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ON THE GROUP $\varepsilon(K(\pi, 1)\times X)$ FOR 1-CONNECTED CW-COMPLEXES X

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§ 0. Introduction

Let $\varepsilon(Y)$ be the group of based homotopy classes of based self-homotopy equivalences of Y. This group has been studied by many authors since Arcowitz and Curjel's paper [1] was published in 1964. Most of results which have been obtained are seemed to depend on the assumption "Y is 1-connected", and there are only a few known results in the case of non-simply connected spaces ([2], [3], [4], [5] and [7]). In this paper, as a special case of the latter, we study the case of $Y=K(\pi, 1)\times X$ for a 1-connected CW-complex X. Our method is very simple, in fact, it is essential only to consider $\varepsilon(Y)$ as the group consisting of invertible elements of the semi-group $[Y, Y]_0$ having the multiplication given by the composition of maps. Our main result is the following

THEOREM. *There is a split extension*:

$$\{1\}\longrightarrow \mathrm{Inv}[K(\pi, 1), k_0; X^X, 1_X]\longrightarrow \varepsilon(K(\pi, 1)\times X)\longrightarrow \mathrm{Aut}(\pi)\times\varepsilon(X)\longrightarrow\{1\}$$

where X^X denotes the space of self-maps $X\to X$ with the compact-open topology.

If the group of integers is taken as π we have

COROLLARY A. *There is a split extension*:

$$\{1\}\longrightarrow \pi_1(X^X, 1_X)\longrightarrow \varepsilon(S^1\times X)\longrightarrow Z_2\times\varepsilon(X)\longrightarrow\{1\}.$$

And moreover if we take the complex projective space CP^n as X, the following corollary follows from Proposition 1.2 of [6].

COROLLARY B. *There is a split extension*:

$$\{1\}\longrightarrow Z_{n+1}\longrightarrow \varepsilon(S^1\times CP^n)\longrightarrow Z_2\times Z_2\longrightarrow\{1\}.$$

Specially, we have a split extension:

$$\{1\}\longrightarrow Z_2\longrightarrow \varepsilon(S^1\times S^2)\longrightarrow Z_2\times Z_2\longrightarrow\{1\}.$$

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Remark. Corollary B is different from Theorem 8.8 of [5] which should be corrected. In fact, it was noted by Prof. Y. Nomura that Cup Product Theorem of W. Rutter should be applied to the proof of Theorem 8.8.

COROLLARY C. *If X is the suspension of a* 1*-connected CW-complex up to homotopy, then there is a split extension*:

$$\{1\} \longrightarrow [SX, X]_0 \longrightarrow \varepsilon(S^1\times X) \longrightarrow \varepsilon(X)\times Z_2 \longrightarrow \{1\}.$$

Specially if $X=S^n$ $(n\geqq 3)$ we have a split extension (Theorem 7.9 *of* [5]):

$$\{1\} \longrightarrow Z_2 \longrightarrow \varepsilon(S^1\times S^n) \longrightarrow Z_2\oplus Z_2 \longrightarrow \{1\}.$$

Throughout this paper we use following notations:

$(Y^Y)_0$=the space of continuous maps: $(Y, y_0)\to(Y, y_0)$ with the compact-open topology.

$\pi_0(Y)$=the set of path-connected components of the space Y.

Inv G=the group consisting of invertible elements of a semi-group G.

pr_X: the projection $X\times Y\to X$.

$K=K(\pi, 1)$.

§ 1. Lemmas

Let $P: (K\times X^{K\times X})_0\to(K^{K\times X})_0\times(X^{K\times X})_0$ be the map defined by

$$P(f)=(pr_K\circ f,\ pr_X\circ f).$$

Then the following lemma easily follows from definitions.

LEMMA 1.1. *P is an isomorphism (homeomorphism and homomorphism of semi-groups), where the multiplication of the space $(K^{K\times X})_0\times(X^{K\times X})_0$* with the *unit (pr_K, pr_X) is given by*

$$(f_1, g_1)\times(f_2, g_2)=(f_1\circ(f_2, g_2),\ g_1\circ(f_2, g_2)).$$

Consider a map $r: (K^{K\times X})_0\to(K^K)_0$ defined by $r(f)=f\circ i_K$, where i_K denotes the inclusion: $K\to K\times x_0\subset K\times X$.

