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TWO RESULTS ASSOCIATED WITH SPREAD RELATION

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0. Introduction.

Let $u=u_1-u_2$ be nonconstant, where u_1 and u_2 are subharmonic in the plane. For such a function u, we will write

$$N(r, u)=\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta .$$

Then the Nevanlinna characteristic of u is defined by

$$T(r)\equiv T(r, u)=N(r, u^+)+N(r, u_2).$$

With this $T(r, u)$, the order and lower order of u are defined as follows:

$$\rho=\overline{\lim_{r\to\infty}}\frac{\log T(r, u)}{\log r} \qquad \text{(the order of } u\text{)},$$

$$\mu=\underline{\lim_{r\to\infty}}\frac{\log T(r, u)}{\log r} \qquad \text{(the lower order of } u\text{)}.$$

Further, we use the notation

$$\delta(\infty)\equiv\delta(\infty, u)=1-\overline{\lim_{r\to\infty}}\frac{N(r, u_2)}{T(r, u)} .$$

(Throughout this paper, $|E|$ denotes the one-dimensional Lebesgue measure of the set E.)

Assume that $\mu<\infty$. Then, Baernstein [3] proved the following inequality.

Spread Relation: For any fixed $b\in(-\infty, \infty)$,

$$\overline{\lim_{r\to\infty}} |\{\theta : u(re^{i\theta})>b\}| \geqq \min\left\{\frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\infty)}{2}\right)^{1/2}, 2\pi\right\} .$$

In connection with this relation, we show the following result.

THEOREM 1. *Let $u=u_1-u_2$ be nonconstant, where u_1 and u_2 are subharmonic in the plane. Suppose $\mu<\infty$ and $\delta(\infty)>0$. Let λ be a number satisfying*

$$\lambda>\mu , \quad \frac{4}{\lambda}\sin^{-1}\left(\frac{\delta(\infty)}{2}\right)^{1/2}\leqq 2\pi .$$

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Then, for any fixed $b \in (-\infty, \infty)$,

$$\overline{\log \text{ dens}}\left\{r : |\{\theta : u(re^{i\theta}) > b\}| \geqq \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta(\infty)}{2}\right)^{1/2}\right\} \geqq 1 - \frac{\mu}{\lambda}.$$

In order to show Theorem 1, it is sufficient to prove the following

THEOREM 2. *Let* $u = u_1 - u_2$ *be nonconstant, where* u_1 *and* u_2 *are subharmonic in the plane. Suppose* $\mu < \infty$. *Let* δ *and* λ *be numbers satisfying*

$$\lambda > \mu, \quad 0 < \delta \leqq 1, \quad \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta}{2}\right)^{1/2} \leqq 2\pi.$$

Further, suppose that there exists a $r_0 > 0$ *such that* $r \geqq r_0$ *implies*

$$N(r, u_2) \leqq (1-\delta) T(r, u) + O(1).$$

Then, for any fixed $b \in (-\infty, \infty)$,

$$\overline{\log \text{ dens}}\left\{r : |\{\theta : u(re^{i\theta}) > b\}| \geqq \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}\right\} \geqq 1 - \frac{\mu}{\lambda}.$$

If we assume that $\rho < \infty$, we have the following

THEOREM 3. *Let* $u = u_1 - u_2$ *be nonconstant, where* u_1 *and* u_2 *are subharmonic in the plane. Suppose* $\rho < \infty$ *and* $\delta(\infty) > 0$. *Let* λ *be a number satisfying*

$$\lambda > \rho, \quad \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta(\infty)}{2}\right)^{1/2} \leqq 2\pi.$$

Then, for any fixed $b \in (-\infty, \infty)$,

$$\underline{\log \text{ dens}}\left\{r : |\{\theta : u(re^{i\theta}) > b\}| \geqq \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta(\infty)}{2}\right)^{1/2}\right\} \geqq 1 - \frac{\rho}{\lambda}.$$

Again, we prove the following Theorem 4 in place of Theorem 3.

