this and Lemma 2 that

$$(25)\qquad T(r,\ f)\leqq\frac{2N}{N+1}C_q\Big\{qr^q\int_0^r\frac{N(t)}{t^{q+1}}dt+(q+1)r^{q+1}\int_r^\infty\frac{N(t)}{t^{q+2}}dt\Big\}+O(r^q+\log r).$$

If $N(r)$ has order less than λ, we deduce from (25) that $T(r,\ f)$ has order less than λ, a contradiction. Thus $N(r)$ has order λ. Hence, by a growth lemma of Pólya (cf. [4, Lemma 4.7]), there exists, for small $\varepsilon>0$, a positive sequence $\{\nu_n\}$ tending to ∞ such that

$$(26)\qquad \frac{N(t)}{N(\nu_n)}\leqq\Big(\frac{t}{\nu_n}\Big)^{\lambda-\varepsilon}\quad(0<t\leqq\nu_n),\qquad \frac{N(t)}{N(\nu_n)}\leqq\Big(\frac{t}{\nu_n}\Big)^{\lambda+\varepsilon}\quad(\nu_n\leqq t<\infty),$$

$$\lim_{n\to\infty}\frac{N(\nu_n)}{\nu_n^{\lambda-\varepsilon}}=\infty\,.$$

Choose $\varepsilon>0$ such that $\lambda-\varepsilon>q$. Substituting (26) into (23), (24) with $r=\nu_n$, we obtain

$$\sum_{j=0}^{N}|\gamma_m^{(j)}(\nu_n)|\begin{cases}\leqq N(\nu_n)\left\{\dfrac{m(m-\varepsilon)}{(m-\varepsilon)^2-\lambda^2}-1\right\} & (m\geqq q+1)\\[2mm] \leqq N(\nu_n)\left\{\dfrac{m^2}{(\lambda-\varepsilon)^2-m^2}+1\right\} & (1\leqq m\leqq q)\,.\end{cases}$$

Hence by (22) and (26) we have

$$\overline{\lim_{n\to\infty}}\left\{\frac{m_2(\nu_n,f)}{N(\nu_n)}\right\}^2\leqq N^2\left\{1+2\sum_{m=1}^{\infty}\frac{\lambda^4}{(\lambda^2-m^2)^2}\right\}=N^2\left(\frac{\pi\lambda}{\sin\pi\lambda}\right)^2\left\{\frac{1}{2}+\frac{\sin 2\pi\lambda}{4\pi\lambda}\right\},$$

which implies

$$\overline{\lim_{r\to\infty}}\frac{N(r)}{m_2(r,f)}\geqq\frac{1}{N}\frac{|\sin\pi\lambda|}{\pi\lambda}\left\{\frac{2}{1+(\sin 2\pi\lambda)/(2\pi\lambda)}\right\}^{1/2}.$$

Case 2) Assume next that $\mu_*<\lambda_*$. Let $\rho\in(\mu_*,\lambda_*)$ be nonintegral and choose $a=a(\rho)\in(0,1)$ such that

(27) $$a^{\rho}<\rho\log 2/2^{\rho}\,,\qquad a^{-\rho}-1+(2^{\rho}/\log 2)\log a>0\,.$$

Let $q=[\rho]$,

$$f_n^{(l,j)}(z)=\frac{\prod_{s_n<|z_k^{(l,j)}|<aR_n}E\left(\dfrac{z}{z_k^{(l,j)}},q\right)}{\prod_{s_n<|w_k^{(l,j)}|<aR_n}E\left(\dfrac{z}{w_k^{(l,j)}},q\right)}\qquad(n=1,2,\cdots),$$

where $\{z_k^{(l,j)}\}$, $\{w_k^{(l,j)}\}$ are the sequences of the zeros and poles of f_l/f_j, and s_n, R_n are the same as in Lemma 9. Here we introduce an entire function associated with $f_n^{(l,j)}(z)$:

$$\hat{f}_n^{(l,j)}(z)=\prod_{s_n<|z_k^{(l,j)}|<aR_n}E\left(\frac{z}{|z_k^{(l,j)}|},q\right)\prod_{s_n<|z_k^{(l,j)}|\leqq aR_n}E\left(\frac{z}{|w_k^{(l,j)}|},q\right).$$

Further let $\{z_k^{(l)}\}$ be the zeros of $f_l(z)$ and let

$$\hat{f}_n^{(l)}(z)=\prod_{s_n<|z_k^{(l)}|<aR_n}E\left(\frac{z}{|z_k^{(l)}|},q\right).$$

