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KODAI MATH. J.
4 (1981), 399—417

QUATERNION CR-SUBMANIFOLDS OF QUATERNION MANIFOLDS

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1. Introduction.

A quaternion manifold (or quaternion Kaehlerian manifold [10]) is defined as a Riemannian manifold whose holonomy group is a subgroup of $Sp(1)$. $Sp(m)=Sp(1)\times Sp(m)/\{\pm 1\}$. The quaternion projective space QP^m, its noncompact dual and the quaternion number space Q^m are three important examples of quaternion manifolds. It is well-known that on a quaternion manifold M, there exists a 3-dimensional vector bundle E of tensors of type (1, 1) with local cross-section of almost Hermitian structures satisfying certain conditions (see § 2 for details). A submanifold N in a quaternion manifold M is called a *quaternion* (respectively, *totally real*) submanifold if each tangent space of N is carried into itself (respectively, the normal space) by each section in E. It is known that every quaternion submanifold in any quaternion manifold is always totally geodesic. So it is more interesting to study a more general class of submanifolds than quaternion submanifolds. The main purpose of this paper is to establish the general theory of quaternion CR-submanifolds in a quaternion manifold which generalizes the theory of quaternion submanifolds and the theory of totally real submanifolds. It is proved in section 3 that such submanifolds are characterized by a simple equation in terms of the curvature tensor of a quaternion-space-form.

In section 4 we shall study the integrability of the two natural distributions on a quaternion CR-submanifold.

In section 5 we obtain some basic lemmas for quaternion CR-submanifolds. In particular, we shall obtain two fundamental lemmas which play important rôle in this theory. Several applications of the fundamental lemmas are given in section 6.

In section 7 we study quaternion CR-submanifolds which are foliated by totally geodesic, totally real submanifolds.

In the last section we give an example of a quaternion CR-submanifold of an almost quaternion metric manifold on which the totally real distribution is not integrable.

Received February 25, 1980

2. Quaternion Manifolds.

Let M be a $4m$-dimensional quaternion manifold with metric tensor $\langle\,,\,\rangle$. Then there exists a 3-dimensional vector bundle E of tensors of type (1, 1) with local basis of almost Hermitian structures I, J, K such that

(a) $IJ=-JI=K$, $JK=-KJ=I$, $KI=-IK=J$

(b) for any local cross-section ϕ of E and any vector X tangent to M, $\tilde{\nabla}_X\phi$ is also a local cross-section of E, where $\tilde{\nabla}$ denotes the covariant differentiation on M.

Condition (b) is equivalent to the following condition (b′) there exist local 1-forms p, q and r such that

$$\left\{\begin{aligned} &\tilde{\nabla}_X I=r(X)J-q(X)K,\\ &\tilde{\nabla}_X J=-r(X)I+p(X)K,\\ &\tilde{\nabla}_X K=q(X)I-p(X)J. \end{aligned}\right. \tag{2.1}$$

Let X be a unit vector tangent to the quaternion manifold M. Then X, IX, JX and KX form an orthonormal frame. We denote by $Q(X)$ the 4-plane spanned by them. We call $Q(X)$ the quaternion section determined by X. For any two vectors X, Y tangent to M, the plane $X\wedge Y$ spanned by X, Y is said to be *totally real* if $Q(X)$ and $Q(Y)$ are orthogonal. Any plane in a quaternion section is called a *quaternion plane*. The sectional curvature of a quaternion plane is called a *quaternion sectional curvature*. A quaternion manifold is called a *quaternion-space-form* if its quaternion sectional curvatures are equal to a constant. We shall denote $M(c)$ (or $M^m(c)$) a (real) $4m$-dimensional quaternion-space-form with quaternion sectional curvature c.

It is well-known that a quaternion manifold M is a quaternion-space-form with constant quaternion sectional curvature c if and only if the curvature tensor $\tilde{R}$ of M is of the following form [10]

$$\tilde{R}(X, Y)Z=\frac{c}{4}\Big\{\langle Y, Z\rangle X-\langle X, Z\rangle Y+\sum_{r=1}^{3}[\langle\phi_r Y, Z\rangle\phi_r X -\langle\phi_r X, Z\rangle\phi_r Y+2\langle X, \phi_r Y\rangle\phi_r Z]\Big\}, \tag{2.2}$$

where $\phi_1=I$, $\phi_2=J$ and $\phi_3=K$.

Let $\tilde{K}(X, \phi_r X)$ denotes the quaternion sectional curvature of the quaternion plane $X\wedge(\phi_r X)$. The quaternion-mean-curvature $m(X)$ associated with a unit vector X is defined by

$$m(X)=\frac{1}{3}\{\tilde{K}(X, \phi_1 X)+\tilde{K}(X, \phi_2 X)+\tilde{K}(X, \phi_3 X)\}. \tag{2.3}$$

3. Quaternion CR-submanifolds.

Let N be a Riemannian manifold isometrically immersed in a quaternion manifold M. A distribution $\mathcal{D}: x \to \mathcal{D}_x \subseteqq T_x N$ is called a *quaternion distribution* if we have $\phi_r(\mathcal{D}) \subseteqq \mathcal{D}$, $r=1, 2, 3$. In other words, $\mathcal{D}$ is a quaternion distribution if $\mathcal{D}$ is carried into itself by its quaternion structure.

DEFINITION 3.1. A submanifold N in a quaternion manifold M is called a *quaternion CR-submanifold* if it admits a differentiable quaternion distribution $\mathcal{D}$ such that its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is totally real, i.e., $\phi_r(\mathcal{D}_x^{\perp}) \subseteqq T_x^{\perp} N$, $r=1, 2, 3$, for any $x \in N$, where $T_x^{\perp} N$ denotes the normal space of N in M at x.

A submanifold N in a quaternion manifold M is called a *quaternion submanifold* (respectively, a *totally real submanifold*) if $\dim \mathcal{D}_x^{\perp}=0$ (respectively, $\dim \mathcal{D}_x=0$). A quaternion CR-submanifold is said to be *proper* if it is neither totally real nor quaternionic.

The following result gives a characterization of quaternion CR-submanifolds in a quaternion-space-form.

PROPOSITION 3.2. *Let N be a submanifold of a quaternion-space-form $M(c)$, $c \neq 0$, and $\mathcal{D}_x \equiv T_x N \cap I(T_x N) \cap J(T_x N) \cap K(T_x N)$, $x \in N$. Then N is a quaternion CR-submanifold of M if and only if either N is totally real or $\mathcal{D}$ defines a differentiable distribution of positive dimension such that*

$$\tilde{R}(\mathcal{D}, \mathcal{D}; \mathcal{D}^{\perp}, \mathcal{D}^{\perp})=0,$$

where $\mathcal{D}^{\perp}$ is the orthogonal complementary distribution of $\mathcal{D}$.

This proposition can be proved in a similar way as the proof of Theorem 6.1 of [3].