THEOREM 4. *Let* $u = u_1 - u_2$ *be nonconstant, where* u_1 *and* u_2 *are subharmonic in the plane. Suppose* $\rho < \infty$. *Let* δ *and* λ *be numbers satisfying*

$$\lambda > \rho, \quad 0 < \delta \leqq 1, \quad \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta}{2}\right)^{1/2} \leqq 2\pi.$$

Further, suppose that there exists a $r_0 > 0$ *such that* $r \geqq r_0$ *implies*

$$N(r, u_2) \leqq (1-\delta) T(r, u) + O(1).$$

Then, for any fixed $b \in (-\infty, \infty)$,

$$\underline{\log \text{ dens}}\left\{r : |\{\theta : u(re^{i\theta}) > b\}| \geqq \frac{4}{\lambda} \sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}\right\} \geqq 1 - \frac{\rho}{\lambda}.$$

Theorems 1 and 3 may be regarded as analogues of Barry's results [5], [4] on the $\cos \pi\rho$ theorem. The proof of Theorems 2 and 4 is accomplished by combining Barry's method with Baernstein's "star function" method.

For $u=u_1-u_2$, we define

$$u^*(re^{i\theta})=\sup_E \frac{1}{2\pi}\int_E u(re^{i\omega})d\omega \qquad (r>0,\ 0\leqq\theta\leqq\pi),$$

where the sup is taken over all sets $E\subset[-\pi, \pi]$ with $|E|=2\theta$, and define

$$u^{\#}(re^{i\theta})=u^*(re^{i\theta})+N(r, u_2).$$

In [2], Baernstein proved that $u^{\#}(z)$ is subharmonic in the upper half plane.

Note that we may prove Theorems 2 and 4 under the following additional conditions.

(i) u_1 and u_2 are harmonic in a neighborhood of 0.
(ii) $b=0$.
(iii) $u_1(z)\geqq u_2(z)$ for all z, $u_1(0)=u_2(0)=0$.

For the details, see [3, p. 89].

1. Preliminaries for the proof of Theorem 2.

1.1. A function $h(z)$. Set β and γ as follows:

$$\beta=\frac{2}{\lambda}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}, \quad \gamma=\frac{\beta}{\pi}.$$

Then

$$\alpha\equiv\gamma\lambda=\frac{2}{\pi}\sin^{-1}\left(\frac{\delta}{2}\right)^{1/2}\leqq 1/2. \tag{1.1}$$

Fix $R>0$ and define

$$B(t)=\begin{cases} T(t^\gamma) & (0\leqq t\leqq R), \\ T_1(R^\gamma-)\log\dfrac{t}{R}+T(R^\gamma) & (R\leqq t<\infty), \end{cases}$$

where $T_1(t^\gamma)$ denotes the logarithmic derivative of the function $t\to T(t^\gamma)$. Then $B(t)$ is a convex nondecreasing function of $\log t$, and the Poisson integral

$$h(z)=\frac{1}{\pi}\int_0^\infty \frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}B(t)dt \qquad (z=re^{i\theta})$$

is harmonic in the slit plane $|\arg z|<\pi$, is zero on the positive axis and tends to $B(r)$ as $\theta\to\pi-$. Further,

$$h_\theta(re^{i\theta})=\frac{1}{\pi}\int_0^\infty \log\left|1+\frac{r}{t}e^{i\theta}\right|dB_1(t) \qquad (|\theta|<\pi) \tag{1.2}$$

and

$$(1.3)\qquad \lim_{\theta\to\pi-}\frac{B(r)-h(re^{i\theta})}{\pi-\theta}=\lim_{\theta\to\pi-}h_\theta(re^{i\theta})=\frac{1}{\pi}\int_0^\infty \log\left|1-\frac{r}{t}\right|dB_1(t)$$

hold, where $B_1(t)$ is the logarithmic derivative of the logarithmically convex nondecreasing function $B(t)$, which were established in § 3 of [1].

Since $u(z)$ is nonconstant (cf. [3, p. 83, b)]), there exists a number $r_2>0$ such that $B_1(r_2)>0$, provided that $R(>2r_2)$ is large enough.

