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ON A CERTAIN MINIMAL IMMERSION OF A RIEMANNIAN MANIFOLD INTO A SPHERE

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Dedicated to Professor I. Mogi on his 60th birthday

Introduction. Minimal immersions of spheres into a sphere have been completely determined by do Carmo and Wallach [2]. Let $H^{r,n}$ be the space of all spherical harmonic polynomials of degree r on an n-dimensional sphere S^n, where $\dim H^{r,n}=(n+2r-1)(n+r-2)!/r!(n-1)! =: N(r)+1$. For an orthonormal basis $\{f_1, \cdots, f_{N(r)+1}\}$ of $H^{r,n}$, we define an immersion ι_r of S^n into an $(N(r)+1)$-dimensional Euclidean space $\boldsymbol{R}^{N(r)+1}$ by $\iota_r(x)=(f_1(x), \cdots, f_{N(r)+1})/(N(r)+1)^{1/2}$, which is called a *standard immersion*. Then the image by ι_r is contained in the unit sphere $S^{N(r)}(1)$ in $\boldsymbol{R}^{N(r)+1}$, and by means of a theorem of Takahashi [2] it is seen that ι_r is a minimal isometric immersion and $\iota_r(S^n)$ is not contained in the great sphere of $S^{N(r)}(1)$. With regard to the degree of the immersion in the sense of Wallach [9], they showed that the degree of ι_r is equal to r and if $r\leqq 3$, then ι_r is rigid.

On the other hand, Hong [3] introduced recently a notion of planar geodesic immersions. Let M and $\tilde{M}$ be complete connected Riemannian manifolds of dimension n and $n+p$, respectively. An isometric immersion ι of M into $\tilde{M}$ is called a *planar geodesic immersion* if each geodesic in M is mapped locally under the immersion into a 2-dimensional totally geodesic submanifold of $\tilde{M}$. Planar geodesic immersions of M into $S^{n+p}(c)$ have been classified by Little [5] and Sakamoto [8], independently, who stated that M is a compact symmetric space of rank one and the second fundamental form is parallel. The so-called Veronese manifold can be considered as one of examples determined by the planar geodesic immersion, while it can be regarded as the case of degree 2 in the ambient space.

When one pays attension to the rigidness of the standard immersion ι_r, it seems to be important to study the structure of the immersion with lower degree. As a matter of fact, the local version and the characterization of the Veronese manifold which is essentially an easiest model in our situation are investigated from variously different viewpoints. Furthermore, the local version concerning the immersion ι_3 of S^n into $S^{N(3)}$ has been treated by the author and Itoh [6]. In this paper, we shall be concerned with the characterization of the standard immersion ι_3 of S^n into $S^{N(3)}$. So as to do so, the notion of planar geodesic

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immersions is generalized. The purpose of this paper is to prove Theorem 8.1.

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1. Preliminaries. First of all, we recall the general theory of Riemannian submanifolds immersed into a Riemannian manifold to fix our notation and state several properties which are subsequently useful. We denote by $M^n(c)$ an $n(\geqq 3)$-dimensional connected Riemannian manifold of constant curvature c, which is called a real space form of constant curvature c. Let M be an $n(\geqq 3)$-dimensional connected and orientable Riemannian manifold with the Riemannian metric g and let ι be an isometric immersion of M into $\tilde{M}=M^{n+p}(c)$. When the argument is only considered in the local version, M need not be distinguished from $\iota(M)$. So, in order to simplify the discussion, we identify any point x in M with the point $\iota(x)$, and moreover any tangent vector u at x with the tangent vector $d\iota_x(u)$.

Now we choose an orthonormal local frame field $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ of $\tilde{M}$ in such a way that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and hence the others are normal to M. Restricting the canonical forms and the connection forms on $\tilde{M}$ with respect to these frames to M, we denote them by ω_A, ω_{AB}, where $A, B=1, \cdots, n, n+1, \cdots, n+p$. We then have

$$\omega_\alpha=0\,. \tag{1.1}$$

Here and in the sequel, we make use of the following convention on the range of indices:

$$\iota, \jmath, \cdots=1, \cdots, n\,,$$

$$\alpha, \beta, \cdots=n+1, \cdots, n+p\,.$$

The metric induced from the Riemannian metric in the ambient space $\tilde{M}$ under the immersion ι is given by $g=2\sum_\iota \omega_\iota\cdot\omega_\iota$. Then $\{e_1, \cdots, e_n\}$ are also the orthonormal frame field with respect to the induced metric, and $\{\omega_1, \cdots, \omega_n\}$ are the dual field with respect to $\{e_1, \cdots, e_n\}$. It follows from (1.1) and Cartan's lemma that we have

$$\omega_{\alpha\iota}=\sum_\jmath h^\alpha_{\iota\jmath}\omega_\jmath\,, \qquad h^\alpha_{\iota\jmath}=h^\alpha_{\jmath\iota}\,. \tag{1.2}$$

The quadratic form $\sum_{\iota,\jmath} h^\alpha_{\iota\jmath}\omega_\iota\omega_\jmath$ is called the *second fundamental form* of the immersion ι on M in the direction of e_α. At any point x of M, we denote by X and Y (resp. ξ and η) vector fields on a neighborhood of x tangent (resp. normal) to M. The second fundamental form σ of M can be written as

$$\sigma(X, Y)=\sum_{\alpha,\iota,\jmath} h^\alpha_{\iota\jmath}\omega_\iota(X)\omega_\jmath(Y)e_\alpha \tag{1.3}$$

for any vector fields X and Y. By means of the structure equations of the ambient space, we have the following structure equations of the submanifold M:

$$d\omega_\iota+\sum_\jmath \omega_{\iota\jmath}\wedge\omega_\jmath=0\,, \qquad \omega_{\iota\jmath}+\omega_{\jmath\iota}=0\,, \tag{1.4}$$

(1.5) $$d\omega_{ij}+\sum_k \omega_{ik}\wedge\omega_{kj}=\Omega_{ij},$$

$$\Omega_{ij}=-\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l,$$

where ω_{ij} (resp. Ω_{ij}) denotes the connection form (resp. the curvature form) on M. Moreover, we have

(1.6) $$d\omega_{\alpha\beta}+\sum_r \omega_{\alpha r}\wedge\omega_{r\beta}=\Omega_{\alpha\beta}, \qquad \Omega_{\alpha\beta}=-\frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k\wedge\omega_l,$$

where $\omega_{\alpha\beta}$ defines the connection induced in the normal bundle on M and $\Omega_{\alpha\beta}$ is called the normal curvature form on M. From (1.2) and (1.5) we have the equation of Gauss

(1.7) $$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+\sum_\alpha (h^\alpha_{il}h^\alpha_{jk}-h^\alpha_{ik}h^\alpha_{jl}).$$

For the second fundamental form σ the vector $\sigma(e_i, e_i)$ is called a *normal curvature vector* in the direction of a unit vector e_i. If every normal curvature vector has the same length for any unit vector u at x, then the immersion ι is said to be *isotropic* at x. If ι is isotropic at any point on M, namely, the length of a normal curvature vector depends only on the initial point, then the immersion is said to be *isotropic*. In particular, if the length is equal to λ, the immersion is said to be λ-isotropic, and the isotropy λ is a continuous function defined on M, the square of which is smooth. The immersion ι is λ -isotropic at a point x if and only if the second fundamental form satisfies

(1.8) $$\mathfrak{S}_3\langle\sigma(u_1, u_2), \sigma(u_3, v)\rangle=\lambda^2\mathfrak{S}_3\langle u_1, u_2\rangle\langle u_3, v\rangle,$$

where u_i ($i=1, 2, 3$) and v are unit vectors at x and $\mathfrak{S}_m$ denotes the cyclic sum with respect to vectors $u_1, \cdots, u_m$. The condition is equivalent to

(1.9) $$\langle\sigma(u, u), \sigma(u, v)\rangle=0$$

for any orthonormal vectors u and v at x [7].