For a submanifold N in a quaternion manifold M we denote by $\langle\, , \rangle$ the metric tensor of M as well as that induced on N. Let ∇ be the induced covariant differentiation on N. The Gauss and Weingarten formulas for N are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{3.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{3.2}$$

for any vector fields X, Y tangent to N and any vector field ξ normal to N, where σ, A_ξ and D are the second fundamental form, the second fundamental tensor associated with ξ and the normal connection, respectively. Moreover, we have

$$\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle, \tag{3.3}$$

For the second fundamental form σ, we define the covariant differentiation

$\bar{\nabla}$ with respect to the connection in $TN \oplus T^{\perp}N$ by

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{3.4}$$

for X, Y, Z tangent to N. The Gauss, Codazzi and Ricci equations of N are then given by

$$R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle, \tag{3.5}$$

$$(R(X, Y)Z)^{\perp} = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z), \tag{3.6}$$

$$\tilde{R}(X, Y; \xi, \eta) = R^{\perp}(X, Y; \xi, \eta) - \langle [A_{\xi}, A_{\eta}]X, Y \rangle \tag{3.6}$$

for X, Y, Z, W tangent to N and ξ, η normal to N, where R and $R^{\perp}$ are the curvature tensors associated with ∇ and D respectively, $R(X, Y; Z, W) = \langle R(X, Y)Z, W \rangle, \cdots$, etc, and $\perp$ in (3.6) denotes the normal component.

The mean curvature vector H of N in M is defined by

$$H = \frac{1}{n} \operatorname{trace} \sigma, \tag{3.8}$$

where n denotes the dimension of N. If we have

$$\sigma(X, Y) = \langle X, Y \rangle H \tag{3.9}$$

for any X, Y tangent to N, N is called a *totally umbilical* submanifold. In particular, if $\sigma = 0$ identically, N is called a *totally geodesic* submanifold.

We mention the following known result for later use.

LEMMA 3.3. ([4], [8]). *Every quaternion submanifold of a quaternion manifold is totally geodesic.*

From this lemma, it is more interesting to study more general submanifolds, for example, quaternion CR-submanifolds in a quaternion manifold than quaternion submanifolds.

LEMMA 3.4. *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then for any vector fields U, V tangent to M, X in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$ we have*

$$\tilde{R}(U, V; \phi_r X, \phi_r Z) = \tilde{R}(U, V; X, Z). \tag{3.10}$$

Proof. From (2.1) we may prove that

$$\tilde{R}(U, V)IX = 2\{dr + p \wedge q\}(U, V)JX - 2\{dq - p \wedge r\}(U, V)KX + I(\tilde{R}(U, V)X),$$

Since $\langle JX, IZ \rangle = \langle KX, IZ \rangle = 0$, this implies (3.10) for $r=1$. A similar argument gives (3.10) for $r=2$ and 3.

DEFINITION 3.5. Let N be a quaternion CR-submanifold in a quaternion manifold M. Then N is called a *QR-product* if locally N is the Riemannian product of a quaternion submanifold and a totally real submanifold of M.

4. Integrability.

In this section we discuss the integrability of the totally real distribution $\mathcal{D}^{\perp}$ and the quaternion distribution $\mathcal{D}$.

By using (3.1), (3.2) and (3.3) we have the following

LEMMA 4.1. *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then we have $A_{\phi Y}Z=A_{\phi Z}Y$ for any $Y, Z\in\mathcal{D}^{\perp}$ and any section ϕ in E.*

By using this lemma we can obtain the following integrability Theorem for quaternion CR-submanifolds similar to the integrability Theorem of Chen [5], [6].

THEOREM 4.2. (Integrability of $\mathcal{D}^{\perp}$). *The totally real distribution $\mathcal{D}^{\perp}$ of a quaternion CR-submanifold N in a quaternion manifold M is always integrable.*

Similarly, by using Lemma 3.3, (3.1) and (3.2) we also have the following

THEOREM 4.3. (Integrability of $\mathcal{D}$). *The quaternion distribution $\mathcal{D}$ of a quaternion CR-submanifold N in a quaternion manifold M is integrable if and only if $\sigma(\mathcal{D}, \mathcal{D})=0$.*

5. Fundamental Lemmas.

In the following, we denote by ν the subbundle of the normal bundle $T^{\perp}N$ which is the orthogonal complement of $I\mathcal{D}^{\perp}\oplus J\mathcal{D}^{\perp}\oplus K\mathcal{D}^{\perp}$, i. e.

$$T^{\perp}N=I\mathcal{D}^{\perp}\oplus J\mathcal{D}^{\perp}\oplus K\mathcal{D}^{\perp}\oplus\nu\,, \qquad \langle\nu, \phi_r\mathcal{D}^{\perp}\rangle=0\,. \tag{5.1}$$

We give the following lemmas for later use

LEMMA 5.1. *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then we have*

$$\langle\sigma(\mathcal{D}, \mathcal{D}), \nu\rangle=0\,, \tag{5.2}$$

$$\langle\sigma(\phi_r X, Z), \xi\rangle=\langle D_X(\phi_r Z), \xi\rangle=\langle\phi_r\sigma(X, Z), \xi\rangle\,, \tag{5.3}$$

$$\langle D_{\phi_r X}(\phi_s Z), \xi\rangle=\langle D_X(\phi_r\phi_s Z), \xi\rangle\,, \quad r\neq s\,, \quad r, s=1, 2, 3, \tag{5.4}$$

for any vector fields X in $\mathcal{D}$, Z in $\mathcal{D}^{\perp}$ and ξ in ν.

Proof. From (2.1) and (3.1) we have, for any vector fields X, Y in $\mathcal{D}$, Z in $\mathcal{D}^{\perp}$ and ξ in ν

$$\langle\sigma(X, Y), \phi_r\xi\rangle=\langle\tilde{\nabla}_X Y, \phi_r\xi\rangle=-\langle\tilde{\nabla}_X(\phi_r Y), \xi\rangle=-\langle\sigma(X, \phi_r Y), \xi\rangle .$$

Hence we have

$$\langle\sigma(\phi_s X, \phi_r Y), \xi\rangle=\langle\sigma(X, Y), \phi_s\phi_r\xi\rangle=\langle\sigma(X, Y), \phi_r\phi_s\xi\rangle , \qquad r\neq s .$$

Since $\phi_s\phi_r=-\phi_r\phi_s$, this implies (5.2).

Moreover from (2.1) and (3.1) we also have

$$\begin{aligned}\langle\sigma(\phi_r X, Z), \xi\rangle&=\langle\tilde{\nabla}_Z(\phi_r X), \xi\rangle=\langle\phi_r\tilde{\nabla}_Z X, \xi\rangle\\&=-\langle\tilde{\nabla}_Z X, \phi_r\xi\rangle=-\langle\tilde{\nabla}_X Z, \phi_r\xi\rangle\\&=\langle\tilde{\nabla}_X(\phi_r Z), \xi\rangle=\langle D_X(\phi_r Z), \xi\rangle .\end{aligned}$$

Moreover we have

$$\langle\sigma(\phi_r X, Z), \xi\rangle=-\langle\sigma(X, Z), \phi_r\xi\rangle=\langle\phi_r\sigma(X, Z), \xi\rangle .$$

These prove (5.3). Equation (5.4) follows from (5.3).

For any vectors fields X, Y in $\mathcal{D}$, we put

$$\nabla_X Y=\mathring{\nabla}_X Y+\mathring{\sigma}(X, Y) , \tag{5.5}$$

where $\mathring{\nabla}_X Y$ and $\mathring{\sigma}(X, Y)$ are the $\mathcal{D}$- and $\mathcal{D}^{\perp}$-components of $\nabla_X Y$ respectively.

