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NONLINEAR CONSTRACTIONS IN ABSTRACT SPACES

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I. Introduction.

Recently, Eisenfeld J. & Lakshmikantham V. [4, 5, 6], Bolen J. C. & Williams B. B. [1], Heikkila S. & Seikkala S. [7, 8], Chung K. J. [3], Kwapisz M. [10] and Ważewski T. [11] proved some fixed point theorems in abstract cones which extend and generalize many known results. In this paper, we extend some main results of Boyd D. W. & Wong J. S. W. [2] to cone-valued metric spaces.

II. Definitions.

Let E be a normed space. A set $K\subset E$ is said to be a cone if (i) K is closed (ii) if $u, v\in K$ then $\alpha u+\beta v\in K$ for all $\alpha, \beta\geqq 0$, (iii) $K\cap(-K)=\{\mathcal{O}\}$ where $\mathcal{O}$ is the zero of the space E, and (iv) $K^0\neq\emptyset$ where K^0 is the interior of K. We say $u\geqq v$ if and only if $u-v\in K$, and $u>v$ if and only if $u-v\in K$ and $u\neq v$. The cone K is said to be strongly normal if there is $\delta>0$ such that if $z=\sum_{i=1}^{n} b_i x_i$, $x_i\in K$, $\|x_i\|=1$, $\sum_{i=1}^{n} b_i=1$, $b_i\geqq 0$ implies $\|z\|>\delta$. The cone K is said to be normal if there is $\delta>0$ such that $\|f_1+f_2\|>\delta$ for $f_1, f_2\in K$ and $\|f_1\|=\|f_2\|=1$. The norm in E is said to be semimonotone if there is a numerical constant M such that $\mathcal{O}\leqq x\leqq y$ implies $\|x\|\leqq M\|y\|$ (where the constant M does not depend on x and y).

Let X be a set and K a cone. A function $d: X\times X\rightarrow K$ is said to be a K-metric on X if and only if (i) $d(x, y)=d(y, x)$, (ii) $d(x, y)=\mathcal{O}$ if and only if $x=y$, and (iii) $d(x, y)\leqq d(x, z)+d(z, y)$. A sequence $\{x_n\}$ in a K-metric space X is said to converge to x_0 in X if and only if for each $u\in K^0$ there exists a positive integer N such that $d(x_n, x_0)\leqq u$ for $u\geqq N$. A sequence $\{x_n\}$ in X is Cauchy if and only if for each $u\in K^0$ there exists a positive integer N such that $d(x_n, x_m)\leqq u$ for $n, m\geqq N$. The K-metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges.

Throughout the rest of this paper we assume that K is strongly normal, that E is a reflexive Banach space, that (X, d) is a complete K-metric space, that $P=\{d(x, y); x, y\in X\}$, that $\bar{P}$ denotes the weak closure of P, and that $P_1=\{z; z\in\bar{P}$ and $z\neq\mathcal{O}\}$.

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III. Preliminary results.

In this section we list Mazur Lemma and needed properties of the cone K and the related K-metric space which will be used in our theorem.

(a) Strongly normal is normal.

(b) A necessary and sufficient condition for the cone K to be normal is that the norm be semimonotone (cf. [9]).

(c) If the sequence $\{u_n\}$ in E converges (in norm) to u, the sequence $\{v_n\}$ in E converges (in norm) to v and $u_n \leqq v_n$ for each n, then $u \leqq v$.

(d) If $\{x_n\}$ is a sequence in the K-metric space X that has a limit in X, then the limit is unique.

(e) If $u \in K^0$, then there exists a positive number c such that if $v \in \{p\,;\, \|p\| < c\} \cap K$ then $v \leqq u$.

(f) If h is an element in the Banach space E, $h_n \in K$ for each n, $h \leqq h_n$ for each n and $\{h_n\}$ converges (in norm) to $\mathcal{O}$ in E, then $-h \in K$.

(g) If $u \in K^0$ and $\{h_n\}$ is a sequence in K which converges (in norm) to $\mathcal{O}$ in E, then there exists a positive integer N such that $h_n \leqq u$ for $n \geqq N$.

(h) If $\{x_n\}$ is a sequence in the K-metric space X that is convergent to x in X then $\{d(x_n, x)\}$ converges (in norm) to $\mathcal{O}$ in E.