By (1.2),

$$\begin{aligned}\pi h_\theta\Big(\frac{R}{2}\Big)&=\int_0^R \log\Big(1+\frac{R}{2t}\Big)dB_1(t)\\ &=B_1(R-)\log\frac{3}{2}+B(R)\frac{1}{3}+\int_0^R\frac{R/2}{(t+R/2)^2}B(t)dt\\ &\leqq B_1(R-)\frac{3}{2}+B(R)\leqq\int_R^{Re}\frac{B_1(t)}{t}dt+B(R)\\ &\leqq B(Re)+B(R)\leqq 2B(Re)=2\{T_1(R^\gamma-)+T(R^\gamma)\}\\ &\leqq 2\Big\{\int_R^{Re}\frac{T_1(t^\gamma)}{t}dt+T(R^\gamma)\Big\}\leqq 2\{T(R^\gamma e^\gamma)+T(R^\gamma)\}\leqq 4T(R^\gamma e^\gamma),\quad \text{i.e.}\end{aligned}$$

$$(1.4)\qquad h_\theta\Big(\frac{R}{2}\Big)\leqq\frac{4}{\pi}T(R^\gamma e^\gamma).$$

Also,

$$(1.5)\qquad \begin{aligned}\pi h_\theta(r_2)&=\int_0^R\log\Big(1+\frac{r_2}{t}\Big)dB_1(t)\geqq\int_0^{r_2}\log\Big(1+\frac{r_2}{t}\Big)dB_1(t)\\ &=B_1(r_2)\log 2+\int_0^{r_2}B_1(t)\frac{r_2}{r_2+t}\frac{dt}{t}\geqq B_1(r_2)\log 2>0.\end{aligned}$$

1.2. A function $h_1(z)$. Let D be the disk $\{z:|z|<R\}$ and let $h_1(z)$ be the bounded harmonic function in D defined by

$$\begin{aligned}h_1(re^{i\theta})=&\frac{1}{2\pi}\int_0^\pi T(R^\gamma)\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}dt\\ &+\frac{1}{2\pi}\int_\pi^{2\pi}(-T(R^\gamma))\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-t)}dt.\end{aligned}$$

Then it is easy to see that

$$(1.6)\qquad h_1(re^{i\theta})\begin{cases}>0 & (0<\theta<\pi),\\ =0 & (\theta=0,\ \pi),\\ <0 & (\pi<\theta<2\pi).\end{cases}$$

By Proposition 6 of [1],

$$(1.7)\qquad |(h_1)_\theta(z)|\leqq A\frac{|z|}{R}T(R^\gamma)\qquad (|z|\leqq R/2),$$

where $A(>0)$ is an absolute constant.

From (1.6) we deduce that

$$(h_1)_\theta(-t)\leqq 0 \qquad (0<t<R). \tag{1.8}$$

1.3. A function $F(t)$. By (1.1), $\alpha\leqq 1/2$. Then, for $0<R_1<R_2$,

$$\begin{aligned}&\int_{R_1}^{R_2}\left\{\log\left|1-\frac{t}{s}\right|-\cos\pi\alpha\cdot\log\left(1+\frac{t}{s}\right)\right\}\frac{dt}{t^{1+\alpha}}\\ &\qquad >C(\alpha)\frac{\log(1+R_1/s)}{R_1{}^\alpha}-K(\alpha)\frac{\log(1+R_2/s)}{R_2{}^\alpha},\end{aligned}\tag{1.9}$$

where $C(\alpha)=(1-\cos\pi\alpha)/\alpha$ and $K(\alpha)$ is some number greater than $C(\alpha)$ (cf. [7, (21), (22)]). We take the integral $dB_1(s)$ of each term in (1.9). Using (1.2) and (1.3), we have

$$\int_{R_1}^{R_2}\{h_\theta(-t)-\cos\pi\alpha h_\theta(t)\}\frac{dt}{t^{1+\alpha}}>C(\alpha)\frac{h_\theta(R_1)}{R_1{}^\alpha}-K(\alpha)\frac{h_\theta(R_2)}{R_2{}^\alpha}.$$

so that

$$\int_{R_1}^{R_2}\left\{h_\theta(-t)-h_\theta(t)+C(\alpha)t\frac{\partial}{\partial t}h_\theta(t)\right\}\frac{dt}{t^{1+\alpha}}+(K(\alpha)-C(\alpha))\frac{h_\theta(R_2)}{R_2{}^\alpha}>0. \tag{1.10}$$

Here we put

$$F(t)=h_\theta(-t)+(h_1)_\theta(-t)-h_\theta(t)+C(\alpha)t\frac{\partial}{\partial t}h_\theta(t).$$