For any vector Z in $\mathcal{D}^{\perp}$, (2.1) and (5.5) give

$$\begin{aligned}\langle\mathring{\sigma}(X, \phi_r Y), Z\rangle&=\langle\tilde{\nabla}_X(\phi_r Y), Z\rangle=\langle\phi_r(\tilde{\nabla}_X Y), Z\rangle\\&=-\langle\sigma(X, Y), \phi_r Z\rangle=\langle\phi_r\sigma(X, Y), Z\rangle \qquad r=1, 2, 3,\end{aligned} \tag{5.6}$$

for any vector fields X, Y in $\mathcal{D}$. Consequently we have

$$\begin{aligned}-\langle\mathring{\sigma}(X, X), Z\rangle&=\langle\mathring{\sigma}(X, \phi_r^2 X), Z\rangle=\langle\phi_r\sigma(X, \phi_r X), Z\rangle\\&=\langle\phi_r\sigma(\phi_r X, X), Z\rangle=\langle\mathring{\sigma}(\phi_r X, \phi_r X), Z\rangle .\end{aligned}$$

Hence we obtain $\mathring{\sigma}(X, X)=-\mathring{\sigma}(\phi_r X, \phi_r X)$ for $r=1, 2, 3$. Therefore

$$\mathring{\sigma}(X, X)=-\mathring{\sigma}(KX, KX)=-\mathring{\sigma}(IJX, IJX)=\mathring{\sigma}(JX, JX) .$$

Since we already have $\mathring{\sigma}(X, X)=-\mathring{\sigma}(JX, JX)$, this implies the following

LEMMA 5.2. *Let N be a quaternion CR-submanifold in a quaternion manifold M. Then for any vector field X in $\mathcal{D}$, we have $\mathring{\sigma}(X, X)=0$,* i.e., $\nabla_X X\in\mathcal{D}$.

Remark. $\mathring{\sigma}$ is not symmetric in general. In fact, $\mathring{\sigma}(X, Y)$ is symmetric in X and Y if and only if the distribution $\mathcal{D}$ is integrable.

In the following we shall denote by σ' the second fundamental form of a maximal integral submanifold $N^{\perp}$ of $\mathcal{D}^{\perp}$ in N. For any vector fields X in $\mathcal{D}$, Z, W in $\mathcal{D}^{\perp}$, we have

$$(5.7) \qquad \begin{aligned}\langle\sigma(X, Z), \phi_r W\rangle &= -\langle\tilde{\nabla}_Z(\phi_r W), X\rangle = \langle\tilde{\nabla}_Z W, \phi_r X\rangle \\ &= \langle\nabla_Z W, \phi_r X\rangle = \langle\sigma'(Z, W), \phi_r X\rangle .\end{aligned}$$

This implies the following

LEMMA 5.3. *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then the leaf $N^{\perp}$ of $\mathcal{D}^{\perp}$ is totally geodesic in N if and only if*

$$\langle\sigma(\mathcal{D}, \mathcal{D}^{\perp}), \phi_r\mathcal{D}^{\perp}\rangle = 0\,, \qquad r=1, 2, 3.$$

From (5.7) we may also obtain the following

LEMMA 5.4. *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then for any vector fields X in $\mathcal{D}$ and Z, W in $\mathcal{D}^{\perp}$, we have*

$$(5.8) \qquad \langle\sigma(\phi_r X, Z), \phi_s W\rangle + \langle\sigma(\phi_s X, Z), \phi_r W\rangle = 0\,,$$

for $r\neq s$, r, $s=1, 2, 3$.

LEMMA 5.5. *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then for any vector fields X in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$ we have*

$$\langle(A_{\phi_s Z} - A_{\phi_r Z}\phi_t)X, \mathcal{D}^{\perp}\rangle = 0$$

where $\phi_t = \phi_s\phi_r$, $r\neq s$.

Lemma 5.5 follows from Lemma 5.4.

Now, we give the following

LEMMA 5.6. (First Fundamental Lemma). *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then for any vector fields X in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$ we have*

$$(5.9) \qquad A_{\phi_s Z}X = A_{\phi_r Z}\phi_t X\,,$$

$$(5.10) \qquad A_{\phi_s Z}\phi_r X = -A_{\phi_r Z}\phi_s X\,,$$

$$(5.11) \qquad A_{\phi_s Z}\phi_s X = A_{\phi_r Z}\phi_r X$$

for $r\neq s$, where $\phi_t = \phi_s\phi_r$.

Proof. From (5.6) we obtain

$$\begin{aligned}\langle A_{\phi_r Z}\phi_s Y, X\rangle &= -\langle\phi_r\sigma(X, \phi_s Y), Z\rangle = -\langle\hat{\sigma}(X, \phi_r\phi_s Y), Z\rangle \\ &= \langle\phi_s\sigma(X, \phi_r Y), Z\rangle = -\langle A_{\phi_s Z}\phi_r Y, X\rangle\end{aligned}$$

for $r \neq s$ and for any vector fields X, Y in $\mathcal{D}$, and Z in $\mathcal{D}^\perp$. Replacing Y by $\phi_r Y$ we get $\langle A_{\phi_s Z} Y, X\rangle = \langle A_{\phi_r Z}\phi_t Y, X\rangle$. Combining this with Lemma 5.5 we obtain (5.9).

Replacing X by $\phi_r X$ in (5.9) we obtain (5.10). And replacing X by $\phi_r\phi_s X$ in (5.10) we have (5.11).

For each $A_{\phi_s Z}$ we define an endomorphism

$$\tilde{A}_{\phi_s Z} : \mathcal{D} \to \mathcal{D}$$

to be the $\mathcal{D}$-component of $A_{\phi_s Z}$, i. e., $\tilde{A}_{\phi_s Z} X \in \mathcal{D}$ with

$$\langle \tilde{A}_{\phi_s Z} X, Y\rangle = \langle A_{\phi_s Z} Y, X\rangle \tag{5.12}$$

for any X, Y in $\mathcal{D}$. Then it is clear that $\tilde{A}_{\phi_s Z}$ is a self-adjoint endomorphism of $\mathcal{D}$.