Now, we denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M (resp. $\tilde{M}$). For any vector fields X and $Y=\sum_i y_i e_i$, we have

(1.10) $$\nabla_X Y=\sum_k\Big\{dy_k(X)-\sum_i y_i\omega_{ik}(X)\Big\}e_k,$$

(1.11) $$\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X, Y)=\nabla_X Y+\sum_{\alpha,i,j} h^\alpha_{ij}\omega_i(X)\omega_j(Y)e_\alpha.$$

For any vector fields ξ and X on a neighborhood of x in M, we may express

(1.12) $$\tilde{\nabla}_X\xi=-H_\xi X+\nabla^\perp_X\xi,$$

where $-H_\xi X$ and $\nabla^\perp_X\xi$ denote the components of $\tilde{\nabla}_X\xi$ tangent and normal to M,

respectively. Then $\nabla^{\perp}_X$ becomes the covariant differentiation with respect to the induced connection in the normal bundle $N(M)$. The tangent component $H_\xi X$ is related to the second fundamental form σ as follows: $\langle H_\xi X, Y\rangle=\langle\sigma(X, Y), \xi\rangle$ for any vector field Y. Then H_ξ is a symmetric transformation of the tangent space of M. In particular, we have

$$\tilde{\nabla}_X e_\alpha=-\sum_k \omega_{\alpha k}(X)e_k-\sum_\beta \omega_{\alpha\beta}(X)e_\beta\,. \tag{1.13}$$

We define the covariant derivative h^α_{ijk} of h^α_{ij} by

$$\sum_k h^\alpha_{ijk}\omega_k=dh^\alpha_{ij}-\sum_l h^\alpha_{lj}\omega_{il}-\sum_l h^\alpha_{il}\omega_{lj}-\sum_\beta h^\beta_{ij}\omega_{\beta\alpha}\,. \tag{1.14}$$

Since $\tilde{M}$ is of constant curvature, we get

$$h^\alpha_{ijk}-h^\alpha_{ikj}=0\,.$$

Now, suppose that $h^\alpha_{i_1\cdots i_m}$ are defined for some $m\geqq 2$. Inductively we define the covariant derivative $h^\alpha_{i_1\cdots i_m j}$ of $h^\alpha_{i_1\cdots i_m}$ by

$$\sum_j h^\alpha_{i_1\cdots i_m j}\omega_j=dh^\alpha_{i_1\cdots i_m}-\sum_{r=1}^{m}\sum_j h^\alpha_{i_1\cdots i_{r-1}ji_{r+1}\cdots i_m}\omega_{ji_r}-\sum_\beta h^\beta_{i_1\cdots i_m}\omega_{\beta\alpha}\,. \tag{1.15}_m$$

From the equation above we have the following Ricci formula:

$$\begin{aligned}
&h^\alpha_{i_1\cdots i_m jk}-h^\alpha_{i_1\cdots i_m kj}\\
&=c\sum_{r=1}^{m}(h^\alpha_{i_1\cdots i_{r-1}ji_{r+1}\cdots i_m}\delta_{i_rk}-h^\alpha_{i_1\cdots i_{r-1}ki_{r+1}\cdots i_m}\delta_{i_rj})\\
&+\sum_{r=1}^{m}\sum_{\beta,l}h^\alpha_{i_1\cdots i_{r-1}li_{r+1}\cdots i_m}(h^\beta_{i_rk}h^\beta_{lj}-h^\beta_{i_rj}h^\beta_{lk})\\
&+\sum_{\beta,l}h^\beta_{i_1\cdots i_m}(h^\alpha_{lk}h^\beta_{lj}-h^\alpha_{lj}h^\beta_{lk})\,.
\end{aligned} \tag{1.16}_m$$

We define the covariant differentiation ∇' on the Whitney sum $T(M)\oplus N(M)$ as follows: For any $N(M)$-valued tensor field T of type $(0, k)$, we define

$$(\nabla'_X T)(Y_1, \cdots, Y_k)=\nabla^{\perp}_X(T(Y_1, \cdots, Y_k))-\sum_{r=1}^{k}T(Y_1, \cdots, \nabla_X Y_r, \cdots, Y_k) \tag{1.17}$$

and $\nabla' T$ ia also defined by $(\nabla' T)(Y_1, \cdots, Y_k, X)=(\nabla'_X T)(Y_1, \cdots Y_k)$, which is an $N(M)$-valued tensor field of type $(0, k+1)$. We denote by $\nabla'^2 T$ the covariant derivative of $\nabla' T$. Furthermore, we can inductively define a covariant derivative $\nabla'^m T$. Then, for the second fundamental form σ, it follows

$$\begin{aligned}
&(\nabla'^m\sigma)(Y, Z, X_1, \cdots, X_m)\\
&\quad=\sum_{\alpha,j,k,i_1,\cdots,i_m}h^\alpha_{jki_1\cdots i_m}\omega_j(Y)\omega_k(Z)\omega_{i_1}(X_1)\cdots\omega_{i_m}(X_m)e_\alpha\,.
\end{aligned} \tag{1.18}$$

2. Degrees of isometric immersions. In this section, let ι be an isometric immersion of M into $\tilde{M}=M^{n+p}(c)$. We consider the decomposition of the normal bundle of the immersion. For any point x in M the normal space of M is simply denoted by N_x. Namely, we have $N_x=(d\iota_x(M_x))^N$, where $(.)^N$ means taking the orthogonal complement into the tangent space $T_{\iota(x)}(\tilde{M})$ in the ambient space. The second fundamental form σ of ι at x is a bilinear map of $M_x\times M_x$ into N_x defined by $\sigma(X, Y)=(\tilde{\nabla}_X Y)^N$, where X and Y are vector fields on a neighborhood of x in M. For convenience' sake, we put $\sigma_{2x}=\sigma_x$, and so σ_{2x} can be regarded as a linear map of a symmetric square $S^2(M_x)$ of M_x into the normal space N_x. We set $N_x^1:=\sigma_{2x}(S^2(M_x))$, which is called a *first normal space* of ι at x. This means that N_x^1 is the linear subspace of N_x spanned by normal vectors $\sigma(u, v)$ for any vectors u and v at x, so we see that $\dim N_x^1\leqq {}_nH_2=n(n+1)/2$. In particular, if ι is minimal, then $\dim N_x^1\leqq n(n+1)/2-1$. The point x of M is said to be 2-*regular* if N_x^1 is of maximal dimension with respect to basic points. Let M_2 be the set consisting of all 2-regular points of M. Then M_2 is open in M. For any point x in M_2, we set $O_x^2:=d\iota_x(M_x)\oplus N_x^1$, which is called the *second osculating space* of ι at x. Now, for the 2-regular point x we define a trilinear map σ_{3x} of $M_x\times M_x\times M_x$ into $(O_x^2)^N$ by $\sigma_{3x}(X, Y, Z):=(\tilde{\nabla}_X(\sigma_{2x}(Y, Z)))^{N_2}$ for any tangent vecor fields X, Y and Z, where $(.)^{N_2}$ denotes the orthogonal projection into $(O_x^2)^N$. σ_{3x} is well defined and it is symmetric, because $\tilde{M}$ is of constant curvature, so it induces a linear map $\sigma_{3x}: S^3(M_x)\to(O_x^2)^N$ of the symmetric third power of M_x into $(O_x^2)^N$. We call σ_{3x} the *third fundamental form* of ι at x and the linear subspace $N_x^2:=\sigma_{3x}(S^3(M_x))$ is called the *second normal space* of ι at x. The second normal space N_x^2 at x is the orthogonal projection of the linear subspace spanned by $\nabla'\sigma(v, w, u)$ for any vectors u, v and w at x, so $\dim N_x^2\leqq {}_nH_3=n(n+1)(n+2)/6$. In particular, if ι is minimal, then $\dim N_x^2\leqq n(n+1)(n+2)/6-n$. The point x in M_2 is said to be 3-*regular* if N_x^2 is of maximal dimension with respect to basic points. Let M_3 be the set consisting of all 3-regular points of M_2. Then M_3 is also open in M_2. We put $O_x^3:=O_x^2\oplus N_x^2$, which is called the *third osculating space* of ι at x.