Then it follows from (1.4), (1.7) and (1.10) that, for $0<r\leqq R/2$,

$$\begin{aligned}\int_r^{R/2}F(t)\frac{dt}{t^{1+\alpha}}&>-(K(\alpha)-C(\alpha))\frac{(4/\pi)T(R^\gamma e^\gamma)}{(R/2)^\alpha}-A\frac{T(R^\gamma)}{R}\int_r^{R/2}\frac{dt}{t^\alpha}\\ &>-\left\{(K(\alpha)-C(\alpha))\frac{2^{2+\alpha}}{\pi}+\frac{A}{(1-\alpha)2^{1-\alpha}}\right\}\frac{T(R^\gamma e^\gamma)}{R^\alpha}\\ &\equiv -K_1(\alpha)\frac{T(R^\gamma e^\gamma)}{R^\alpha}.\end{aligned}\tag{1.11}$$

1.4. A function $H(z)$. Consider the harmonic function $H(z)$ in $D^+=\{z:z\in D,\ 0<\arg z<\pi\}$ defined by

$$H(re^{i\theta})=h(re^{i\theta})+\cos\pi\alpha h(re^{i(\pi-\theta)})+h_1(re^{i\theta})+A_1(\pi-\theta),$$

where A_1 is a large positive constant independent of R. The boundary values of H satisfy

$$\begin{aligned}&H(-r)=T(r^\gamma) \qquad (0\leqq r<R),\\ &H(r)=\cos\pi\alpha T(r^\gamma)+A_1\pi \qquad (0<r<R),\\ &H(Re^{i\theta})\geqq T(R^\gamma) \qquad (0<\theta<\pi).\end{aligned}\tag{1.12}$$

Now, set

$$v(z)=u^{\#}(z^{\gamma}).$$

Then by (1.12) we have

$$v(-r)=u^{\#}(r^{\gamma}e^{i\beta})\leqq T(r^{\gamma})=H(-r) \qquad (0\leqq r<R),$$

$$v(r)=u^{\#}(r^{\gamma})=N(r^{\gamma},\ u_2)\leqq(1-\delta)T(r^{\gamma})+C=\cos\pi\alpha T(r^{\gamma})+C$$

$$(r\geqq r_1\equiv r_0^{1/\gamma},\ C:\ \text{a positive constant}),$$

$$v(Re^{i\theta})=u^{\#}(R^{\gamma}e^{i\gamma\theta})\leqq T(R^{\gamma})<H(Re^{i\theta}) \qquad (0<\theta<\pi).$$

Hence, if we choose $A_1\pi=\max\{C,\ N(r_0,\ u_2)\}$, then

$$v(z)\leqq H(z) \qquad (z\in D^+). \tag{1.13}$$

2. Proof of Theorem 2.

We define, for $0<r\leqq R/2$,

$$H_1(r)=K_1(\alpha)\frac{T(R^{\gamma}e^{\gamma})}{R^{\alpha}}r^{\alpha}+r^{\alpha}\int_0^{R/2}F(t)\frac{dt}{t^{1+\alpha}}. \tag{2.1}$$

By (1.11), $H_1(r)>0$ for $0<r\leqq R/2$. Set

$$G_1(r)=H_1(r)+C(\alpha)h_{\theta}(r). \tag{2.2}$$

Then (2.1) implies

$$rG_1'(r)=\alpha H_1(r)+h_{\theta}(r)-h_{\theta}(-r)-(h_1)_{\theta}(-r) \tag{2.3}$$

a.e. in $0<r\leqq R/2$. By (1.2), $h_{\theta}(r)-h_{\theta}(-r)\geqq 0$ for $0<r\leqq R/2$, and by (1.8), $-(h_1)_{\theta}(-r)\geqq 0$ for $0<r\leqq R/2$. Hence, (2.3) shows that

$$rG_1'(r)>0 \qquad \text{a.e. in } 0<r\leqq R/2. \tag{2.4}$$

Since $G_1(r)$ is absolutely continuous in r and its derivative is positive a.e., we know that $G_1(r)$ increases with r, and hence $\log G_1(r)$ is absolutely continuous. By (1.5) and (2.2),

$$G_1(r_2)>C(\alpha)h_{\theta}(r_2)\geqq C(\alpha)B_1(r_2)\frac{\log 2}{\pi}>0. \tag{2.5}$$