From (5.9) we have

$$\tilde{A}_{\phi_s Z} = \tilde{A}_{\phi_r Z}\phi_t \qquad r \neq s\,, \quad \phi_t = \phi_s\phi_r\,. \tag{5.13}$$

Since $\tilde{A}_{\phi_s Z}$ is self-adjoint and ϕ_t satisfies $\langle \phi_t X, Y\rangle = -\langle X, \phi_t Y\rangle$ for any X, Y in $\mathcal{D}$, we have

$$\tilde{A}_{\phi_r Z}\phi_t = -\phi_t \tilde{A}_{\phi_r Z}\,. \tag{5.14}$$

Consequently, we have

$$\tilde{A}_{\phi_r Z}\phi_r = \tilde{A}_{\phi_r Z}\phi_t\phi_s = \phi_t\phi_s\tilde{A}_{\phi_r Z} = \phi_r\tilde{A}_{\phi_r Z}\,, \tag{5.15}$$

where $\phi_t = \phi_s\phi_r$. Hence we have the following

LEMMA 5.7. (Second Fundamental Lemma). *Let N be a quaternion CR-submanifold of a quaternion manifold M. Then for any vectors X, Y in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$ we have*

$$\langle A_{\phi_r Z}\phi_t X, Y\rangle = -\langle \phi_t A_{\phi_r Z} X, Y\rangle\,, \qquad r \neq t\,, \tag{5.16}$$

$$\langle A_{\phi_r Z}\phi_r X, Y\rangle = \langle \phi_r A_{\phi_r Z} X, Y\rangle\,. \tag{5.17}$$

As a corollary of Lemma 5.7 we have the following

COROLLARY 5.8. *Let N be a quaternion CR-submanifold of a quaternion manifold M. We have*

$$\sigma(X, X) + \sum_{r=1}^{3} \sigma(\phi_r X, \phi_r X) = 0 \tag{5.18}$$

for any vector X in $\mathcal{D}$.

Proof. From Lemma 5.7 we have

$$\langle\sigma(\phi_t X,\ Y),\ \phi_r Z\rangle=\langle\sigma(X,\ \phi_t Y),\ \phi_r Z\rangle\,,\qquad r\neq t\,,$$

$$\langle\sigma(\phi_r X,\ Y),\ \phi_r Z\rangle=-\langle\sigma(X,\ \phi_r Y),\ \phi_r Z\rangle\,.$$

Thus we have

$$\langle\sigma(X,\ X),\ \phi_r Z\rangle=\langle\sigma(\phi_r X,\ \phi_r X),\ \phi_r Z\rangle=-\langle\sigma(\phi_t X,\ \phi_t X),\ \phi_r Z\rangle\,,\quad r\neq t\,. \tag{5.19}$$

Combining this with Lemma 5.1 we obtain (5.18).

6. Some applications of Fundamental Lemmas.

In this paper we shall apply the fundamental lemmas repeatly.
In this section we shall apply them to obtain the following

THEOREM 6.1. *Every totally umbilical proper quaternion CR-submanifold in a quaternion manifold is totally geodesic.*

Proof. If N is a proper quaternion CR-submanifold and N is totally umbilical, then we have

$$\sigma(Y,\ Z)=\langle Y,\ Z\rangle H \tag{6.1}$$

for any vectors Y, Z tangent to N. Hence from Lemma 5.1, H lies in $\sum_{r=1}^{3}\phi_r\mathcal{D}^{\perp}$.

Assume that N is not totally geodesic. Then there exist a ϕ_s, $s=1$, 2 or 3 and a unit vector Z in $\mathcal{D}^{\perp}$ such that

$$\lambda\equiv\langle\phi_s Z,\ H\rangle\neq 0\,. \tag{6.2}$$

From which together with the fundamental lemmas and (6.1) we get

$$\lambda=\langle A_{\phi_s Z}X,\ X\rangle=\langle A_{\phi_r Z}\phi_t X,\ X\rangle=-\langle\phi_t A_{\phi_r Z}X,\ X\rangle=\langle X,\ \phi_t X\rangle\langle H,\ \phi_r Z\rangle=0\,.$$

This contradicts (6.2).

PROPOSITION 6.2. *Let N be a totally geodesic quaternion CR-submanifold in a quaternion manifold M. Then N is locally the Riemannian product of a totally geodesic quaternion submanifold $N^{\top}$ and a totally geodesic totally real submanifold $N^{\perp}$.*

The proposition follows from Lemma 3.3, Theorems 4.2, 4.3 and Lemma 5.3.

Let N be a quaternion CR-submanifold in a quaternion manifold M. Then N is said to be of *minimal codimension* if the subbundle v is trivial, i.e., $T^{\perp}N=I\mathcal{D}^{\perp}\oplus J\mathcal{D}^{\perp}\oplus K\mathcal{D}^{\perp}$.

We suppose that N is a totally geodesic proper quaternion CR-submanifold of minimal codimension in a quaternion manifold M. Then for any U, V, W tangent to N, Z in $\mathcal{D}^{\perp}$ and using the equation of Codazzi we have

$$\tilde{R}(U, V, W, \phi_r Z)=0 \qquad r=1, 2, 3. \tag{6.3}$$

On the other hand, for any vector fields X, Y in $\mathcal{D}$ and Z, W in $\mathcal{D}^\perp$, the equation of Bianchi, Lemma 3.4 and (6.3) give

$$\begin{aligned}\tilde{R}(X, Y; \phi_r Z, \phi_s W)&=\tilde{R}(\phi_r Z, X; \phi_s Y, W)+\tilde{R}(Y, \phi_r Z; \phi_s X, W)\\ &=-\tilde{R}(\phi_s Y, W, X, \phi_r Z)+\tilde{R}(\phi_s X, W; Y, \phi_r Z)=0.\end{aligned} \tag{6.4}$$

As N is totally geodesic in M, the equation of Gauss gives

$$\tilde{R}(X, Y; Z, W)=0. \tag{6.5}$$

Let $N^\top$ be any leaf of $\mathcal{D}$. Then from Lemma 3.3, $N^\top$ is totally geodesic in M. So, the equation of Ricci of $N^\top$ in M is given by $\tilde{R}(X, Y; \xi, \eta)=R_\top^\perp(X, Y; \xi, \eta)$ for X, Y in $\mathcal{D}$ and ξ, η in $T^\perp N^\top=\mathcal{D}^\perp \oplus I\mathcal{D}^\perp \oplus J\mathcal{D}^\perp \oplus K\mathcal{D}^\perp$. Then from (6.4) and (6.5) we have that the normal connection of $N^\top$ in M is flat, i.e., $R_\top^\perp \equiv 0$, and using [1], we obtain that M and $N^\top$ are Ricci flat.

From this and Theorem 6.1 we obtain the following

THEOREM 6.3. *The only quaternion manifolds which admit totally umbilical proper quaternion CR-submanifolds of minimal codimension are Ricci flat quaternion manifolds.*

THEOREM 6.4. *Let N be a quaternion CR-submanifold of a quaternion-space-form $M(c)$. Then the quaternion mean curvature of N satisfies*

$$m(X) \leqq c \tag{6.6}$$

for any unit vector X in $\mathcal{D}$. The equality of (6.6) *holds for any unit vector X in $\mathcal{D}$ if and only if the quaternion distribution $\mathcal{D}$ is integrable.*

Proof. From (5.2) of Lemma 5.1 and the equation of Gauss, we obtain

$$K(X, \phi_r X)=\tilde{K}(X, \phi_r X)+\sum_{s=1}^{3}\sum_{\alpha=1}^{p}\langle A_{\phi_s Z_\alpha}X, X\rangle\langle A_{\phi_s Z_\alpha}\phi_r X, \phi_r X\rangle-\|\sigma(X, \phi_r X)\|^2,$$

where K denotes the sectional curvature on N, X is a unit vector in $\mathcal{D}$ and $Z_1, \cdots, Z_p$ an orthonormal basis of $\mathcal{D}^\perp$. Hence by (5.16) and (5.17) of Lemma 5.7 we have

$$K(X, \phi_r X)=c+\sum_{\alpha=1}^{p}\langle A_{\phi_r Z_\alpha}X, X\rangle^2-\sum_{\alpha=1}^{p}\sum_{s\neq r}\langle A_{\phi_s Z_\alpha}X, X\rangle^2-\|\sigma(X, \phi_r X)\|^2. \tag{6.7}$$

Therefore the quaternion mean curvature of N satisfies

$$m(X)=c-\frac{1}{3}\sum_{\alpha=1}^{p}\sum_{r=1}^{3}\langle A_{\phi_r Z_\alpha}X, X\rangle^2-\frac{1}{3}\sum_{r=1}^{3}\|\sigma(X, \phi_r X)\|^2 \leqq c. \tag{6.8}$$

Combining Theorem 4.3 and (6.8) we see that $m(X)=c$ for all unit vector X in

$\mathcal{D}$ if and only if $\mathcal{D}$ is integrable.