LEMMA 1.2. *The induced correspondence $r_*: \pi_0((K^{K\times X})_0)\to\pi_0((K^K)_0)$ is bijective for any* 1*-connected CW-complex X.*

Proof. For a map $g: K\to K$ $(\in(K^K)_0)$, define a map f $(\in K^{K\times X})_0)$ by $f=g\circ pr_K$, then clearly r_* is surjective since $r(f)=g$. Next, suppose $r(f_1)=r(f_2)$. Since we may consider that this means $f_1|K\times x_0=f_2|K\times x_0$ the injectivity follows from the 1-connectedness of X and $\pi_i(K)=0$ $(i>1)$.

Moreover we make the set $\pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0)$ into a semi-group with a twisted multiplication which is defined by

$$(\alpha, \beta)\times(\alpha', \beta')=(\alpha\circ\alpha', \beta(\alpha'\circ pr_K, \beta')), \qquad \mathrm{unit}=(1_K, pr_X).$$

Now, consider the correspondence

$$\psi: \pi_0((K^{K\times X})_0\times(X^{K\times X})_0) \longrightarrow \pi_0((K^{K\times X})_0)\times\pi_0((X^{K\times X})_0)$$

$$\xrightarrow[r_*\times id]{} \pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0),$$

where the first arrow denotes the natural decomposition.

LEMMA 1.3. *ψ is an isomorphism if X is a 1-connected CW-complex.*

Proof. Clearly ψ is a (1—1)-correspondence by Lemma 1.2. For any element (f, g) $(\in\pi_0((K^{K\times X})_0\times(X^{K\times X})_0))$ we can regard f as a map $f'\circ pr_K$ for a map f' $(\in(K^K)_0)$ by Lemma 1.2. Then we have

$$\begin{aligned}\psi((f_1, g_1)\times(f_2, g_2)) &= \psi((f_1'\circ pr_K, g_1)(f_2'\circ pr_X, g_2))\\ &= \psi(f_1'\circ pr_K(f_2'\circ pr_K), g_2), g_1(f_2'\circ pr_K, g_2))\\ &= \psi(f_1'f_2'\circ pr_K, g_1(f_2'\circ pr_K, g_2))\\ &= (f_1'\circ f_2', g_1(f_2'\circ pr_K, g_2))=(f_1', g_1)\times(f_2', g_2)\\ &= \psi((f_1, g_1))\times\psi((f_2, g_2)).\end{aligned}$$

We denote by G the sub-group $\mathrm{Inv}\,(\pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0))$ of the semi-group $\pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0)$ with the twisted multiplication. Since the projection: $\pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0)\to\pi_0((K^K)_0)$ is a homomorphism, it induces a homomorphism

$$\varepsilon(K\times X)=G \longrightarrow \mathrm{Inv}\ \pi_0((K^K)_0)=\varepsilon(K).$$

Clearly this is surjective, i.e. we have an exact sequence:

1.4 $$\{1\}\longrightarrow\Delta\longrightarrow G\longrightarrow\varepsilon(K)\longrightarrow\{1\},$$

where Δ denotes the kernel of the above homomorphism.

LEMMA 1.5. *Δ is isomorphic to* $\mathrm{Inv}\ \pi_0((X^{K\times X})_0)$.

Proof. Consider the restriction on Δ of the projection

$$pr|\Delta: \pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0) \longrightarrow \pi_0((X^{K\times X})_0).$$

Since Δ is contained in the semi-subgroup $\{(1_K, \pi_0((X^{K\times X})_0)\}$ $(\subset\pi_0((K^K)_0)\times\pi_0((X^{K\times X})_0))$, $pr|\Delta$ is a homomorphism. Hence $pr|\Delta$ induces a homomorphism $\Delta\to\mathrm{Inv}\ \pi_0((X^{K\times X})_0)$. Then it is easy to show that this is an isomorphism.

Let $p:(X^{K\times X})_0\to(X^X)_0$ be the fibring defined by $p(f)=f\mid k_0\times X$, and consider the part of the homotopy exact sequence of the fibering:

1.6
$$\pi_1((X^{K\times X})_0,\ pr_X)\longrightarrow\pi_1((X^K)_0,\ 1_X)\longrightarrow\pi_0(p^{-1}(1_X))$$
$$\longrightarrow\pi_0((X^{K\times X})_0)\longrightarrow\pi_0((X^{\times})_0)\,.$$

Then, from definitions, we have

LEMMA 1.7. *$p^{-1}(1_X)$ is a semi-group ($\subset(X^{K\times X})_0$) and all arrows are homomorphisms in the sequence* 1.6.

