S. KOBAYASHI AND N. SUITA
KODAI MATH. J.
3 (1980), 315—320

ON SUBORDINATION OF SUBHARMONIC FUNCTIONS

BY SHŌJI KOBAYASHI AND NOBUYUKI SUITA

Introduction. In the present paper we are concerned with analytic maps from a Riemann surface into another which preserve the least harmonic majorant of a subharmonic function.

Let R denote an open Riemann surface. Let $S(R)$ be the class of all functions subharmonic on R which admit harmonic majorants on R and $S^+(R)=\{f\in S(R): f$ is bounded below on $R\}$. We denote by $\hat{f}$ the least harmonic majorant of f for any $f\in S(R)$. Let R_j $(j=1, 2)$ be open Riemann surfaces and ϕ be an analytic map from R_1 into R_2. Littlewood's subordination theorem (see [3, p. 10]) shows that $f\circ\phi\in S(R_1)$ and

$$\hat{f}\circ\phi\geqq\widehat{f\circ\phi} \tag{1}$$

on R_1 for any $f\in S(R_2)$. In this paper we deal with the problem when equality holds in (1).

In the case where $R\in O_G$, it is well known that there exist no positive superharmonic functions but the constants on R (see for example [1, p. 204]). Therefore we easily see that $\hat{f}-f\equiv 0$ for any $f\in S(R)$, which means that $S(R)$ reduces to the harmonic functions. It is easily verified that if $R_1\in O_G$ and $R_2\notin O_G$ there exist no nonconstant analytic maps from R_1 into R_2. Hence, if one of R_j $(j=1, 2)$ is of class O_G, equality always holds in (1) for any $f\in S(R_2)$.

From now on we assume that $R_j\notin O_G$ for $j=1, 2$. Let $G_j(z, t)$ denote the Green's function of R_j with pole at t. Following Heins [4], we say that ϕ is of *type B1* when $G_2(\phi(z), t)$ majorates no positive bounded harmonic functions for some $t\in R_2$, or equivalently for every $t\in R_2$ (see Theorem 4.1 of [4, p. 446]), and we say that ϕ is of *type $B1_1$* when $G_2(\phi(z), t)$ majorates no positive harmonic functions for every $t\in R_2$. Let U denote the open unit disc and π_j be a universal covering map of R_j. By applying the monodromy theorem, we can define an analytic function ψ in U which is bounded by 1 such that

$$\phi\circ\pi_1=\pi_2\circ\psi\,. \tag{2}$$

An *inner function* is any function ψ analytic in U with the properties $|\psi(z)|\leqq 1$ in U and $|\psi^*(e^{i\theta})|=1$ a.e. on ∂U, where ψ^* denotes the Fatou's boundary function of ψ.

Received May 28, 1979

1. Main results. First of all we state our results.

THEOREM 1. *Let ϕ be an analytic map from R_1 into R_2, then the following statements are equivalent:*
(a) $\hat{f}\circ\phi=\widehat{f\circ\phi}$ *for every* $f\in S^+(R_2)$.
(b) $\hat{f}\circ\phi=\widehat{f\circ\phi}$ *for some* $f\in S(R_2)$ *which is not harmonic on whole* R_2.
(c) *There exists an inner function* ψ *such that* $\phi\circ\pi_1=\pi_2\circ\psi$.
(d) ϕ *is of type B1.*

Remark. Theorem 1 is a generalization of a theorem of Ryff (Theorem 3 of [7, p. 351]) which states the invariance of H_p norm of an analytic function in U under composition by any inner function ψ with $\psi(0)=0$.

THEOREM 2. *Let ϕ be as in Theorem 1, then the following statements are equivalent:*
(a) $\hat{f}\circ\phi=\widehat{f\circ\phi}$ *for every* $f\in S(R_2)$.
(b) *There exists an inner function* ψ *such that* $(\psi(z)-\alpha)/(1-\bar{\alpha}\psi(z))$ *is a Blaschke product for every* $\alpha\in U$ *and such that* $\phi\circ\pi_1=\pi_2\circ\psi$.
(c) ϕ *is of type* $B1_1$.

2. Proof of the theorems. First we need a lemma.

LEMMA 1. *Let $R\notin O_G$ and π be a universal covering map of R, then $\hat{f}\circ\pi=\widehat{f\circ\pi}$ holds for any $f\in S(R)$.*

Proof. Since $\hat{f}\circ\pi$ is a harmonic majorant of $f\circ\pi$, we easily see that $\hat{f}\circ\pi\geqq\widehat{f\circ\pi}$. We must show the inverse inequality. Let Γ be the cover transformation group under which π is invariant. Since $\widehat{f\circ\pi}\circ T$ is a harmonic majorant of $f\circ\pi\circ T=f\circ\pi$ for every $T\in\Gamma$, we see that $\widehat{f\circ\pi}\leqq\widehat{f\circ\pi}\circ T$. By composing T^{-1} from right, we obtain the inverse inequality $\widehat{f\circ\pi}\geqq\widehat{f\circ\pi}\circ T$. Thus we see that $\widehat{f\circ\pi}$ is invariant under Γ. Therefore we can define a single-valued harmonic function on R by $\widehat{f\circ\pi}\circ\pi^{-1}$, which is a harmonic majorant of f. Then we see that $\widehat{f\circ\pi}\circ\pi^{-1}\geqq\hat{f}$, and hence $\widehat{f\circ\pi}\geqq\hat{f}\circ\pi$ as desired.

Remark. Lemma 1 was essentially proved by Rudin [5, p. 48].