PROPOSITION 6.5. *Let N be a quaternion CR-submanifold of a quaternion manifold M. If the leaves of $\mathcal{D}^\perp$ are minimal in M, then N is minimal in M.* This proposition follows from Lemma 5.1 and Corollary 5.8.

We may use the fundamental lemmas to give the following estimate for the length of curvature tensor $R^\perp$ on the normal bundle.

THEOREM 6.6 *Let N be a quaternion CR-submanifold of a quaternion-space-form $M(c)$, $c \geqq 0$. Then we have*

$$\|R^\perp\|^2 \geqq 3pqc^2, \tag{6.9}$$

where $q=\dim_Q \mathcal{D}$ and $p=\dim_R \mathcal{D}^\perp$. If the equality of (6.9) *holds, then N is a QR-product, i.e., N is locally the Riemannian product of a quaternion submanifold $N^\top$ and a totally real submanifold $N^\perp$ of $M(c)$.*

Proof. Let X and Z be unit vectors in $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively. Then, for $r \neq s$, the equation (3.7) of Ricci implies

$$\begin{aligned} \frac{c}{2} = \tilde{R}(X, \phi_r X; \phi_s Z, \phi_t Z) &= R^\perp(X, \phi_r X; \phi_s Z, \phi_t Z) \\ &\quad - \langle [A_{\phi_s Z}, A_{\phi_t Z}]X, \phi_r X\rangle, \qquad \phi_t = \phi_s \phi_r . \end{aligned} \tag{6.10}$$

Thus by Lemma 5.6 we have

$$R^\perp(X, \phi_r X; \phi_s Z, \phi_t Z) = \frac{c}{2} + \|A_{\phi_s Z}X\|^2 + \|A_{\phi_t Z}X\|^2 \geqq \frac{c}{2}. \tag{6.11}$$

Thus the length of the normal curvature tensor $R^\perp$ satisfies

$$\begin{aligned} \|R^\perp\|^2 &\geqq \sum_{i,j=1}^{4q} \sum_{A,B=1}^{N} \{R^\perp(X_i, X_j; \xi_A, \xi_B)\}^2 \\ &\geqq \sum_{\alpha=1}^{p} \sum_{i=1}^{4q} \sum_{r,s,t=1}^{3} \{R^\perp(X_i, \phi_r X_i; \phi_s Z_\alpha, \phi_t Z_\alpha)\}^2, \qquad \phi_t = \phi_s \phi_r \end{aligned} \tag{6.12}$$

where $\{X_1, \cdots, X_{4q}\}$, $\{Z_1, \cdots, Z_p\}$ and $\{\xi_1, \cdots, \xi_N\}$ are orthonormal bases of $\mathcal{D}$, $\mathcal{D}^\perp$ and $T^\perp N$ respectively. Combining (6.11) and (6.12) we obtain (6.9).

If the equality sign of (6.9) holds, then we have

$$A_{\phi_s Z}X = 0 \tag{6.13}$$

for any X in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$. Thus by Theorem 4.3 and Lemma 5.1, we may conclude that the quaternion distribution is integrable, and each leaf $N^\top$ is totally geodesic in $M(c)$ by Lemma 3.3. So in particular, $N^\top$ is totally geodesic in N. Therefore N is a QR-product by Theorem 4.2, Lemma 5.3 and (6.13).

7. Quaternion CR-submanifolds foliated by totally geodesic, totally real leaves.

Let N be a quaternion CR-submanifold in a quaternion manifold M. Then $\mathcal{D}^{\perp}$ is always integrable. In this section we shall study the case in which the leaves of totally real distribution $\mathcal{D}^{\perp}$ are totally geodesic in N. For this case, Lemma 5.3 gives

(7.1) $$\langle\sigma(\mathcal{D}, \mathcal{D}^{\perp}), \phi_r\mathcal{D}^{\perp}\rangle=0, \qquad r=1, 2, 3.$$

In others words, we have

(7.2) $$A_{\phi_r Z}X\in\mathcal{D}, \quad A_{\phi_r Z}W\in\mathcal{D}^{\perp}$$

for any vectors X in $\mathcal{D}$ and Z, W in $\mathcal{D}^{\perp}$.

For any unit vector fields X, Y in $\mathcal{D}$ and Z, W in $\mathcal{D}^{\perp}$, equation (3.6) of Codazzi gives

(7.3) $$\begin{aligned}\tilde{R}(X, Y; Z, \phi_rW)=&\langle D_X\sigma(Y, Z)-\sigma(\nabla_XY, Z)-\sigma(Y, \nabla_XZ), \phi_rW\rangle\\ &-\langle D_Y\sigma(X, Z)-\sigma(\nabla_YX, Z)-\sigma(X, \nabla_YZ), \phi_rW\rangle.\end{aligned}$$

From (2.1), (3.1), (3.2) and (7.1) we have

(7.4) $$\begin{aligned}\langle D_X\sigma(Y, Z), \phi_rW\rangle&=-\langle\sigma(Y, Z), \tilde{\nabla}_X(\phi_rW)\rangle=-\langle\sigma(Y, Z), \phi_r\tilde{\nabla}_XW\rangle\\ &=-\langle\sigma(Y, Z), \phi_r\sigma(X, W)\rangle.\end{aligned}$$

So, in particular, we have from (5.3) that

(7.5) $$\langle D_X\sigma(\phi_rX, Z), \phi_rZ\rangle=-\langle\sigma(\phi_rX, Z), \phi_r\sigma(X, Z)\rangle=-\|\sigma(X, Z)\|^2$$

Similarly, we may also prove that

(7.6) $$\langle D_{\phi_rX}\sigma(X, Z), \phi_rZ\rangle=\|\sigma(X, Z)\|^2.$$

Moreover, from (2.1), (3.1), (7.2) and Lemma 5.7 we have

(7.7) $$\begin{aligned}\langle\sigma(\nabla_XY, Z), \phi_rW\rangle&=\langle A_{\phi_rW}Z, \tilde{\nabla}_XY\rangle=\langle\phi_rA_{\phi_rW}Z, \tilde{\nabla}_X(\phi_rY)\rangle\\ &=\langle\phi_rA_{\phi_rW}Z, \sigma(X, \phi_rY)\rangle=\langle A_{\phi_rU}\phi_rY, X\rangle=-\langle A_{\phi_rU}Y, \phi_rX\rangle\\ &=-\langle\sigma(Y, \phi_rX), \phi_rA_{\phi_rW}Z\rangle,\end{aligned}$$

where $U=A_{\phi_rW}Z$. In particular, we have from (7.2) and Lemma 5.7 that

(7.8) $$\langle\sigma(\nabla_X\phi_rX, Z), \phi_rZ\rangle=-\langle\sigma(X, X), \phi_rA_{\phi_rZ}Z\rangle$$