We now proceed inductively and suppoese that the $(j-1)$-th osculating space O_x^{j-1} of ι at the $(j-1)$-regular point x is defined. Then we see that it is possible to define a linear map σ_{jx} of the symmetric j-th power of M_x into $(O_x^{j-1})^N$ by

$$\sigma_{jx}(X_1, \cdots, X_j):=(\tilde{\nabla}_{X_1}(\sigma_{j-1x}(X_2, \cdots, X_j)))^{N_{j-1}}$$

for any vector fields $X_1, \cdots, X_j$, where $(.)^{N_{j-1}}$ denotes the orthogonal projection into $(O_x^{j-1})^N$. We call σ_{jx} the *j-th fundamental form* of ι at x, $N_x^{j-1}:=\sigma_{jx}(S^j(M_x))$ the *(j−1)-th* normal space of ι at x, and $O_x^j:=O_x^{j-1}\oplus N_x^{j-1}$ the *j-th osculating* space of ι at x. Clearly, the process must be eventually stopped, because of $\dim O_x^j\leqq \dim \tilde{M}$. Thus there exists a first integer q for which $\sigma_j=0$ for $j>q$ and $\sigma_q\neq 0$. Then q is called the *degree* of ι and the set M_q is open in M.

We shall be here concerned with the decomposition above of Riemannian homogeneous spaces and then standard immersions of Riemannian symmetric spaces into a sphere. Let $M=G/K$ be a Riemannian homogeneous space, where

G is a compact connected Lie group which acts effectively on M and K is a closed subgroup of G, and let $\langle , \rangle$ be the G-invariant Riemannian metric on M. The isometric immersion ι of M into $\tilde{M}$ is said to be *equivalent*, if there is a continuous group homomorphism $\rho : G \to I(\tilde{M})$ such that $\iota(gx)=\rho(g)\iota(x)$ for any element g in G and any point x in M, where $I(\tilde{M})$ denotes the group of isometries of $\tilde{M}$. Assume that ι is equivalent. It follows from the equvalence of ι that the dimension of the j-th normal space N_x^j is independent of the basic point x, so M_j coincides with M itself. We put $N_x^0=d\iota_x(M_x)$ for convenience' sake. This gives a decomposition of the restriction $T_M(\tilde{M})$ to $\iota(M)$ of the tangent bundle $T(\tilde{M})$ of $\tilde{M}$ into the Whitney sum $T_M(\tilde{M})=\sum_{j=0}^{q-1} N^j \oplus (O^q)^N$, where the fiber of N^j at x is N_x^j and the fiber of $(O^q)^N$ at x is $(O_x^q)^N$. Let o be the origin of $M=G/K$. The linear isotropy representation is the homomorphism of K into the group of linear transformations of $T_o(M)$ which assign to each $k \in K$ the differential of k at o. We denote by K^* the image of K by the linear isotropy representation, which is called the linear isotropy group at o. Then we remark that the action of K^* on $M_o=T_o(M)$ may be extended to $S^j(M_o)$ by the usual tensor action. Consequently, we have $\sigma_j(ku)=k\sigma_j(u)$ for any element u in $S^j(M_o)$, k in K, and $j=2, \cdots, q$, by means of the equivalence of ι. This shows that we have $\|\sigma_2(u, u)\|=\|\sigma_2(ku, ku)\|$ for any vector u in M_o. Thus the immersion ι is constant isotropic, because of the equivalence of ι.

In particular, let $M=G/K$ be a compact symmetric space where the isotropy aciton of K is irreducible, and let Δ be the Laplace operator for $(M, \langle , \rangle)$, where $\langle , \rangle$ is some G-invariant Riemannian structure up to scalar multilple. For any $\lambda \in R$, let V_λ be the eigenspace with the eigenvalue λ of Δ, and for any real valued functions f_1 and f_2 on M, let $(f_1, f_2) := \int_M f_1 f_2 dM$. Then V_λ is a vector space over R endowed with the inner product $(\ ,\ )$. Let $p_\lambda+1 := \dim V_\lambda$. For each non-zero eigenvalue λ, let $f_1, \cdots f_{p_\lambda+1}$ be an orthonormal basis of V_λ. We define $\iota_\lambda : M \to R^{p_\lambda+1}$ by

$$\iota_\lambda(x)=(f_1(x), \cdots, f_{p_\lambda+1}(x))/(p_\lambda+1)^{1/2}.$$

Then it is seen in [9] that ι_λ is a minimal isometric immersion of $(M, \langle , \rangle)$ into a p_λ-dimensional sphere $S^{p_\lambda}(1)$ of constant curvature 1. We call such ι_λ the *standard immersion* of M. By a theorem of Wallach [9], if the degree of the standard immersion ι_λ is less than or equal to 3, then ι_λ is rigid. Evidently, a rigid immersion of a homogeneous space is equivalent.

3. Cubic geodesic immersions. Let ι be an isotropic immersion of an n-dimensional Riemannian manifold M into an $(n+p)$-dimensional real space form $\tilde{M}=M^{n+p}(c)$ of constant curvature c, where $p \geqq 2$. For a regular curve $C : (a, b) \to M$, if there exist an open interval I_t of each t in (a, b) and a 3-dimensional totally geodesic submanifold M_t in $\tilde{M}$ such that $I_t \subset (a, b)$ and $\iota(C(I_t)) \subset M_t$, then the curve is said to be locally *cubic*. The immersion ι is called a *cubic geodesic*

immersion, if any geodesic in M is locally cubic. This can be regarded as the generalization of the concept of planar geodesic immersions, though it seems to be quite probable that isotropicness is the superfluous condition. In particular, a cubic geodesic immersion which is not planar geodesic is said to be *proper*.

Now, we replace isotropicness in the condition of cubic geodesic immersions with another one related much more to cubic curves themselves. From now on, we assume that the isometric immersion of M into $\tilde{M}=M^{n+p}(c)$ satisfies only the condition that each geodesic on M is locally a cubic curve. For any point x and any unit vector u at x, let γ_u be a geodesic in M passing through $x=\gamma_u(0)$ with the initial vector $\gamma_u'(0)=u$ and parametrized by arc length s, which is called a *normal* geodesic. Let $X(s)=\gamma_u'(s)$. Then (1.11) is reduced to

$$\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X, Y)=\nabla_X Y+\sum_{\alpha, i, j} h_{ij}^{\alpha}\omega_i(X)\omega_j(Y)e_\alpha . \tag{3.1}$$

It follows from (1.12) and (1.17) that we have

$$\begin{aligned}\tilde{\nabla}_X(\sigma(Y, Z))=&(\nabla'\sigma)(Y, Z, X)+\sigma(\nabla_X Y, Z)+\sigma(Y, \nabla_X Y)\\ &-\sum_i\langle\sigma(Y, Z), \sigma(X, e_i)\rangle e_i\end{aligned} \tag{3.2}$$

for any vector fields Y and Z along γ_u tangent to M. Since γ_u is locally contained in a 3-dimensional totally geodesic submanifold M_0 in $\tilde{M}$, local vector fields X, $\sigma(X, X)$ and $\tilde{\nabla}_X(\sigma(X, X))$ are tangent to M_0 at $\gamma_u(s)$ for a sufficiently small s.

LEMMA 3.1. *The immersion ι is cubic geodesic if and only if the orthogonal complement of the vector $\gamma_u'(0)$ in the tangent space T_xM_0 is a subspace of the normal space N_x for any point x and any unit vector u at x.*

Proof. We suppose that the orthogonal complement of T_xM_0 is contained in N_x and then it suffices to show that ι is isotropic. We may suppose that $\sigma(u, u)\neq 0$ for a unit vector u. For a normal geodesic $\gamma_u: (-a, a)\to M$, there is a sufficiently small interval I of 0 in $(-a, a)$ such that $\iota\circ\gamma_u|I$ is contained in a 3-dimensional totally geodesic submanifold M_0 and moreover we have $\sigma(X, X)\neq 0$, where $X(s)=\gamma_u'(s)$. Accordingly, there exists a unit vector ξ_1 along $\gamma_u|I$ such that $\sigma(X, X)=k\xi_1$, where $k(s)=\|\sigma(X, X)\|$ is a positive smooth function. Since $\tilde{\nabla}_X(\sigma(X, X))$ is tangent to M_0, it is spanned by three mutually orthogonal vector fields X, ξ_1 and ξ along $\gamma_u|I$. Namely,

$$\tilde{\nabla}_X(\sigma(X, X))=fX+g\xi_1+\xi , \tag{3.3}$$

where f and g are smooth functions. It follows from (3.2), (3.3) and the assumption that we have $\langle\sigma(X, X), \sigma(X, Y)\rangle+f\langle X, Y\rangle=0$ along $\gamma_u|I$, which yields $\langle\sigma(u, u), \sigma(u, v)\rangle=0$ for any orthonormal vectors u and v at x. Thus ι is isotropic, because of (1.9).