Now, let $\alpha:S^1\times X\to X$ be a map such that $\alpha(s_0,\ x)=x$, $\alpha(s,\ x_0)=x_0$ and define a map $\hat{\alpha}:S^1\times K\times X\to X$ by $\hat{\alpha}(s,\ k,\ x)=\alpha(s,\ x)$. Since $\hat{\alpha}(s_0,\ k,\ x)=\alpha(s_0,\ x)$ $=x=pr_X(k,\ x)$ and $\hat{\alpha}(s,\ k_0,\ x_0)=x_0$, $\hat{\alpha}$ defines an element of $\pi_1((X^{K\times X})_0),\ pr_X)$. Then it follows from $\hat{\alpha}(s,\ k_0,\ x)=\alpha(s,\ x)$ that the homomorphism

$$\pi_1((X^{K\times X}),\ pr_X)\longrightarrow\pi_1((X^X),\ 1_X)$$

is surjective. Thus, by using Lemma 1.7, the sequence 1.6 is transformed into the exact sequence:

1.8
$$\{1\}\longrightarrow\mathrm{Inv}\ \pi_0(P^{-1}(1_X))\longrightarrow\mathrm{Inv}\ \pi_0((X^{K\times X})_0)\longrightarrow\mathrm{Inv}\ \pi_0((X^X)_0)\longrightarrow\{1\}\,.$$

§2. Proof of Theorem and Corollaries

First we note that $\pi_0(p^{-1}(1_X))$ is isomorphic to $[K,\ k_0;\ X^X,\ 1_X]$ because we have equalities

$$\pi_0(p^{-1}(1_X))=\{f:K\times X\longrightarrow X,\ f(k_0,\ x)=x\}/\text{homotopic relative to } k_0\times X$$
$$=[K,\ k_0;\ X^X,\ 1_X].$$

Next, by combining the sequence 1.4 and 1.8, we obtain from Lemma 1.5 the following diagram:

$$\begin{tikzcd}
 & \{1\} & & & \\
 & \varepsilon(X) \arrow[u] \arrow[dr, shift left, "c_2"] & & & \\
\{1\} \arrow[r] & \Delta \arrow[u] \arrow[r] & \varepsilon(K\times X) \arrow[ul, shift left, "c_3"] \arrow[r, shift right] & \varepsilon(K) \arrow[l, shift right, "c_1"'] \arrow[r] & \{1\} \\
 & \mathrm{Inv}\,[K,\ k_0;\ X^X,\ 1_X] \arrow[u] & & & \\
 & \{1\} \arrow[u] & & &
\end{tikzcd}$$

where homomorphisms c_i $(i=1, 2, 3)$ are defined as follows:

$$c_1(f)(k, x)=(f(k), x) \qquad (f\in\varepsilon(K)),$$

$$c_2(g)(k, x)=(k, g(x)) \qquad (g\in\varepsilon(X)),$$

$$c_3(h)(k, x)=pr_X\circ h(k_0, x) \qquad (h\in\varepsilon(K\times X)).$$

Note that any map: $k_0\times X\to K\times X$ is homotopic to a map: $k_0\times X\to k_0\times X\subset K\times X$ by the assumption "X is 1-connected", and this fact asserts that c_3 is a homomorphism. Thus Theorem follows from commutativities in the above diagram:

(1) $(\Delta\longrightarrow\varepsilon(K\times X)\xrightarrow[c_3]{}\varepsilon(X))=(\Delta\longrightarrow\varepsilon(X))$,

(2) $c_3\circ c_2=\text{identity}$,

(3) $(\varepsilon(K)\xrightarrow[c_1]{}\varepsilon(K\times X)\longrightarrow\varepsilon(K))=\text{identity}$.

In the case of $K=K(Z, 1)=S^1$ we have

$$\operatorname{Inv}[K, k_0; X^X, 1_X]=\pi_1(X^X, 1_X).$$

Hence Corollary A follows from Theorem. At last, let X be the suspension of a 1-connected CW-complex Y. By considering the fibering

$$(X^X)_0\longrightarrow X^X\xrightarrow[p]{}X, \qquad p(f)=f(x_0) \qquad (f\in X^X),$$

we obtain an exact sequence

$$\pi_2(X, x_0)\longrightarrow\pi_1((X^X)_0, 1_X)\longrightarrow\pi_1(X^X, 1_X)\longrightarrow\pi_1(X, x_0),$$

i.e. $\pi_1((X^X)_0, 1_X)\cong\pi_1(X^X, 1_X)$. On the other hand, it follows from $X=SY$ that $\pi_1((X^X)_0, 1_X)$ is isomorphic to $\pi_1((X^X)_0, *)$, where $*$ denotes the constant map $X\to x_0$. Then Corollary C is obtained from $\pi_1((X^X)_0, *)\cong[SX, X]_0$.

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