(7.9) $$\langle\sigma(\nabla_{\phi_rX}X, Z), \phi_rZ\rangle=\langle\sigma(X, X), \phi_rA_{\phi_rZ}Z\rangle.$$

From (2.1), (3.1) and (7.2) we also have

$$\langle\sigma(Y,\nabla_X Z),\ \phi_r W\rangle=\langle A_{\phi_r W}Y,\ \tilde{\nabla}_X Z\rangle=\langle\phi_r A_{\phi_r W}Y,\ \tilde{\nabla}_X(\phi_r Z)\rangle \tag{7.10}$$
$$=-\langle\phi_r A_{\phi_r W}Y,\ A_{\phi_r Z}X\rangle\,.$$

Hence, in particular, from (7.2) and Lemma 5.7 we obtain

$$\langle\sigma(\phi_r X,\ \nabla_X Z),\ \phi_r Z\rangle=\|A_{\phi_r Z}X\|^2\,, \tag{7.11}$$

$$\langle\sigma(X,\ \nabla_{\phi_r X}Z),\ \phi_r Z\rangle=-\|A_{\phi_r Z}X\|^2\,. \tag{7.12}$$

Combining (7.3), (7.5), (7.6), (7.8), (7.9), (7.11) and (7.12) we get

$$\tilde{R}(X,\ \phi_r X\,;\ Z,\ \phi_r Z)=-2\|\sigma(X,\ Z)\|^2-2\|A_{\phi_r Z}X\|^2+2\langle\sigma(X,\ X),\ \phi_r A_{\phi_r Z}Z\rangle\,. \tag{7.13}$$

Let $\{X_1,\ \cdots,\ X_q,\ X_{q+1}=IX_1,\ \cdots,\ X_{2q+1}=JX_1,\ \cdots,\ X_{3q+1}=KX_1,\ \cdots,\ X_{4q}=KX_q\}$ be an orthonormal basis of $\mathcal{D}$. Then by Corollary 5.8 and (7.13) we get

$$\sum_{i=1}^{4q}\tilde{R}(X_i,\ \phi_r X_i\,;\ Z,\ \phi_r Z)=-2\sum_{i=1}^{4q}\{\|\sigma(X_i,\ Z)\|^2+\|A_{\phi_r Z}X_i\|^2\}\,. \tag{7.14}$$

On the other hand, by equation of Bianchi and Lemma 3.4 we have

$$\tilde{R}(X,\ \phi_r X\,;\ Z,\ \phi_r Z)=-\tilde{K}(X,\ Z)-\tilde{K}(X,\ \phi_r Z)\,. \tag{7.15}$$

Thus (7.14) and (7.15) imply

$$\sum_{i=1}^{4q}\{\|\sigma(X_i,\ Z)\|^2+\|A_{\phi_r Z}X_i\|^2\}=\frac{1}{2}\sum_{i=1}^{4q}\{\tilde{K}(X_i,\ Z)+\tilde{K}(X_i,\ \phi_r Z)\}\,. \tag{7.16}$$

From this we obtain the following

THEOREM 7.1. *Let N be a quaternion CR-submanifold in a non-positively curved quaternion manifold M. If the leaves of $\mathcal{D}^\perp$ are totally geodesic in N, then we have*

(1) $\tilde{K}(X,\ Z)=0$ *for any vectors X in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$,*
(2) *N is a QR-product, and*
(3) $\sigma(\mathcal{D},\ \mathcal{D}^\perp)=0$, *i.e., N is mixed totally geodesic.*

This theorem follows immediately from Lemma 3.3, Theorem 4.2 and equation (7.16).

From Theorem 7.1 we obtain the following

COROLLARY 7.2. *Let N be a quaternion CR-submanifold in a quaternion-space-form $M(c)$, $c\leqq 0$. If the leaves of $\mathcal{D}^\perp$ are totally geodesic in N, then $c=0$ and N is a QR-product. In particular, locally N is the Riemannian product of a totally geodesic quaternion submanifold and a totally real submanifold.*

Let $QP^m(4)$ be the quaternion projective space of quaternion sectional curvature 4. If N is a quaternion CR-submanifold of $QP^m(4)$ such that the leaves of $\mathcal{D}^\perp$ are totally geodesic in N, then (2.2) and (7.13) imply

(7.17) $$\|\sigma(X, Z)\|^2+\|A_{\phi_r Z}X\|^2=1+\langle\sigma(X, X), \phi_r A_{\phi_r Z}Z\rangle$$

for any unit vectors X in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$.

On the other hand, from Lemma 5.1 and (7.1) we have

(7.18) $$\|\sigma(\phi_t X, Z)\|^2=\|\sigma(X, Z)\|^2 .$$

From Lemma 5.7 and (7.2) we have

(7.19) $$\|A_{\phi_r Z}X\|^2=\|A_{\phi_r z}\phi_t X\|^2 .$$

Moreover, from (5.19) and (7.2) we also have

(7.20) $$\langle\sigma(X, X)+\sigma(\phi_t X, \phi_t X), \phi_r A_{\phi_r Z}Z\rangle=0 , \qquad r\neq t .$$

Combining (7.17), (7.18), (7.19) and (7.20) we obtain the following

LEMMA 7.3. *Let N be a quaternion CR-submanifold in $QP^m(4)$. If the leaves of $\mathcal{D}^\perp$ are totally geodesic in N, then we have*

(7.21) $$\|\sigma(X, Z)\|^2+\|A_{\phi_r Z}X\|^2=1 ,$$

(7.22) $$\langle\sigma(X, X), \phi_r A_{\phi_r Z}W\rangle=0$$

for any unit vectors X, Y in $\mathcal{D}$ and Z, W in $\mathcal{D}^\perp$.

(7.22) follows from (7.17), (7.21), linearity and Lemma 4.1.

LEMMA 7.4. *Under the same hypothesis of Lemma 7.3 we have*

(7.23) $$\langle\sigma(\mathcal{D}, \mathcal{D}), \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp)\rangle=0 .$$

Proof. For each $r=1, 2, 3$ we put $\mathcal{V}_r=\{A_{\phi_r Z}W/Z, W\in\mathcal{D}_x^\perp\}$. Then $\mathcal{V}_r$, $r=1, 2, 3$ are linear subspaces of $\mathcal{D}_x^\perp$ by (7.2). Let $\mathcal{V}_r^\perp$ be the orthogonal complementary subspace of $\mathcal{V}_r$ in $\mathcal{D}_x^\perp$. Then Lemma 7.3 implies

(7.24) $$\sigma(\mathcal{D}, \mathcal{D})\subseteq\phi_1\mathcal{V}_1^\perp\oplus\phi_2\mathcal{V}_2^\perp\oplus\phi_3\mathcal{V}_3^\perp .$$

On the other hand, by Lemma 4.1 we have

(7.25) $$0=\langle Z_r, A_{\phi_r V}W\rangle=\langle A_{\phi_r Z_r}V, W\rangle=\langle\sigma(V, W), \phi_r Z_r\rangle$$

for any vectors Z_r in $\mathcal{V}_r^\perp$ and V, W in $\mathcal{D}_x^\perp$.