Conversely, since ι is λ-isotropic, it follows from (3.2) that (3.3) is reduced to $\nabla'\sigma(X, X, X)-\lambda^2X=fX+g\xi_1+\xi$, which implies $\langle\xi, Y\rangle=0$ for any vector field Y orthogonal to X. Therefore ξ is normal to M. Q. E. D.

We shall here give an important example of a proper cubic geodesic immersion. We are concerned with the standard immersion ι_3 of $S^n(1)$ into $S^{N(3)}(c)$ mentioned in the introduction, where $N(3)+1=(n+5)(n+1)!/3!(n-1)!$ and $c=n/3(n+2)$. Then it is seen in [2, 9] that the degree of ι_3 is equal to 3. For orthonormal elements u and v of $S^n(1)$ in $\boldsymbol{R}^{n+1}$, we identify v with a unit tangent vector at u. Then $\gamma: t\to u\cos t+v\sin t$ is a geodesic in $S^n(1)$ with the initial vector v and moreover, by the construction of the immersion, we have

$$\iota_3(\gamma(t))=u_1\cos t+v_1\sin t+u_3\cos 3t+v_3\sin 3t\,,$$

where $u_\imath$ and $v_\imath$ ($\imath=1, 3$) are fixed elements in $\boldsymbol{R}^{N(3)+1}$. This means that the curve $\iota_3\circ\gamma$ is contained in the 4-dimensional linear subspace spanned by the vectors $u_\imath$ and $v_\imath$ in $\boldsymbol{R}^{N(3)+1}$, and therefore every geodesic in $S^n(1)$ is mapped in a 3-dimensional totally geodesic submanifold of $S^{N(3)}(c)$ under the immersion ι_3. It implies that ι_3 is properly cubic geodesic.

4. Properties of cubic geodesic immersions. In this section, let ι be a cubic geodesic immersion of M into $\tilde{M}=M^{n+p}(c)$. By using notations in the previous section, general properties of cubic geodesic immersions are studied. We denote by $\kappa(s)$ the isotropy at the point $\gamma_u(s)$. Namely, $k(s)=\|\sigma(X, X)\|$, the square of which is a smooth function along γ_u. We suppose that k is positive. Then k is also smooth and there exists a unit vector field ξ_1 along γ_u normal to M such that

$$\sigma(X, X)=k\xi_1\,. \tag{4.1}$$

Since $\tilde{\nabla}_X\xi_1$ is also tangent to M_o, there is a vector field ξ normal to M and mutually orthogonal to X and ξ_1 such that

$$\tilde{\nabla}_X\xi_1=-kX+\xi\,. \tag{4.2}$$

By means of (3.2), (4.1) and (4.2) we have

$$(Xk)\xi_1+k\xi=\nabla'\sigma(X, X, X)\,, \tag{4.3}$$

because ι is k-isotropic. By virtue of (1.12) and (1.17), we get

$$\begin{aligned}\tilde{\nabla}_X(\nabla'\sigma(Y, Z, W))=&\nabla'^2\sigma(Y, Z, W, X)+\nabla'\sigma(\nabla_XY, Z, W)\\&+\nabla'\sigma(Y, \nabla_XZ, W)+\nabla'\sigma(Y, Z, \nabla_XW)\\&-\sum_\imath\langle\nabla'\sigma(Y, Z, W), \sigma(X, e_\imath)\rangle e_\imath\,.\end{aligned} \tag{4.4}$$

Differentiating (4.3) in the direction of X and using (4.4), we have

$$(4.5)\qquad -k(Xk)X+(X^2k)\xi_1+2(Xk)\xi+k\tilde{\nabla}_X\xi$$
$$=\nabla'^2\sigma(X, X, X, X)-\sum_\iota\langle\nabla'\sigma(X, X, X), \sigma(X, e_\iota)\rangle e_\iota .$$

Since X, ξ_1 and ξ are mutually orthogonal, $\tilde{\nabla}_X\xi$ is orthogonal to X, and therefore it is only spanned by ξ_1 and ξ, because $\tilde{\nabla}_X\xi$ is also tangent to M_0. It implies together with (4.5) that we have

$$(4.6)\qquad 2\langle\nabla'\sigma(X, X, X), \sigma(X, Y)\rangle=(Xk^2)\langle X, Y\rangle .$$

Making use of this equation, we have

LEMMA 4.1. *ι is constant isotropic.*

Proof. We may suppose that there is a point x such that the isotropy k at the point is positive. Then we can choose a normal coordinate neighborhood U of x in M such that $k>0$ on U. For any unit vectors u and v at x, let γ_u and γ_v be normal geodesics in U passing through $x=\gamma_u(0)=\gamma_v(0)$. Let $X=\gamma_u'$ and $Y=\gamma_v'$. Differentiating $\langle\sigma(X, X), \sigma(X, Z)\rangle=k^2\langle X, Z\rangle$ obtained by (1.8) in the direction of X and using (3.2), we have $\langle\nabla'\sigma(X, X, X), \sigma(X, Z)\rangle+\langle\sigma(X, X), \nabla'\sigma(X, X, Z)\rangle=(Xk^2)\langle X, Z\rangle$ for any vector field Z along γ_u, because $\nabla'\sigma$ is symmetric. On the other hand, differentiating $\|\sigma(W, W)\|^2=k^2$ in the direction of Y, we have $2\langle\sigma(W, W), \nabla'\sigma(W, W, Y)\rangle=Yk^2$ for any vector field W along γ_v. Combining these two equations with (4.6), we have $Yk^2=(Xk^2)\langle u, v\rangle$ at the point x. This means that k^2 is constant along any geodesic γ_v, and therefore it is constant on U. Consequently, if there exists a point x such that $k(x)>0$, then k is constant on M. Q. E. D.

By means of this lemma, we may assume that the isotropy k is positive. Now, taking account of (4.2), we see that ι is the planar geodesic immersion, provided that ξ vanishes identically. Therefore, we may suppose that there exist a point x and a unit vector u at x so that the normal vector field ξ defined by (4.2) is not a zero vector at x. It follows from the continuity of ξ that we may locally suppose that ξ has no zero points along the geodesic passing through x with the direction of u. We put $l=\|\xi\|$ and $\xi_2=\xi/l$. Then the function l is smooth along γ_u, and it is easily seen that X, ξ_1 and ξ_2 are the Frenet frames for γ_u which satisfy the Frenet formulas. Making use of (4.3), (4.5) and (4.6), we have

$$(4.7)\qquad kl\xi_2=\nabla'\sigma(X, X, X),$$

$$(4.8)\qquad \langle\nabla'\sigma(X, X, X), \sigma(X, Y)\rangle=0,$$

$$(4.9)\qquad l^3\sigma(X, X)-(Xl)\nabla'\sigma(X, X, X)+l\nabla'^2\sigma(X, X, X, X)=0$$

for any vector field Y along γ_u. Moreover, since ι is constant isotropic, we have

$$(4.10)\qquad \langle\sigma(X, X), \nabla'\sigma(X, X, Y)\rangle=0 .$$

For any fixed point x we can regard l as the function on the tangent unit sphere S_x at x. Concerning this function, we find

LEMMA 4.2. *l is independent of any unit vector at x.*

Proof. We suppose that there exists a unit vector u in S_x such that $l(u)>0$. We take a normal coordinate neighborhood U of x. Differentiating (4.8) in the direction of X and making use of (4.4), (4.8) and (4.9), we get

$$\langle\nabla'\sigma(X, X, X), \nabla'\sigma(X, X, Y)\rangle=(kl)^2\langle X, Y\rangle \tag{4.11}$$

for any vector field Y.