(i) Mazur Lemma: Let E be a normed space and $\{u_n\}$ a sequence converging weakly to u, then there is a sequence of convex combinations $\{v_n\}$ such that

$$v_n = \sum_{i=n}^{N} b_i u_i \text{ where } \sum_{i=n}^{N} b_i = 1, \text{ and } b_i \geqq 0,\ n \leqq i \leqq N$$

which converges to u in norm.

(j) Let the sequence $\{u_n\}$ in E be weakly convergent to v, if $u_n \geqq \mathcal{O}$ for each $n \geqq 1$ then $v \geqq \mathcal{O}$.

IV. Examples and main results:

Example 1. Let $E = R$ (all real numbers) and $K = R^+$ (all nonnegative real numbers), then K is strongly normal and semimonotone, and K satisfies the law of trichotomy.

Example 2. Let $E = R^2$ and $K = \{z \in R^2\,;\, 0 < a \leqq \operatorname{Arg} z \leqq b < \pi/2\} \cup \{\mathcal{O}\}$, where the symbol Arg z denotes the argument of the complex number z. Although K is strongly normal, semimonotone, K doesn't satisfy the law of trichotomy.

The mapping $\phi : P_1 \to K$ is said to be upper semicontinuous if $\{u_n\}$ and $\{\phi u_n\}$ are both weakly convergent, then $\lim \phi u_n \leqq \phi(\lim u_n)$. Let G be a family of mappings ϕ such that $\phi : P_1 \to K$, ϕ is upper semicontinuous on P_1.

The property of the law of trichotomy of the set R has been used in the proof of [2], but it can not be used in our Theorem 1 (cf. Example 2). The proof of Theorem 1 differs from that of Theorem 1 [2].

THEOREM 1. *Let f be a self-mapping of X. Suppose that there exists $\phi\in G$ such that for all $x, y\in X$:*

(1) $$d(fx, fy)\leqq\phi(d(x, y)),$$

where ϕ satisfies the condition: for any $t\in P_1$,

(2) $$\phi(t)<t.$$

Then, f has a unique fixed point x_0 and $f^n x\to x_0$ for each x in X.

Proof. Let $x_0\in X$. We define the sequence $\{x_n\}$ by $x_1=fx_0$, $x_2=fx_1, \cdots, x_{2n+1}=fx_{2n}, \cdots$. Let $d_n=d(x_n, x_{n+1})\neq\mathcal{O}$. It follows, by (1), that, for each positive integer n,

(3) $$d_{n+1}=d(fx_n, fx_{n+1})\leqq\phi(d(x_n, x_{n+1}))\leqq d_n\leqq d_1.$$

Therefore $\{d_n\}$ is decreasing and bounded.

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there is an $\varepsilon\in K^0$ such that for every integer i, there exist integers $n(i)$ and $m(i)$ with $i\leqq n(i)<m(i)$ such that

(4) $$d(x_{n(i)}, x_{m(i)})\nleqq\varepsilon.$$

Let, for each integer i, $m(i)$ be the least integer exceeding $n(i)$ satisfying (4); that is

(5) $$d(x_{n(i)}, x_{m(i)})\nleqq\varepsilon \quad\text{and}\quad d(x_{n(i)}, x_{m(i)-1})\leqq\varepsilon.$$

Since K is semimonotone, the sequence $\{d(x_{n(i)}, x_{m(i)-1})\}$ is norm-bounded. Consequently the sequence $\{d(x_{n(i)}, x_{m(i)})\}$ is norm-bounded.

Since E is a reflexive Banach space, for convenience, we suppose

(A) $$\begin{cases} \{d(x_{n(i)}, x_{m(i)})\} & \text{is weakly convergent to } z_1, \\ \{d(x_{n(i)}, x_{m(i)-1})\} & \text{is weakly convergent to } z_2, \end{cases}$$

where z_1 and z_2 are in K. Since

(6) $$d(x_{n(i)}, x_{m(i)})+d(x_{m(i)}, x_{m(i)-1})\geqq d(x_{n(i)}, x_{m(i)-1}),$$

(7) $$d(x_{n(i)}, x_{m(i)-1})+d(x_{m(i)-1}, x_{m(i)})\geqq d(x_{n(i)}, x_{m(i)}),$$