Combining (7.24) and (7.25) we obtain (7.23).

THEOREM 7.5. *Let N be a quaternion CR-submanifold in $QP^m(4)$. If the leaves of $\mathcal{D}^\perp$ are totally geodesic in N, then for any unit vectors X in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$ we have*

$$K(X, Z) \geqq 0 \tag{7.26}$$

The equality sign holds if and only if the quaternionic distribution $\mathcal{D}$ is integrable.

Proof. From the equations of Gauss we have

$$K(X, Z)=1+\langle\sigma(X, X), \sigma(Z, Z)\rangle-\|\sigma(X, Z)\|^{2}.$$

Thus by Lemma 7.3 we have

$$K(X, Z)=\langle\sigma(X, X), \sigma(Z, Z)\rangle+\|A_{\phi_r Z}X\|^{2}.$$

Combining this with Lemma 7.4 we obtain

$$K(X, Z)=\|A_{\phi_r Z}X\|^{2} \geqq 0. \tag{7.27}$$

It is clear that $K(\mathcal{D}, \mathcal{D}^{\perp})=0$ if and only if $A_{\phi_r \mathcal{D}^{\perp}}\mathcal{D}=0$. Thus by Lemma 5.1 and (7.2), $K(\mathcal{D}, \mathcal{D}^{\perp})=0$ if and only if $\sigma(\mathcal{D}, \mathcal{D})=0$. Therefore the equality of (7.26) holds if and only if $\mathcal{D}$ is integrable by Theorem 4.3.

As a immediate consequence of Theorem 7.5 we obtain the following

COROLLARY 7.6. *Let N be a proper quaternion CR-submanifold of $QP^{m}(4)$. If N is negatively curved, then N is not foliated by totally geodesic totally real submanifolds.*

Now we shall apply Lemma 7.3 to obtain the following result for QR-products.

THEOREM 7.7. *Let N be a QR-product in $QP^{m}(4)$. Then we have*

(1) $\|\sigma(X, Z)\|=1$ *for any unit vectors X in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$,*

(2) $m \geqq p+q+pq$, *where* $p=\dim_R \mathcal{D}_x^{\perp}$, $q=\dim_Q \mathcal{D}_x$.

Proof. If N is a QR-product in $QP^{m}(4)$, then both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable and the leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are totally geodesic in N. Thus we have $A_{\phi_r Z}X=0$ for any unit vectors X in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$. From Lemma 7.3 we obtain

$$\langle\sigma(X, Z), \sigma(X, Z)\rangle=1. \tag{7.28}$$

Thus by linearity we see that for any orthonormal vectors X, Y in $\mathcal{D}$ and Z, W in $\mathcal{D}^{\perp}$

$$\langle\sigma(X, Z), \sigma(Y, Z)\rangle=\langle\sigma(X, Z), \sigma(X, W)\rangle=0. \tag{7.29}$$

Therefore we also have

$$\langle\sigma(X, Z), \sigma(Y, W)\rangle+\langle\sigma(X, W), \sigma(Y, Z)\rangle=0. \tag{7.30}$$

On the other hand, by equations of Gauss and (2.2) we obtain

(7.31) $$\langle\sigma(X, Z), \sigma(Y, W)\rangle-\langle\sigma(X, W), \sigma(Y, Z)\rangle=0.$$

Combining (7.28), (7.29), (7.30) and (7.31) we see that, for orthonormal bases $\{X_1, \cdots, X_{4q}\}$ and $\{Z_1, \cdots, Z_p\}$ of $\mathcal{D}$ and $\mathcal{D}^{\perp}$,

$$\{\sigma(X_i, Z_\alpha)/i=1, \cdots, 4q, \alpha=1, \cdots, p\}$$

are orthonormal vectors in $T^{\perp}N$. On the other hand, (7.1) shows that these vectors are perpendicular to $\sum_{r=1}^{3}\phi_r\mathcal{D}^{\perp}$. From these we conclude that the quaternion dimension of $QP^m(4)$ is greater than or equal to $p+q+pq$.

8. A Counterexample.

An *almost quaternion metric manifold* [10] is a Riemannian manifold with a 3-dimensional vector bundle F of tensors of type (1, 1) with local basis of almost Hermitian structures I, J, K satisfying $IJ=-JI=K$, $JK=-KJ=I$ and $KI=-IK=J$.

The purpose of this section is to give an example of a quaternion CR-submanifold of an almost quaternion metric manifold on which $\mathcal{D}^{\perp}$ is not integrable.

Let $\bar{\nabla}$ be a symmetric connection on a differentiable manifold $\bar{M}$ and X a vector field on $\bar{M}$. Let f be an arbitrary C^{∞} function on $\bar{M}$ and Z in $T\bar{M}$. We define the horizontal lift of X to $T\bar{M}$ to be the vector field X^H on $T\bar{M}$ given by [11] $(X^H df)(Z)=(\bar{\nabla}_X df)Z$, where on the right $\bar{\nabla}_X$ is acting on the 1-form df and on the left df is regarded as a function on $T\bar{M}$. The vertical lift X^V of X is independent of the connection and is simply defined by $X^V\omega=\omega(X)\cdot\Pi$, Π is the natural projection from $T\bar{M}$ onto $\bar{M}$ and ω a 1-form on $\bar{M}$.

For a tensor field Ψ of type (1, 1) on $\bar{M}$ its horizontal lift Ψ^H may be defined by $\Psi^H X^V=(\Psi X)^V$ and $\Psi^H X^H=(\Psi X)^H$.

Recall the connection map $\bar{K}: TT\bar{M}\to T\bar{M}$ given by $\bar{K}(X_Z^V)=X_{\Pi(Z)}$, $\bar{K}X^H=0$, [9]. If G is a Riemannian metric on $\bar{M}$ and $\bar{\nabla}$ its Levi-Civita connection, we define the Sasaki metric g on $T\bar{M}$ by $g(X, Y)=G(\Pi_*X, \Pi_*Y)+G(\bar{K}X, \bar{K}Y)$ for any vectors X, Y tangent to $T\bar{M}$. The Levi-Civita connection $\hat{\nabla}$ of g is given in terms of $\bar{\nabla}$ and the curvature tensor $\bar{R}$ of $\bar{M}$ by

$$(\hat{\nabla}_{X^H}Y^H)_Z=(\bar{\nabla}_X Y)_Z^H-\frac{1}{2}(\bar{R}(X, Y)Z)^V,$$

$$(\hat{\nabla}_{X^H}Y^V)_Z=(\bar{\nabla}_X Y)_Z^V-\frac{1}{2}(\bar{R}(Y, Z)X)^H,$$

$$(\hat{\nabla}_{X^V}Y^H)_Z=-\frac{1}{2}(\bar{R}(X, Z)Y)^H, \quad (\hat{\nabla}_{X^V}Y^V)=0.$$

Now, let $\bar{M}$ be a quaternion manifold with quaternion structure $(G, \phi_1, \phi_2, \phi_3)$, where $\phi_1=I$, $\phi_2=J$ and $\phi_3=K$. Let ϕ_r^H be the horizontal lift of ϕ_r to $T\bar{M}$ and g the Sasaki metric on $T\bar{M}$. It is easy to check that $(g, \phi_1^H, \phi_2^H, \phi_3^H)$ is an almost quaternion metric structure on $T\bar{M}$.