In spite of the value of the function l, as a direct consequence of (4.3) and the equation above, we have $\langle\nabla'\sigma(v, v, v), \nabla'\sigma(v, v, w)\rangle=0$ for any orthonormal vectors v and w. We put $v_1=av+bw$, $w_1=bv-aw$ and $v_2=bv+aw$, $w_2=av-bw$, where a and b are real numbers such that $a^2+b^2=1$. Then pairs (v_1, w_1) and (v_2, w_2) are both orthonormal, and so we have $\langle\nabla'\sigma(v_i, v_i, v_i), \nabla'\sigma(v_i, v_i, w_i)\rangle=0$ $(i=1, 2)$. Summing up these equations, we obtain

$$\begin{aligned} &ab(a^4+b^4)(\|\nabla'\sigma(v, v, v)\|^2-\|\nabla'\sigma(w, w, w)\|^2)-ab(a^4-4a^2b^2+b^4)\\ &\quad\cdot\{3(\|\nabla'\sigma(v, v, w)\|^2-\|\nabla'\sigma(v, w, w)\|^2)+2(\langle\nabla'\sigma(v, v, v), \nabla'\sigma(v, w, w)\rangle\\ &\quad-\langle\nabla'\sigma(w, w, w), \nabla'\sigma(w, v, v)\rangle)\}=0. \end{aligned}$$

If we put $a^2=(3+\sqrt{3})/6$ and $b^2=(3-\sqrt{3})/6$, then this yields $\|\nabla'\sigma(v, v, v)\|^2=\|\nabla'\sigma(w, w, w)\|^2$ By means of (4.3) and (4.7), we have $(kl(v))^2=(kl(w))^2$, which means that

$$l(v)=l(w). \tag{4.12}$$

For any vector v in S_x, let $V(v)$ be a subset in S_x consisting of vectors w such that $l(v)=l(w)$, and let L be a subspace of M_x spanned by $e_2, \cdots, e_n$ for any orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ in M_x such that $e_1=v$. Then (4.12) shows that the intersection of L and S_x is contained in $V(v)$. For any vector w in S_x which does not belong to L, there exist two orthonormal vectors v_1 and v_2 in L such that $w=av_1+bv_2$, because of $\dim M\geqq 3$. Since w is also orthogonal to v_2, $l(w)=l(v_2)$, and therefore $l(v_2)=l(v)$. Thus we see that w belongs to $V(v)$, which means that the subset $V(v)$ coincides with S_x itself. Accordingly, the function l is constant on S_x. Q. E. D.

Now, let ϕ be an m-form on a vector space V. For any vectors $v_1, \cdots, v_m$ and any permutation τ in a symmetric group S_m of order m, we put $\tau\phi(v_1, \cdots, v_m)=\phi(v_{\tau(1)}, \cdots, v_{\tau(m)})$and we define the symmetrizer $\mathcal{S}_m$ by $\mathcal{S}_m\phi=\sum_{\tau\in S_m}\tau\phi$. By means of Lemma 4.2, (4.7) is true for any unit vector at a fixed point x, and therefore it follows from (4.7), (4.8), (4.10) and the linearity of the forms σ and $\nabla'\sigma$ that we have the following equations at x;

$$\mathcal{S}_6\langle\nabla'\sigma(u_1, u_2, u_3), \nabla'\sigma(u_4, u_5, u_6)\rangle=(kl)^2\mathcal{S}_6\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle, \tag{4.7)'}$$

(4.8)′ $$\mathfrak{S}_4\langle\nabla'\sigma(u_1, u_2, u_3), \sigma(u_4, v)\rangle=0,$$

(4.10)′ $$\mathfrak{S}_4\langle\sigma(u_1, u_2), \nabla'\sigma(u_3, u_4, v)\rangle=0.$$

Since $\nabla'\sigma$ is symmetric with respect to all elements, (4.8)′ is reduced to

(4.8)″ $$\mathfrak{S}_4\langle\nabla'\sigma(u_1, u_2, u_3), \sigma(u_4, v)\rangle=0.$$

Combining (4.10)′ with (4.8)″, we have

(4.13) $$\langle\sigma(v, w), \nabla'\sigma(u_1, u_2, u_3)\rangle=\mathfrak{S}_3\langle\nabla'\sigma(v, w, u_1), \sigma(u_2, u_3)\rangle.$$

We define here a helix in an m-dimensional Riemannian manifold N. Let $C: c=c(s)$ be a regular curve in N, where s is the arc length. The curve C is said to be of *order* r ($\leqq m$), if it has the Frenet frames $\{c(s), \xi_0, \cdots, \xi_r\}$ along C and the following Frenet formulas along C are satisfied:

(4.14) $$\begin{cases}\dfrac{dc}{ds}=\xi_0=: X,\\ \nabla_X\xi_j=-k_j(s)\xi_{j-1}+k_{j+1}(s)\xi_{j+1}\end{cases}$$

for $j=0, \cdots, r$, where ∇ denotes the covariant differentiation in N and $k_0(s)=k_{j+1}(s)=0$ and $k_j(s)$ ($j=1, \cdots, r$) is positive along C. Then $k_j(s)$ (resp. ξ_j) is called the *j-th curvature* (resp. the *j-th normal vector*) of C. For example, a geodesic is a curve of order 0 and, a circle is a curve of order 1 and moreover the first curvature k_1 is constant. A curve C is called a *helix* if C is of order 2 and k_1 and k_2 are both constant along C. In particular, ξ_1 is called a *principal vector* and ξ_2 is called a *binormal vector*. Let $C: c=c(s)$ be a helix. Then the components satisfy a system of differential equations, because of the Frenet formulas for C. According to the fundamental theory of differential equations, we see that there exists a unique solution satisfying the given initial condition in a sufficiently small interval of $s=0$. Namely, for any point p in N and any orthonormal vectors u, v and w at p, there exists locally a helix passing through p with a tangent vector u, which satisfies certain conditions.

Now, coming back to our situation, it follows from (4.1) and the Frenet formulas that any geodesic γ_u in M is at most of order 2, and k and l are the first and second curvature of γ_u, respectively. Moreover, k is constant along γ_u. Concerning with geodesics in M, we have

PROPOSITION 4.3. *If a cubic geodesic immersion ι of M into $\tilde{M}=M^{n+p}(c)$ is proper, then any geodesic in M is a helix in the ambient space $\tilde{M}$.*

Proof. In order to prove this property, it suffices to show that the second curvature l along any geodesic in M is positive constant. Since ι is not planar geodesic, we may suppose that there exists a point x such that $l(x)>0$. For any unit vectors u and v at x, let γ_u and γ_v be normal geodesics passing through

$x=\gamma_u(0)=\gamma_v(0)$ with the tangent vector $u=\gamma'_u(0)$, $v=\gamma'_v(0)$, respectively. Then we take a normal coordinate neighborhood U of x in M, on which l is positive. Differentiating (4.11) in the direction of $X=\gamma'_u$, we have

$$\langle\nabla'^2\sigma(X, X, X, X), \nabla'\sigma(X, X, Y)\rangle+\langle\nabla'\sigma(X, X, X), \nabla'^2\sigma(X, X, Y, X)\rangle$$
$$=k^2(Xl^2)\langle X, Y\rangle .$$

Again, differentiating (4.7)′ with $u_\iota=W$ ($\iota=1, \cdots, 6$) in the direction of $Z=\gamma'_v$, we get

$$2\langle\nabla'\sigma(W, W, W), \nabla'^2\sigma(W, W, W, Z)\rangle=k^2(Zl^2) .$$

Since the Ricci formula $(1.16)_2$ shows $\nabla'^2\sigma(W, W, W, Z)-\nabla'^2\sigma(W, W, Z, W)$ can be expressed as a linear combination of vectors $\sigma(W, \ )$, the above equations and (4.8) give

$$2\langle\nabla'^2\sigma(u, u, u, u), \nabla'\sigma(u, u, v)\rangle=k^2(2Xl^2\langle u, v\rangle-Zl^2)$$

at x.

On the other hand, combining with (4.9), (4.10) and (4.11), we have $l^2\langle\nabla'^2\sigma(u, u, u, u), \nabla'\sigma(u, u, v)\rangle=(kl)^2Xl^2\langle u, v\rangle/2$ at x. Thus we see

$$(kl)^2(Xl^2\langle u, v\rangle-Zl^2)=0 ,$$

which implies l is constant along γ_v. This means that we can prove the fact that the second curvature is constant on U. The assertion is thereby proved.

Q. E. D.

Remark. Helices in a Riemannian manifold are studied by Ikawa [4].