From (6), (7) and (B), we see that $z_1\geqq z_2$, $z_2\geqq z_1$ and $z_1=z_2=z$ (say). We see that

(8) $$\begin{aligned} d(x_{n(i)}, x_{m(i)}) &\leqq d(x_{n(i)}, x_{n(i)+1})+d(x_{n(i)+1}, x_{m(i)+1})+d(x_{m(i)+1}, x_{m(i)}) \\ &\leqq 2d_{n(i)}+\phi(d(x_{n(i)}, x_{m(i)})). \end{aligned}$$

Since E is a reflexive Banach space, for convenience, we suppose

$$\text{(B)} \qquad \begin{cases} \{d_{n(i)}\} & \text{is weakly convergent to } c\,, \\ \{d_{n(i)-1}\} & \text{is weakly convergent to } b\,, \end{cases}$$

where b and c are in K.
From the fact that $d_{n(i)-1} \geqq d_{n(i)} \geqq d_{n(i+1)-1}$, it follows that $b=c$. Since $d_{n+1} \leqq \phi(d_n) \leqq d_n$, we obtain that $\{\phi(d_n)\}$ is bounded. Therefore there exists a subsequence $\{d_{r(i)}\}$ of $\{d_{n(i)}\}$ such that $\{\phi(d_{r(i)-1})\}$ has a weak limit. If $c > \mathcal{O}$, we have $c = \lim d_{r(i)} \leqq \lim \phi(d_{r(i)-1}) \leqq \phi(c) < c$, which is a contradiction. Hence $c=\mathcal{O}$. In fact $\{d_n\}$ is weakly convergent to $\mathcal{O}$.

Since $\mathcal{O} \leqq \phi(d(x_{n(i)}, x_{m(i)})) \leqq d(x_{n(i)}, x_{m(i)})$, and E is reflexive, for convenience, we let $\{\phi(d(x_{n(i)}, x_{m(i)}))\}$ have a weak limit. If $z \neq \mathcal{O}$, we see, by (A), (B), (j) and (8), that $z \leqq \phi(z)$. We obtain that $z=\mathcal{O}$.

By (4) and (g), there exist a positive number s and a subsequence $\{d(x_{p(i)}, x_{q(i)})\}$ of $\{d(x_{n(i)}, x_{m(i)})\}$ such that $\lim_{i\to\infty} \|d(x_{p(i)}, x_{q(i)})\| = s > 0$.

Since the sequence $\{d(x_{p(i)}, x_{q(i)})\}$ is weakly convergent to $\mathcal{O}$, by Mazur Lemma, there is a sequence of convex combinations $\{v_n\}$ such that $v_n = \sum_{i=n}^{N} b_i u_i$ where $\sum_{i=n}^{N} b_i = 1$, $b_i \geqq 0$, $n \leqq i \leqq N$ and $u_i = d(x_{p(i)}, x_{q(i)})$, which converges to $\mathcal{O}$ (in norm). For convenience, we can assume $s=1$. Since K is strongly normal, there exists $\delta > 0$ such that $\|v_n\| > \delta$ for sufficiently large n. Since $\{v_n\}$ converges (in norm) to $\mathcal{O}$, this is a contradition. Therefore $\{x_n\}$ is a Cauchy sequence. By completeness, there is a $u \in X$ such that $\{x_n\}$ converges to u in X. Since f is continuous on X, we obtain that $f(u)=u$. The uniqueness is obvious. This completes the proof.

If E is the set of all real numbers and if K is the set of all nonnegative reals, then, from (4) and (8), Theorem 1 may now be restated in the following form.

THEOREM 2. *Let (X, d) be a complete metric space, f a self-mapping of X such that for all $x, y \in X$.*

$$\text{(C)} \qquad d(fx, fy) \leqq \phi(d(x, y))\,,$$

where ϕ is upper semicontinuous from the right on P_1 (that is: $\limsup_{t\to c^+} \phi(t) \leqq \phi(c)$). Moreover, ϕ satisfies the condition (D).

$$\text{(D)} \qquad \phi(t) < t \qquad \text{for any } t \in P_1\,.$$

Then, f has a unique fixed point x_0 and $f^n x \to x_0$ for each x in X.

Theorem 2 was proved in [2] by Boyd D. W. and Wong J. S. W. but it is a special case of our Theorem 1.

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