Now, define $N_n(t)$ by

$$N_n(t)=\sum_{j=0}^{N}N(t,0,\hat{f}_n^{(j)})\,.$$

It follows from (8) and (9) that

(28) $$N_n(r_n)=(1-o(1))N(r_n)\qquad(n\to\infty)\,.$$

If we set $F(u)=u^{\rho}-1-(2^{\rho}/\log 2)\log u$, (27) implies that $F(u)>0$ for $u>1/a$. Hence, putting $t=(aR_n)u$ $(>R_n)$, we have

$$\left(\frac{R_n}{r_n}\right)^{\rho} a^{\rho}+\frac{1}{\log 2}(2a)^{\rho}\left(\frac{R_n}{r_n}\right)^{\rho} \log\left(\frac{t}{aR_n}\right)\leqq\left(\frac{t}{r_n}\right)^{\rho} \qquad (t>R_n).$$

On the other hand, from (9) it follows that

$$N(aR_n)\leqq N(r_n)\left(\frac{R_n}{r_n}\right)^{\rho} a^{\rho},$$

$$n(aR_n)\leqq\frac{1}{\log 2}N(2aR_n)\leqq\frac{1}{\log 2}N(r_n)(2a)^{\rho}\left(\frac{R_n}{r_n}\right)^{\rho}.$$

Combining above results, we obtain

$$N(aR_n)+n(aR_n)\log(t/aR_n)\leqq N(r_n)\left(\frac{t}{r_n}\right)^{\rho} \qquad (t>R_n).$$

From this and (9) it follows that

$$N_n(t)\leqq N(r_n)\left(\frac{t}{r_n}\right)^{\rho} \qquad (0<t<\infty). \tag{29}$$

Applying Lemma 7 to f_l/f_j $(l\neq j)$ with $|z|=r_n$, we have

$$\log\left|\frac{f_l(z)}{f_j(z)}\right|=\log|f_n^{(l,j)}(z)|+W_n^{(l,j)}(z),$$

$$|W_n^{(l,j)}(z)|\leqq V_q(s_n, r_n, aR_n, f_l/f_j)=o(N(r_n)),$$

where we used Lemma 2, (8) and (10). Hence an easy computation gives

$$m_2(r_n, f_l/f_j)\leqq m_2(r_n, f_n^{(l,j)})+o(N(r_n)).$$

However, since we may assume that $k_2(f)<\infty$, $N(r_n)=O(m_2(r_n, f))$. Thus

$$m_2(r_n, f_l/f_j)\leqq m_2(r_n, f_n^{(l,j)})+o(m_2(r_n, f)) \qquad (n\to\infty).$$

Therefore

$$\{m_2(r_n, f)\}^2=N\sum_{j=0}^{N}\sum_{l>j}\{m_2(r_n, f_l/f_j)\}^2\leqq N\sum_{j=0}^{N}\sum_{l>j}\{m_2(r_n, f_n^{(l,j)})+o(m_2(r_n, f))\}^2,$$

which implies

$$\{m_2(r_n, f)\}^2\leqq(1+o(1))N\sum_{j=0}^{N}\sum_{l>j}\{m_2(r_n, f_n^{(l,j)})\}^2.$$

Further it is known that $m_2(r_n, f_n^{(l,j)})\leqq m_2(r_n, \hat{f}_n^{(l,j)})$. So, we have

$$\{m_2(r_n, f)\}^2\leqq(1+o(1))N\sum_{j=0}^{N}\sum_{l>j}\{m_2(r_n, \hat{f}_n^{(l,j)})\}^2$$

$$=(1+o(1))N\sum_{m=-\infty}^{+\infty}\sum_{j=0}^{N}\sum_{l>j}|\gamma_m(r_n, \hat{f}_n^{(l,j)})|^2 \tag{30}$$

$$\leqq (1+o(1))N^2\Big\{\sum_{m\neq 0}\Big(\sum_{j=0}^{N}|\gamma_m(r_n,\hat{f}_n^{(j)})|\Big)^2+(N_n(r_n))^2\Big\},$$

where we used Lemma 5. By (8), (28) and (29) we have for $m\geqq q+1$,

$$\sum_{j=0}^{N}|\gamma_m(r_n,\hat{f}_n^{(j)})|=\frac{m}{2}\Big\{\int_{s_n}^{r_n}\Big(\frac{t}{r_n}\Big)^m N_n(t)\frac{dt}{t}+\int_{r_n}^{\infty}\Big(\frac{r_n}{t}\Big)^m N_n(t)\frac{dt}{t}\Big\}-N_n(r_n)$$