THEOREM 8.1. *Let $\bar{M}$ be a quaternion manifold. Then $T\bar{M}$ with $(g, \phi_1^H, \phi_2^H, \phi_3^H)$ is a quaternion manifold if and only if $\bar{M}$ is flat.*

Proof. Let $(G, \phi_1, \phi_2, \phi_3)$ be the quaternion structure of $\bar{M}$. Then there exist local 1-forms q_{rs} on $\bar{M}$ such that

$$(\bar{\nabla}_X\phi_r)Y = \sum_{s=1}^{3} q_{rs}(X)\phi_s Y, \qquad r=1, 2, 3,$$

and $q_{rs}+q_{sr}=0$, [10].

Hence, we have

$$(8.1)\quad [(\hat{\nabla}_{X^H}\phi_r^H)Y^H]_Z = \Big(\sum_{s=1}^{3} q_{rs}^V(X^H)\phi_s^H(Y^H)\Big)_Z + \frac{1}{2}\{\phi_r\bar{R}(X, Y)Z - \bar{R}(X, \phi_r Y)Z\}^V,$$

$$(8.2)\quad [(\hat{\nabla}_{X^H}\phi_r^H)Y^V]_Z = \Big(\sum_{s=1}^{3} q_{rs}^V(X^H)\phi_s^H(Y^V)\Big)_Z + \frac{1}{2}\{\phi_r\bar{R}(Y, Z)X - \bar{R}(\phi_r Y, Z)X\}^H,$$

$$(8.3)\quad [(\hat{\nabla}_{X^V}\phi_r^H)Y^H]_Z = \frac{1}{2}\{\phi_r\bar{R}(X, Z)Y - \bar{R}(X, Z)\phi_r Y\}^H,$$

$$(8.4)\quad [(\hat{\nabla}_{X^V}\phi_r^H)Y^V]_Z = 0 \qquad r=1, 2, 3.$$

If $\bar{M}$ is flat, these imply that

$$(8.5)\quad (\hat{\nabla}_{\tilde{X}}\phi_r^H)\tilde{Y} = \sum_{s=1}^{3} q_{rs}^V(\tilde{X})\phi_s^H\tilde{Y} \qquad r=1, 2, 3,$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ tangent to $T\bar{M}$. Thus $T\bar{M}$ is also a quaternion manifold.

Conversely, if $T\bar{M}$ with $(g, \phi_1^H, \phi_2^H, \phi_3^H)$ is a quaternion manifold, then there exist local 1-forms $\tilde{q}_{rs}$ on $T\bar{M}$ such that

$$(\hat{\nabla}_{\tilde{X}}\phi_r^H)\tilde{Y} = \sum_{s=1}^{3} \tilde{q}_{rs}(\tilde{X})\phi_s^H\tilde{Y} \quad \text{and} \quad \tilde{q}_{rs}+\tilde{q}_{sr}=0,$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ tangent to $T\bar{M}$. From (8.1)—(8.4), we obtain

$$(8.6)\quad \Big(\sum_{s=1}^{3} \tilde{q}_{rs}(X^H)\phi_s^H(Y^H)\Big)_Z = \Big(\sum_{s=1}^{3} q_{rs}^V(X^H)\phi_s^H(Y^H)\Big)_Z + \frac{1}{2}\{\phi_r\bar{R}(X, Y)Z - \bar{R}(X, \phi_r Y)Z\}^V,$$

$$(8.7)\quad \Big(\sum_{s=1}^{3} \tilde{q}_{rs}(X^H)\phi_s^H(Y^V)\Big)_Z = \Big(\sum_{s=1}^{3} q_{rs}^V(X^H)\phi_s^H(Y^V)\Big)_Z + \frac{1}{2}\{\phi_r\bar{R}(Y, Z)X - \bar{R}(\phi_r Y, Z)X\}^H,$$

$$\left(\sum_{s=1}^{3} \tilde{q}_{rs}(X^V)\phi_s^H(Y^H)\right)_Z=\frac{1}{2}\{\phi_r\bar{R}(X, Z)Y-\bar{R}(X, Z)\phi_rY\}^H, \tag{8.8}$$

$$\left(\sum_{s=1}^{3} \tilde{q}_{rs}(X^V)\phi_s^H(Y^V)\right)_Z=0 \qquad r=1, 2, 3. \tag{8.9}$$

From (8.6)—(8.9) it follows that $\tilde{q}_{rs}=q_{rs}^V$ and $\bar{R}(X, Y)\phi_rZ=\phi_r\bar{R}(X, Y)Z=\bar{R}(X, \phi_rY)Z$, $r, s=1, 2, 3$. Hence we get

$$\begin{aligned} G(\bar{R}(X, \phi_rY)Z, W)&=G(\bar{R}(X, Y)\phi_rZ, W)=G(\bar{R}(\phi_rZ, W)X, Y)\\ &=G(\bar{R}(Z, W)\phi_rX, Y)=G(\phi_r\bar{R}(Z, W)X, Y)=-G(\bar{R}(Z, W)X, \phi_rY)\\ &=-G(\bar{R}(X, \phi_rY)Z, W). \end{aligned}$$

Since this is true for any vectors X, Y, Z, W tangent to $\bar{M}$, $\bar{M}$ is flat.

Now, let $\bar{M}$ be the quaternion projective space $QP^m(4)$ and N be the real projective space $RP^m(1)$ imbedded in $QP^m(4)$ as a totally geodesic, totally real submanifold. Let $\bar{N}$ be the set of fibres of $T(QP^m(4))$ over the points in $RP^m(1)$. By Theorem 8.1, $T(QP^m(4))$ is an almost quaternion metric manifold which is not a quaternion manifold. Since $RP^m(1)$ is totally real in $QP^m(4)$ and ϕ_r^H acts invariantly on the fibres of $T(QP^m(4))$, $\bar{N}$ is a quaternion CR-submanifold of $T(QP^m(4))$. Let X and Y be tangent to $RP^m(1)$. Then X^H and Y^H are both in $\mathcal{D}^\perp$ of $\bar{N}$. From (2.2) we have

$$\begin{aligned} [X^H, Y^H]_Z&=[X, Y]_Z^H-(\bar{R}(X, Y)Z)^V=[X, Y]_Z^H\\ &-\frac{1}{4}\left\{G(Y, Z)X-G(X, Z)Y+\sum_{r=1}^{3}[G(\phi_rY, Z)\phi_rX-G(\phi_rX, Z)\phi_rY]\right\}^V. \end{aligned}$$

For orthonormal X and Y and $Z=Y_{\Pi(Z)}$ this implies that the vertical part does not vanish. Hence the totally real distribution $\mathcal{D}^\perp$ is not integrable.

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