Thus, we deduce from (2.1), (2.2), (1.4) and (2.5) that

$$\begin{aligned}\int_{r_2}^{R/2}\frac{G_1'(t)}{G_1(t)}dt&=\log G_1\left(\frac{R}{2}\right)-\log G_1(r_2)\\ &<\log\left\{K_1(\alpha)\frac{T(R^{\gamma}e^{\gamma})}{2^{\alpha}}+C(\alpha)\frac{4}{\pi}T(R^{\gamma}e^{\gamma})\right\}-\log\left\{C(\alpha)B_1(r_2)\frac{\log 2}{\pi}\right\}\\ &\equiv C_1+\log T(R^{\gamma}e^{\gamma}),\end{aligned} \tag{2.6}$$

where C_1 is a constant depending only on α, r_2 and $u(z)$.

Let

$$I=\{r : |\{\theta : u(re^{i\theta})>0\}|<2\beta\}$$

and

$$J=\{r : r^{\gamma}\in I\} .$$

Suppose that $r\in I$ and put

$$E(r)=\{\theta : u(re^{i\theta})>0\} .$$

Then, since $u(z)\geqq 0$ everywhere and $|E(r)|<2\beta$, we have

$$u^{\#}(re^{i\beta})\geqq N(r, u^{+})+N(r, u_2)=T(r) ,$$

so that

$$u^{\#}(re^{i\beta})=T(r) \qquad (r\in I) .$$

Hence

$$v(-r)=u^{\#}(r^{\gamma}e^{i\beta})=T(r^{\gamma})=H(-r) \qquad (r\in J\cap(0, R)) . \tag{2.7}$$

It follows from (1.13) and (2.7) that, for $r\in J\cap(0, R)$,

$$H_{\theta}(-r)=\lim_{\theta\to\pi-}\frac{H(-r)-H(re^{i\theta})}{\pi-\theta}\leqq v_{\theta}(-r)=\gamma u_{\theta}^{\#}(r^{\gamma}e^{i\beta}) . \tag{2.8}$$

(Existence of the limit follows from (1.3).) Let $\tilde{u}(re^{i\theta})$ denote the symmetric decreasing rearrangement of $u(re^{i\theta})$ (cf. [2, § 3]). Then

$$u_{\theta}^{\#}(r^{\gamma}e^{i\beta})=\tilde{u}(r^{\gamma}e^{i(\beta-)})\leqq 0 \qquad (r\in J\cap(0, R)) . \tag{2.9}$$

Thus, by (2.8) and (2.9), $H_{\theta}(-r)\leqq 0$ $(r\in J\cap(0, R))$, i. e.

$$h_{\theta}(-r)+(h_1)_{\theta}(-r)\leqq\cos\pi\alpha h_{\theta}(r)+A_1 \qquad (r\in J\cap(0, R)) . \tag{2.10}$$

Combining (2.2), (2.3) and (2.10), we have

$$A_1+rG_1'(r)\geqq\alpha G_1(r) \qquad (r\in J\cap(0, R)) . \tag{2.11}$$

Here we note that

$$\begin{aligned} G_1(r)&>C(\alpha)h_{\theta}(r)>\frac{C(\alpha)}{\pi}\int_0^{\pi}h_{\theta}(re^{i\theta})d\theta=\frac{C(\alpha)}{\pi}h(-r) \\ &=\frac{C(\alpha)}{\pi}B(r)=\frac{C(\alpha)}{\pi}T(r^{\gamma}) \qquad (r\in(0, R)) , \end{aligned} \tag{2.12}$$

by (2.2) and (1.2). Therefore by (2.11) and (2.12)

$$rG_1'(r)>(\alpha-\varepsilon)G_1(r) \qquad (r\in J\cap(r_3, R/2),\ r_3=r_3(\varepsilon)>0) . \tag{2.13}$$

Since $G_1'(r)>0$ a. e. in $0<r\leqq R/2$, we deduce from (2.6) and (2.13) that

$$\int_{J\cap(r_4, R/2)}\frac{dt}{t}<\frac{1}{\alpha-\varepsilon}(C_1+\log T(R^{\gamma}e^{\gamma})) , \tag{2.14}$$

where $r_4=\max(r_2, r_3)$.