5. Degrees of cubic geodesic immersions. As is stated in § 3, it is seen that the degree of the standard immersion ι_3 of $S^n(1)$ into $S^{n+p}(c)$ is equal to 3. In this section, we shall be concerned with degrees of cubic geodesic immersions of M into $\tilde{M}=M^{n+p}(c)$. First of all, let ι be only an isometric immersion of M into $\tilde{M}$, and we assume that every normal goedesic $\gamma: c=c(s)$ in M is a helix in the ambient space. Then the Frenet formulas for γ are given by

$$\text{(5.1)}\qquad \begin{cases} \dfrac{dc}{ds}=: X , \\ \tilde{\nabla}_X X=k\xi_1 , \\ \tilde{\nabla}_X\xi_1=-kX+l\xi_2 , \\ \tilde{\nabla}_X\xi_2=-l\xi_1 , \end{cases}$$

where $(c(s), X, \xi_1, \xi_2)$ is the Frenet frames for γ and k and l are positive constant along γ. It follows from (1.11) and the second equation of (5.1) that we have $\sigma(X, X)=k\xi_1$. Furthermore, by means of (3.2), this equation and the third

equation of (5.1), we get

$$-k^2X+kl\xi_2=\nabla'\sigma(X, X, X)-\sum_i\langle\sigma(X, X), \sigma(X, e_i)\rangle e_i . \tag{5.2}$$

LEMMA 5.1. *The immersion ι is isotropic if and only if the binormal vector ξ_2 is normal to M.*

Proof. If the immersion ι is isotropic, then the property of $\sigma(X, X)$ implies that the isotropy of ι is equal to k. Then it follows from (1.8) that we have $k^2X=\sum_i\langle\sigma(X, X), \sigma(X, e_i)\rangle e_i$, because X is a unit vector field along γ, from which implies, together with (5.2), that we have $\nabla'\sigma(X, X, X)=kl\xi_2$. Consequently, ξ_2 is normal to M.

Conversely, if ξ_2 is normal to M, then (5.2) yields $\langle\sigma(u, u), \sigma(u, v)\rangle=0$ for any orthonormal vectors u and v at any point x, which asserts that the immersion ι is isotropic. Q. E. D.

We suppose that ι is a k-isotropic immersion. For a normal geodesic γ_u passing through $x=\gamma_u(0)$ we can make use of many equations obtained in the preceding section, where k and l are positive constant along γ_u. Thus, (4.9) means

$$\nabla'^2\sigma(X, X, X, X)+l^2\sigma(X, X)=0 \tag{5.3}$$

along γ_u. This means that we have $\nabla'^2\sigma(u, u, u, u)+l^2\sigma(u, u)=0$ for any unit vector u at x, so we see $\mathcal{S}_4\{\nabla'^2\sigma(u_1, u_2, u_3, u_4)+l^2\langle u_1, u_2\rangle\sigma(u_3, u_4)\}=0$, because the forms $\nabla'^2\sigma$ and σ are both linear. By taking account of the fact that the form ∇'^2 is symmetric with respect to the first three elements, the equation is reduced to

$$3!\mathfrak{S}_4\nabla'^2\sigma(u_1, u_2, u_3, u_4)+(2!)^2l^2\mathfrak{S}_3\{\langle u_4, u_1\rangle\sigma(u_2, u_3)+\langle u_1, u_2\rangle\sigma(u_3, u_4)\}=0 ,$$

which gives, together with the property of isotropic immersions and the Ricci formula $(1.16)_2$,

$$\begin{aligned}6\nabla'^2\sigma(u_1, u_2, u_3, v)=\mathfrak{S}_3\{&(-3c+3k^2-l^2)\langle u_1, u_2\rangle\sigma(u_3, v)\\ &+(3c+3k^2-l^2)\langle v, u_1\rangle\sigma(u_2, u_3)-6\sum_i\langle\sigma(e_i, v), \sigma(u_1, u_2)\rangle\sigma(e_i, u_3)\} .\end{aligned} \tag{5.4}$$

This shows that the 4-th fundamental form vanishes identically on M. In other words, the third normal space N_x^3 is trivial. Thus, by the definition of the degree of the immersion, (5.4) says the degree of ι is not greater than 3. On the other hand, by (5.2), $\nabla'\sigma(u, u, u)$ is proportional to the binormal vector ξ_2 which is orthogonal to $\sigma(u, u)=k\xi_1$. Consequently, the second normal space N_x^2 is not empty, which proves the following

PROPOSITION 5.2. *Let ι be an isotropic immersion of M into $\tilde{M}=M^{n+p}(c)$. If any geodesic in M is a helix on $\tilde{M}$, then the degree of ι is equal to* 3.

THEOREM 5.3. *The degree of a cubic geodesic immersion of M into $\tilde{M}=M^{n+p}(c)$ is equal to* 3, *provided that it is proper.*

Remark. We denote by P_mC an m-dimensional complex projective space, and by S_C^m the unit sphere in C^{m+1}. P_mC is considered as the quotient space of $S_C^m=S^{2m+1}(1)$ obtained by identifying x in S_C^m with xa, where $a\in C$ such that $|a|=1$. The canonical Riemannian structure on P_mC is the invariant metric such that the Hopf fibration $\pi: S_C^m\to P_mC$ is a Riemannian submersion. Accordingly, for an eigenfunction f on P_mC with respect to the Laplacian, so is $f\circ\pi$ on S_C^m [9]. If $f\circ\pi$ is a polynomial of degree r with its corresponding eigenvalue $\lambda(r)$, we have $\lambda(r)=r(2m+r)$, and the standard immersion $\iota_r: P_mC\to S^{N(r)}(1)$ is given, where $N(r)+1=\dfrac{m+2r}{m}\left\{\dfrac{(m+r-1)!}{r!(m-1)!}\right\}^2$ is the dimension of the space consisting of all hermitian harmonic polynomials of degree (r, r) on P_mC [1, 2]. Moreover, it is seen in [2, 8] that the degree of ι_1 is equal to 2. As for the standard immersion ι_2 of P_mC into $S^{N(2)}(1)$, the degree is greater than 3. In fact, the first normal space N_x^1 at any point x has the dimension not greater than $2m^2-m-1$, because of the characterization of complex projective spaces, and it implies $N(2)>n+\dim N_x^1+\dim N_x^2$.

Remark. Let ι_0 be a totally umbilical immersion of $S^{N(r)}(c)$ into an $(N(r)+q)$-dimensional real space form $\bar{M}=M^{N(r)+q}(\bar{c})$, where $\bar{c}<c$. For the standard immersion ι_r of $S^n(c)$ or P_mC it is easily seen that the degree of the composition $\iota_0\circ\iota_r$ of M into $\bar{M}$ is equal to that of ι_r, and the immersion $\iota_0\circ\iota_r$ has non-zero mean curvature.

6. Compact cases. In the rest of this paper, let ι be a cubic geodesic immersion of an n-dimensional Riemannian manifold M into an $(n+p)$-dimensional real space form $\tilde{M}=M^{n+p}(c)$ of constant curvature c. Accordingly, any geodesic γ in M is a helix in $\tilde{M}$ and the first curvature k and the second curvature l are positive constant. The present section is devoted to the case where the submanifold M is compact.

Now, for any point x in M and any orthonormal vectors u and w at x, let γ_u (resp. γ_w) be a normal geodesic passing through $x=\gamma_u(0)$ (resp. $\gamma_w(0)$) and let $X=\gamma_u'$ and $W=\gamma_w'$. Then, by virtue of (1.17), it satisfies

$$\begin{aligned}\tilde{\nabla}_X(\nabla'^2\sigma(Y, Z, U, V))&=\nabla'^3\sigma(Y, Z, U, V, X)+\nabla'^2\sigma(\nabla_XY, Z, U, V)+\cdots\\ &+\nabla'^2\sigma(Y, Z, U, \nabla_XV)-\sum_i\langle\nabla'^2\sigma(Y, Z, U, V), (X, e_i)\rangle e_i\end{aligned}\tag{6.1}$$

for any vector fields $Y, Z, U,$ and V along γ_u tangent to M. Next, we put $u_i=u$ $(i=1, 2, 3)$ and $u_4=v$ in (5.4). Then, differentiating it (resp. (5.3)) in the direction of X (resp. W), and making use of (6.1), (4.8)″ and the equation obtained by putting $u_1=e_i$ and $u_i=u$ $(i=2, 3, 4)$ in (4.10)′, we have

$$\nabla'^3\sigma(u, u, u, v, u)=-(3c+3k^2+l^2)\nabla'\sigma(u, u, v)/2$$
$$+6\sum_i\{\langle\sigma(u, v), \sigma(u, e_i)\rangle\nabla'\sigma(u, u, e_i)+2\langle\nabla'\sigma(u, u, v), \sigma(u, e_i)\rangle\sigma(e_i, u)\}$$

where u and v are orthonormal, and

$$\nabla'^3\sigma(u, u, u, u, w)+l^2\nabla'\sigma(u, u, w)=0$$

at x. It follows from these equations and the Ricci formula $(1.16)_3$ that we have

$$(9c+9k^2-l^2)\nabla'\sigma(u, u, v)-30\sum_i\{\langle\sigma(u, v), \sigma(u, e_i)\rangle\nabla'\sigma(u, u, e_i)$$
$$+\langle\nabla'\sigma(u, u, v), \sigma(u, e_i)\rangle\sigma(e_i, u)\}=0. \tag{6.2}$$