$$\leqq\frac{m}{2}\Big\{N(r_n)\int_{s_n}^{r_n}\Big(\frac{t}{r_n}\Big)^m\Big(\frac{t}{r_n}\Big)^\rho\frac{dt}{t}+N(r_n)\int_{r_n}^{\infty}\Big(\frac{r_n}{t}\Big)^m\Big(\frac{t}{r_n}\Big)^\rho\frac{dt}{t}\Big\} \tag{31}$$

$$-(1-o(1))N(r_n)$$

$$=\Big\{\frac{m^2}{m^2-\rho^2}-1+o(1)\Big\}N(r_n)\qquad(n\to\infty),$$

uniformly in m. On the other hand, for $1\leqq m\leqq q$, we have

$$\sum_{j=0}^{N}|\gamma_m(r_n,\hat{f}_n^{(j)})|=\frac{m}{2}\int_{s_n}^{r_n}\Big\{\Big(\frac{r_n}{t}\Big)^m-\Big(\frac{t}{r_n}\Big)^m\Big\}N_n(t)\frac{dt}{t}+N_n(r_n)$$

$$\leqq\frac{m}{2}N(r_n)\int_{s_n}^{r_n}\Big\{\Big(\frac{r_n}{t}\Big)^m-\Big(\frac{t}{r_n}\Big)^m\Big\}\Big(\frac{t}{r_n}\Big)^\rho\frac{dt}{t}+(1-o(1))N(r_n) \tag{32}$$

$$=\Big\{\frac{m^2}{\rho^2-m^2}+1+o(1)\Big\}N(r_n)\qquad(n\to\infty).$$

Substituting (31) and (32) into (30), we have

$$\Big\{\frac{m_2(r_n,f)}{N(r_n)}\Big\}^2\leqq(1+o(1))N^2\Big\{1+2\rho^4\sum_{m=1}^{\infty}\frac{1}{(\rho^2-m^2)^2}\Big\}.$$

Thus

$$\overline{\lim_{r\to\infty}}\frac{N(r)}{m_2(r,f)}\geqq C_N(\rho).$$

This completes the proof of Theorem.

Proof of Corollary 1. By Lemma 6 and Theorem,

$$\frac{k_1(f)}{N+1-k_1(f)}=\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N}N(r,0,f_j)}{(N+1)T(r,f)-\sum_{j=0}^{N}N(r,0,f_j)+O(\log r)}$$

$$\geqq\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N}N(r,0,f_j)}{m_2(r,f)}\geqq C_N(\rho)>\frac{1}{N}\frac{|\sin\pi\rho|}{\pi\rho}\frac{\sqrt{2}}{1+1/4\pi\rho}.$$

Hence

$$k_1(f)\geqq\frac{N+1}{N}\frac{|\sin\pi\rho|}{\pi\rho/\sqrt{2}+1/4\sqrt{2}+|\sin\pi\rho|/N}.$$

Proof of Corollary 2. Let $F(z, y)=A_0 y^N+\cdots+A_N=0$ be the defining equation of $y(z)$. Let $A=(A_0, \cdots, A_N)$ and $F=(F(z, a_0), \cdots, F(z, a_N))$. Then by Lemmas 3 and 4,

$$K_1(y\,;a_0, \cdots, a_N)=\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N} N(r, a_j, y)}{T(r, y)}=\overline{\lim_{r\to\infty}}\frac{\sum_{j=0}^{N} N(r, 0, F(z, a_j))}{T(r, F)}.$$

Corollary 2 follows from this and Corollary 1.

REFERENCES

[1] CARTAN, H., Sur les zéros des combinaisons linéaires de p functions holomorphes données, Mathematica **7** (1933), 5-33.

[2] DRASIN, D. AND SHEA, D.F., Pólya peaks and the oscillation of positive functions, Proc. Amer. Math. Soc. **34** no 2 (1972), 403-411.

[3] EDREI, A. AND FUCHS, W.H.J., On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc. **93** (1959), 292-328.

[4] HAYMAN, W.K., Meromorphic functions, Oxford (1964).

[5] MILES, J. AND SHEA, D.F., An extremal problem in value distribution theory, Quart. J. Math. Oxford (2), **24** (1973), 377-383.

[6] MILES, J. AND SHEA, D. F., On the growth of meromorphic functions having at least one deficient value, Duke Math. J. Vol **43** No 1. (1976), 171-186.

[7] OZAWA, M., On the growth of algebroid functions with several deficient values, II, Kōdai Math. Sem. Rep. **22** (1970), 129-137.

[8] TODA, N., Sur la croissance de fonctions algebroides à valeurs deficientes, Kōdai Math. Sem. Rep. **22** (1970), 324-337.

[9] VALIRON, G., Sur la derivée des fonctions algébroides, Bull. Soc. Math. **59** (1931), 17-39.

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