Now, we choose $R=R_n\uparrow\infty$ such that

$$C_1+\log T(R^\gamma e^\gamma)<(\mu+\varepsilon)\log\left(\frac{R}{2}\right)^\gamma \qquad (\varepsilon>0)$$

Then

$$\frac{1}{\log(R/2)}\int_{J\cap(r_4,(R/2))}\frac{dt}{t}<\frac{\gamma(\mu+\varepsilon)}{\alpha-\varepsilon},$$

so that

$$\frac{1}{\log(R/2)^\gamma}\int_{I\cap(r_4^\gamma,(R/2)^\gamma)}\frac{dt}{t}<\frac{\gamma(\mu+\varepsilon)}{\alpha-\varepsilon}.$$

Hence

$$\underline{\log\ \mathrm{dens}}\, I\leqq\frac{\gamma(\mu+\varepsilon)}{\alpha-\varepsilon},$$

and so, since we may choose ε arbitrarily small.

$$\underline{\log\ \mathrm{dens}}\, I\leqq\frac{\gamma\mu}{\alpha}=\frac{\mu}{\lambda}.$$

This completes the proof of Theorem 2.

3. Proof of Theorem 4.

Define β, γ and α as in § 1. Put $B(t)=T(t^\gamma)$ $(0<t<\infty)$. Since $B(t)=O((t^\gamma)^{\rho+\varepsilon})$ $\leqq O(t^\alpha)$ $(t\to\infty)$, the Poisson integral

$$h(z)=\frac{1}{\pi}\int_0^\infty\frac{r\sin\theta}{t^2+r^2+2tr\cos\theta}B(t)dt \qquad (z=re^{i\theta})$$

is harmonic in the slit plane $|\arg z|<\pi$, is zero on the positive axis and tends to $B(r)$ as $\theta\to\pi-$. Also, (1.2) and (1.3) hold.

Nonconstancy of $u(z)$ implies the existence of a number $r_2>0$ such that $B_1(r_2)>0$. Hence

$$\begin{aligned}\pi h_\theta(r_2)&=\int_0^\infty\log\left(1+\frac{r_2}{t}\right)dB_1(t)\geqq\int_0^{r_2}\log\left(1+\frac{r_2}{t}\right)dB_1(t)\\ &=B_1(r_2)\log 2+\int_0^{r_2}B_1(t)\frac{r_2}{r_2+t}\frac{dt}{t}\geqq B_1(r_2)\log 2>0.\end{aligned}\tag{3.1}$$

By an inequality of Denjoy [6],

$$\int_r^\infty\left\{\log\left|1-\frac{t}{s}\right|-\cos\pi\alpha\log\left(1+\frac{t}{s}\right)\right\}\frac{dt}{t^{1+\alpha}}>C(\alpha)\frac{\log(1+r/s)}{r^\alpha}\tag{3.2}$$

for all $r>0$, where $C(\alpha)=(1-\cos\pi\alpha)/\alpha$. We take the integral $dB_1(s)$ of each term in (3.2). Then, using the Fubini's theorem, we have

$$\int_r^\infty \{h_\theta(-t)-\cos \pi\alpha h_\theta(t)\} \frac{dt}{t^{1+\alpha}} > C(\alpha)\frac{h_\theta(r)}{r^\alpha}\,.$$

On integration by parts this becomes

$$\int_r^\infty F(t)\frac{dt}{t^{1+\alpha}} > 0\,, \tag{3.3}$$

where $F(t)=h_\theta(-t)-h_\theta(t)+C(\alpha)th'_\theta(t)$. We define

$$G_1(r)=r^\alpha\int_r^\infty F(t)\frac{dt}{t^{1+\alpha}}+C(\alpha)h_\theta(r) \qquad (r>0)\,.$$

Then

$$rG'_1(r)=\alpha r^\alpha\int_r^\infty F(t)\frac{dt}{t^{1+\alpha}}+h_\theta(r)-h_\theta(-r) \qquad \text{a.e.}\,.$$