Making use of (6.2), we can prove

LEMMA 6.1. *If the sectional curvature has a minimal value δ, then $l^2\leqq 9\delta$.*

Proof. We suppose that there exist a point x and orthonormal vectors u and v at x so that the sectional curvature $K(u, v)$ attains the minimal value. When we define the curvature transformation K_u by $K_uw=R(w, u)u$ for any vector w at x, v becomes an eigenvector of K_u with its eigenvalue δ. Therefore we have

$$(c-\delta)\langle v, w\rangle+\langle\sigma(u, u), \sigma(v, w)\rangle-\langle\sigma(u, v), \sigma(u, w)\rangle=0$$

by the Gauss equation, from which it follows

$$\langle\sigma(u, v), \sigma(u, w)\rangle=(c+k^2-\delta)\langle v, w\rangle/3. \tag{6.3}$$

Combining (6.3) together with (6.2), we have

$$(10\delta-c-k^2-l^2)\|\nabla'\sigma(u, u, v)\|^2=30\sum_i\langle\nabla'\sigma(u, u, v), \sigma(u, e_i)\rangle^2.$$

Since the right hand side is non-negative, so is the left hand side.

We suppose that $l^2>9\delta$. By means of this condition and (6.3), we have $c+k^2+l^2-10\delta>0$. This and the above equation imply $\nabla'\sigma(u, u, v)=0$. Even though we change a part of u for that of v in the course of this proof, we can assert the same property and we obtain $\nabla'\sigma(u, v, v)=0$. Now, we put $u_i=u$ $(i=1, \cdots, 4)$ and $u_i=v$ $(i=5, 6)$ in (4.7)′. Then it is reduced to

$$2\langle\nabla'\sigma(u, u, u), \nabla'\sigma(u, v, v)\rangle+3\|\nabla'\sigma(u, u, v)\|^2=(kl)^2,$$

which contradicts to the fact that k and l are positive. Consequently, we can prove the conclusion. Q. E. D.

By this lemma, the following property is verified.

PROPOSITION 6.2. *For a proper cubic geodesic immersion of M into $\tilde{M}=M^{n+p}(c)$, if M is compact, then it is of positive curvature.*

7. Minimal immersions. In this section, we prove the following

THEOREM 7.1. *For a cubic geodesic immersion of M into $\tilde{M}=M^{n+p}(c)$, if it is minimal, then M is locally symmetric.*

Since ι is minimal, the mean curvature vector $\eta=\sum_\iota \sigma(e_\iota, e_\iota)/n$ vanishes. By means of the definition of the multi-linear maps $\nabla'\sigma$ and $\nabla'^2\sigma$, we see also $\sum_\iota \nabla'\sigma(e_\iota, e_\iota, u)=0$ and $\sum_\iota \nabla'^2\sigma(e_\iota, e_\iota, u, v)=0$ for any unit vectors u and v at any point x. The last equation and (5.4) give

$$12\sum_{\iota,j}\langle\sigma(u, e_j), \sigma(v, e_\iota)\rangle\sigma(e_\iota, e_j) = \{-3nc+(n+4)(3k^2-l^2)\}\sigma(u, v)\,. \tag{7.1}$$

We define a symmetric matrix H^α of order n by $H^\alpha=(h^\alpha_{ij})$ for a fixed number α, and then another symmetric matrix $A=(A^\alpha_\beta)$ of order p by $A^\alpha_\beta=\mathrm{Tr}\,(H^\alpha H^\beta)$. H^α can be regarded as a symmetric linear transformation on the tangent space defined by $H^\alpha=H_{e_\alpha}$, and A as a positive semi-definite symmetric linear transformation on the normal space.

LEMMA 7.2. *Matrices H^α and A satisfy*

$$\sum_\beta A^\alpha_\beta H^\beta=\{3nc-3nk^2+(n+4)l^2\}H^\alpha/6 \qquad \textit{for any } \alpha\,. \tag{7.2}$$

Proof. By the terminology of these matrices, (7.1) can be rewritten as $\sum_\beta H^\beta H^\alpha H^\beta=\{-3nc+(n+4)(3k^2-l^2)\}H^\alpha/12$ for any index α. On the other hand, the condition of isotropic immersion yields

$$\sum_\beta A^\alpha_\beta H^\beta+2\sum_\beta H^\beta H^\alpha H^\beta=2k^2H^\alpha\,,$$

which, together with the above equation, implies (7.2). Q. E. D.

Lemma 7.2 means that $A^2=LA$ and the matrix A has at most two distinct constant eigenvalues 0 and L, where $L=\{3nc-3nk^2+(n+4)l^2\}/6$. Since A is positive semi-definite, the constant L is non-negative. If $L=0$, then A is a zero matrix on M, which implies that ι is totally geodesic. Thus, without loss of generality, we may assume that it is positive.

Proof of Theorem 7.1. For any point x, we consider a sufficiently small neighborhood U and a normal geodesic γ passing through $x=\gamma(0)$. Let $X=\gamma'$. Then (7.2) is equivalent to $\sum_{\iota,j}\langle\sigma(Y, Z), \sigma(e_\iota, e_j)\rangle\sigma(e_\iota, e_j)=L\sigma(Y, Z)$, where Y and Z are parallel vector fields along γ. Differentiating this equation in the direction of X and taking account of (3.2), we see that the normal component satisfies

$$\sum_{\iota,j}\{(\langle\nabla'\sigma(e_\iota, e_j, X), \sigma(Y, Z)\rangle+\langle\sigma(e_\iota, e_j), \nabla'\sigma(Y, Z, X)\rangle)\sigma(e_\iota, e_j) +\langle\sigma(e_\iota, e_j), \sigma(Y, Z)\rangle\nabla'\sigma(e_\iota, e_j, X)\}=L\nabla'\sigma(X, Y, Z)\,.$$

Since this property holds true for any unit vectors at x, we have the following relation by virtue of the above equation at x and (4.13):

$$(7.3)\qquad \mathfrak{S}_3\sum_{i,j}\langle\sigma(e_i, e_j), \sigma(u_1, u_2)\rangle\nabla'\sigma(e_i, e_j, u_3)$$

$$=3L\nabla'\sigma(u_1, u_2, u_3)-4\sum_{i,j}\langle\sigma(e_i, e_j), \nabla'\sigma(u_1, u_2, u_3)\rangle\sigma(e_i, e_j).$$

Combining this equation together with (7.2), we have

$$\mathfrak{S}_3\sum_{i,j}\langle\sigma(e_i, e_j), \sigma(u_1, u_2)\rangle\langle\nabla'\sigma(e_i, e_j, u_3), \sigma(u, v)\rangle$$

$$+L\langle\sigma(u, v), \nabla'\sigma(u_1, u_2, u_3)\rangle=0,$$

from which it follows that we have

$$3\|\sum_{i,j}\langle\sigma(u, v), \nabla'\sigma(e_i, e_j, w)\rangle\sigma(e_i, e_j)\|^2+L\sum_{i,j}\langle\sigma(u, v), \nabla'\sigma(e_i, e_j, w)\rangle^2=0.$$

Since two terms of the left hand side in this equation are both of non-negative and the coefficient of the second term is positive, we have $\langle\nabla'\sigma(e_i, e_j, w), \sigma(u, v)\rangle=0$. Because σ and $\nabla'\sigma$ are linear forms, it implies that

$$(7.4)\qquad \langle\nabla'\sigma(u_1, u_2, u_3), \sigma(u_4, u_5)\rangle=0$$

for any unit vectors u_i $(i=1, \cdots, 5)$ at x, from which we can obtain the conclusion. Q. E. D.

By the well known properties about symmetric spaces, we have the following theorem as a direct consequence of Proposition 6.2, Theorem 7.1 and a theorem due to Sakamoto [8].

THEOREM 7.3. *Let ι be a cubic geodesic immersion of M into an $(n+p)$-dimensional sphere S^{n+p}. If M is compact and simply connected and ι is minimal, then M is a symmetric space of rank one.*

8. Main theorem. We shall be concerned with the following Main theorem in this paper.

THEOREM 8.1 *Let M be an $n(\geqq 3)$-dimensional compact simply connected Riemannian manifold and let ι be a proper cubic geodesic immersion of M into an $(n+p)$-dimensional sphere $S^{n+p}(c)$, where $p\geqq 2$. If ι is minimal, then the immersion is rigid to the immersion $\iota_0\circ\iota_3$ of S^n into S^{n+p}, where ι_0 is a totally geodesic immersion of $S^{N(3)}(c)$ into $S^{n+p}(c)$ and ι_3 is the standard immersion of S^n into $S^{N(3)}(c)$.*

First of all, we study about the dimension of the first normal space of the submanifold. Let r be the rank of the matrix $A=(A^\alpha_\beta)$. Then it is clear that r

is the dimension of the first normal space, because of the definition of the matrix.

We use here the same notation as that in the course of the proof of Theorem 7.1. Differentiating (5.4) in the direction of X, and taking its normal components, we have

$$
\begin{aligned}
(8.1)\qquad 6\nabla'^3\sigma(u_1, u_2, u_3, v, u)=\mathfrak{S}_3\{&(-3c+3k^2-l^2)\langle u_1, u_2\rangle\nabla'\sigma(u_3, u, v)\\
&+(3c+3k^2-l^2)\langle v, u_1\rangle\nabla'\sigma(u_2, u_3, u)\\
&-6\sum_i\langle\sigma(u_1, u_2), \sigma(v, e_i)\rangle\nabla'\sigma(e_i, u_3, u)\}
\end{aligned}
$$

because M is locally symmetric and it satisfies (7.4). Moreover, since M is minimal and consequently we have $\sum_i\nabla'^3\sigma(e_i, e_i, u, v, w)=0$, it follows from (8.1) that

$$
12\sum_{i,j}\langle\sigma(u, e_i), \sigma(v, e_j)\rangle\nabla'\sigma(w, e_i, e_j)=\{-3nc+(n+4)(3k^2-l^2)\}\nabla'\sigma(u, v, w).
$$

Therefore we have

$$
\begin{aligned}
(8.2)\qquad 12\sum_{i,j,k,l,m}&\langle\nabla'\sigma(e_i, e_j, e_k), \nabla'\sigma(e_l, e_m, e_k)\rangle\langle\sigma(e_i, e_l), \sigma(e_j, e_m)\rangle\\
&=\{-3nc+(n+4)(3k^2-l^2)\}\|\nabla'\sigma\|^2,
\end{aligned}
$$

where $\|\ \|$ denotes the length of the form. When we note that the left hand side in the above equation is non-negative, we can prove the following

LEMMA 8.2. *The rank of the matrix A satisfies*

$$
\frac{n(n+2)}{4}\leqq r\leqq\frac{(n+2)(n-1)}{2}.
$$

Proof. Since ι is k-isotropic and minimal, the square of the length of the second fundamental form is equal to $n(n+2)k^2/2$. On the other hand, by the definition of A, we have $\mathrm{Tr}\,A=rL=\|\sigma\|^2$, where L is the positive eigenvalue of A, so we have

$$
(8.3)\qquad r=3n(n+2)k^2/\{3nc-3nk^2+(n+4)l^2\}.
$$

Furthermore it is seen in [6] that there is the following relation between $\mathrm{Tr}\,A^2$ and $\|\sigma\|$: $\mathrm{Tr}\,A^2\geqq 2\|\sigma\|^4/(n+2)(n-1)$, and the equality is true if and only if M is of constant curvature. Because of $\mathrm{Tr}\,A^2=rL^2=L\|\sigma\|^2$, we get

$$
(8.4)\qquad 3n(n-1)c-3n(n+1)k^2+(n-1)(n+4)l^2\geqq 0.
$$

Combining the non-negativeness of the left hand side in (8.2) with (8.3) and (8.4), we obtain the inequalities. Q. E. D.

Now, due to Theorem 7.3, we may consider the submanifold M as an n-dimensional Riemannian symmetric space of rank one. These spaces contain only

a sphere S^n, a complex projective space PC^n ($n=2m\geqq 4$), a quaternion projective space PQ^n ($n=4m\geqq 8$) and a Cayley projective space PCa^n ($n=16$). Let M^* be an n-dimensional compact simply connected Riemannian symmetric space of rank one. We normalize the Riemannian metric of M^* in such a way that the sectional curvature $K(u, v)$ satisfies $1/4\leqq K(u, v)\leqq 1$. We denote by K_u^* the curvature transformation with respect to u. Then we can regard K_u^* as the transformation of the orthogonal complement to the vector u in M_x^* at any point x. As is well known, distinct eigenvalues of K_u^* are 1 and 1/4. M^* is said to be of *type* s if the maximal eigenvalue of the linear transformation has multiplicity s. For each projective space, s is equal to 1, 3, 7 or $n-1$, according as M^* is a complex projective space, a quaternion projective space, a Cayley projective space or a sphere.

On the other hand, since M has the same situation as that of M^*, we can define the concept of the type of M. Concerning the type of M, we have

LEMMA 8.3. *The type s of M satisfies*

$$(n+s+1)k^2=(n-s-1)c\,,$$

if M is not of constant curvature.

Proof By the definition of the type, the linear transformation K_u on M_x has exactly two distinct eigenvalues and the multiplicity of the maximal one is equal to s. Since M is not of constant curvature, the rank of A is less than $(n+2)(n-1)/2$. When we denote by H_{ij} a normal vector with component ${}^t(h_{ij}^{n+1}, \cdots, h_{ij}^{n+p})$ for any indices i and j, it follows from Lemma 7.2 that we have $AH_{ij}=LH_{ij}$. Then there exist an index ι such that $H_{\iota\iota}$ is a zero vector or distinct indices i and j such that $H_{\iota j}$ is a zero vector, because $r=\operatorname{rank} A \geqq n(n+1)/2-1$ if it is not provided. Since the equation (6.3) holds for the maximal value of the sectional curvatures, we may consider the Riemann metric of M so that eigenvalues of the linear transformation K_u are $4K$ and K, where $4K=c+k^2$. Thus we can choose a suitable orthonormal basis $\{e_1, \cdots, e_n\}$ in such a way that $u=e_1$ and e_ι ($\iota=2, \cdots, s+1$) are contained in the eigenspace corresponding to $4K$ and the others belong to another eigenspace. Thus we have

$$K(e_1, e_\iota)=4K \qquad (\iota=2, \cdots, s+1)\,,$$

$$K(e_1, e_j)=K \qquad \text{for } j\geqq s+2\,.$$

Accordingly, (1.7) and (1.8) imply $\sigma(e_1, e_\iota)=0$ and $\|\sigma(e_1, e_j)\|^2=K$. Using these results and $\sum_\iota\langle\sigma(u, e_\iota), \sigma(v, e_\iota)\rangle=(n+2)k^2\langle u, v\rangle/2$, we have

$$\sum_\iota\langle\sigma(e_1, e_\iota), \sigma(e_1, e_\iota)\rangle=\frac{n+2}{2}k^2=k^2+\frac{(c+k^2)(n-s-1)}{4}\,,$$

from which the necessary equality is given. Q. E. D.

We suppose that M is not of constant curvature. It follows from (8.3) that we have $r<(n+2)k^2/(c-k^2)$, because k and l are positive, which together with Lemma 8.3 implies $r<(n+2)(n-s-1)/2(s+1)$. This contradicts to Lemma 8.2. Consequently, M must be a real space form. Namely, M is an n-dimensional sphere and the degree of the immersion ι is 3. Thus, we can apply Theorem 4.5 in [6] to our case, and the assertion in Theorem 8.1 is thereby proved.

Remark. We note here that the proof of Lemma 3.1 in [6] contains an error and Theornm 4.5 in the paper is true under the additional condition that the mean curvature vector $\mathfrak{h}$ is parallel in the normal bundle.

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