Since $G_1(r)$ is absolutely continuous, this implies that $G_1(r)$ is monotonic increasing, and so $\log G_1(r)$ is absolutely continuous. Therefore

$$\int_{r_2}^r \frac{G'_1(t)}{G_1(t)}dt=\log G_1(r)-\log G_1(r_2)\,. \tag{3.4}$$

Now, by (11) and (12) in [1],

$$\begin{aligned}\pi h_\theta(s)&=\int_0^\infty \log\left(1+\frac{s}{t}\right)dB_1(t)\\ &=\int_0^\infty \frac{s}{t+s}\,dB(t)\\ &=\int_0^\infty \frac{s}{(t+s)^2}B(t)dt=O(s^{\gamma\rho+\varepsilon}) \qquad (s\longrightarrow\infty)\end{aligned}$$

for every $\varepsilon>0$, and therefore

$$\begin{aligned}G_1(r)&=r^\alpha\int_r^\infty \{h_\theta(-t)-\cos \pi\alpha h_\theta(t)\}\frac{dt}{t^{1+\alpha}}\\ &\leqq r^\alpha(1-\cos\pi\alpha)\int_r^\infty h_\theta(t)t^{-1-\alpha}dt=O(r^{\gamma\rho+\varepsilon}) \qquad (r\longrightarrow\infty)\,.\end{aligned} \tag{3.5}$$

Combining (3.1), (3.4) and (3.5), we have

$$\int_{r_2}^r \frac{G'_1(t)}{G_1(t)}dt<(\gamma\rho+o(1))\log r \qquad (r\longrightarrow\infty)\,. \tag{3.6}$$

Next, consider the harmonic function $H(z)$ in the upper half plane defined by

$$H(re^{i\theta})=h(re^{i\theta})+\cos\pi\alpha h(re^{i(\pi-\theta)})+A_1(\pi-\theta)\,,$$

where A_1 is a large positive constant determined later. The boundary values of H satisfy

$$\begin{cases} H(-r)=T(r^{\gamma}) & (0\leqq r<\infty), \\ H(r)=\cos\pi\alpha T(r^{\gamma})+A_1\pi & (0<r<\infty). \end{cases}$$

Set

$$v(z)=u^{\#}(z^{\gamma}).$$

Then

$$\begin{cases} v(-r)=u^{\#}(r^{\gamma}e^{i\beta})\leqq T(r^{\gamma})=H(-r) & (0\leqq r<\infty), \\ v(r)=u^{\#}(r^{\gamma})=N(r^{\gamma},\ u_2)\leqq(1-\delta)T(r^{\gamma})+C & \\ \qquad (r\geqq r_1=r_0^{1/\gamma},\ C\text{: a positive constant}). \end{cases}$$

Hence, if we choose $A_1\pi=\max\{C,\ N(r_0,\ u_2)\}$, then

$$v(z)\leqq H(z)$$

holds on the real axis. Since $v(z)$ and $H(z)$ are both $O(r^{\alpha})$ in the upper half plane as $r\to\infty$, and since $\alpha\leqq 1/2$, we conclude that

$$v(z)\leqq H(z)$$

in the upper half plane.

Now, let I and J be as in § 2. Then the same reasoning as in § 2 gives

$$H_\theta(-r)\leqq 0 \qquad (r\in J).$$

In view of the definition of H, this can be written

$$h_\theta(-r)-\cos\pi\alpha h_\theta(r)\leqq A_1 \qquad (r\in J).$$

Hence

$$A_1+rG_1'(r)\geqq\alpha G_1(r) \qquad (r\in J).$$

Since

$$G_1(r)>C(\alpha)h_\theta(r)=\frac{C(\alpha)}{\pi}\int_0^\infty\frac{r}{(t+r)^2}B(t)dt\geqq\frac{C(\alpha)B(r)}{\pi}\int_r^\infty\frac{r}{(t+r)^2}dt=\frac{C(\alpha)B(r)}{2\pi},$$

we have

$$rG_1'(r)\geqq(\alpha-\varepsilon)G_1(r) \qquad (r\in J\cap(r_3,\ \infty),\ r_3=r_3(\varepsilon)>0). \tag{3.7}$$

Combining (3.6) with (3.7), we have

$$\int_{J\cap(r_4,r)}\frac{dt}{t}<\frac{\gamma\rho+o(1)}{\alpha-\varepsilon}\log r \qquad (r\longrightarrow\infty),$$

where $r_4=\max(r_2,\ r_3)$, and arguing as from (2.14), we deduce that

$$\overline{\log\text{ dens}}\, I\leqq\frac{\gamma\rho}{\alpha}=\frac{\rho}{\lambda}.$$

This completes the proof of Theorem 4.

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