Proof of Theorem 1.
1. (a) implies (d). Suppose that (d) does not hold. Then there exists a positive bounded harmonic function u which is majorated by $G_2(\phi(z), t)$ on R_1 for some $t\in R_2$. Let $g(z)=-\min\{G_2(z, t), M\}$, where $M=\sup\{u(z): z\in R_1\}$, then $g\in S^+(R_2)$ and $\widehat{g\circ\phi}\leqq-u$. On the other hand, we see by (a) that $\widehat{g\circ\phi}=\hat{g}\circ\phi\equiv0$, since $\hat{g}\geqq-\hat{G}_2\equiv0$. This is a contradiction.

2. (d) implies (b). For example, let $f=-\min\{G_2(z, t), 1\}$ for some $t\in R_2$, then we see by (d) that $\widehat{f\circ\phi}\equiv 0\equiv\hat{f}\circ\phi$.

3. (b) implies (c). Suppose that (c) does not hold, then every analytic function ψ in U bounded by 1 such that $\phi\circ\pi_1=\pi_2\circ\psi$ is not inner. Therefore the set $E=\{e^{i\theta}: |\psi^*(e^{i\theta})|\leqq 1-2\delta\}$ is of positive measure for a sufficiently small positive number δ. By Egorov's theorem we can find a compact subset F of E of positive measure on which $\psi(re^{i\theta})$ converges uniformly to $\psi^*(e^{i\theta})$ as $r\to 1$. Then, we have $|\psi(re^{i\theta})|\leqq 1-\delta$ for every $e^{i\theta}\in F$ and every $r\geqq r_0$ for some r_0 with $0<r_0<1$. Let f be as in (b), then we see that $\hat{f}>f$ on R_2, since f is not harmonic on R_2. Let $\varepsilon=\inf\{\hat{f}(\zeta)-f(\zeta): \zeta\in\pi_2(\{|z|\leqq 1-\delta\})\}>0$, then we see that

$$(\hat{f}\circ\pi_2\circ\psi)(re^{i\theta})\geqq(f\circ\pi_2\circ\psi)(re^{i\theta})+\varepsilon \tag{3}$$

if $r\geqq r_0$ and $e^{i\theta}\in F$. Then, by (3), we see for $r\geqq r_0$

$$\begin{aligned}(\hat{f}\circ\phi\circ\pi_1)(0)&=(\hat{f}\circ\pi_2\circ\psi)(0)\\&=\frac{1}{2\pi}\int_0^{2\pi}(\hat{f}\circ\pi_2\circ\psi)(re^{i\theta})d\theta\\&=\frac{1}{2\pi}\left(\int_F+\int_{F^c}(\hat{f}\circ\pi_2\circ\psi)(re^{i\theta})d\theta\right)\\&\geqq\frac{1}{2\pi}\left(\int_F(f\circ\pi_2\circ\psi)(re^{i\theta})+\varepsilon\right)d\theta\\&\qquad+\int_{F^c}(f\circ\pi_2\circ\psi)(re^{i\theta})d\theta)\\&=\frac{1}{2\pi}\int_0^{2\pi}(f\circ\pi_2\circ\psi)(re^{i\theta})d\theta+\varepsilon m(F),\end{aligned} \tag{4}$$

where m denotes the normalized Lebesgue measure. Letting $r\to 1$, we see $(\hat{f}\circ\phi\circ\pi_1)(0)>(\widehat{f\circ\pi_2\circ\psi})(0)$. Then, using Lemma 1, we obtain $\hat{f}\circ\phi>\widehat{f\circ\phi}$, as desired.

4. (c) implies (a). Let $f\in S^+(R_2)$ and we assume that (c) holds. Without loss of generality, we may assume that f is nonnegative on R_2. Let ρ be fixed with $0<\rho<1$ and let $M_\rho=\sup\{(f\circ\pi_2)(z): |z|\leqq\rho\}$. By Egorov's theorem, for every $\varepsilon>0$, there exists an open subset O of the unit circle such that

$$m(O)<\varepsilon/M_\rho \tag{5}$$

and that $\psi(re^{i\theta})$ converges uniformly to $\psi^*(e^{i\theta})$ on O^c. Therefore there exists r with $0<r<1$ such that

$$|\psi(re^{i\theta})|>\rho \tag{6}$$

for $e^{i\theta}\notin O$, since $|\psi^*(e^{i\theta})|=1$ a.e. on ∂U. Let u_ρ be the least harmonic majorant of $f\circ\pi_2$ in $\Delta_\rho=\{z: |z|<\rho\}$. Let $D_\rho=\psi^{-1}(\Delta_\rho)$ and Ω_ρ be the connected component of $D_\rho\cap\Delta_r$ containing 0. Then, by (6), we see that

(7) $$\partial\Omega_\rho \cap \Gamma_r \subset rO \equiv \{re^{i\theta} : e^{i\theta} \in O\}$$

Let ω be the harmonic measure of rO in Δ_r, then by (5)

(8) $$w(0) \leqq \varepsilon / M_\rho .$$

Let $h_\rho = \widehat{f \circ \pi_2 \circ \phi} - u_\rho \circ \phi$, then we easily see that

(9) $$h_\rho \geqq 0 \quad \text{on } \partial\Omega_\rho - \partial\Delta_r ,$$

and

(10) $$h_\rho \geqq -M_\rho \quad \text{on } \partial\Omega_\rho \cap \partial\Delta_r .$$

Therefore, by the maximum principle, we obtain

(11) $$h_\rho \geqq -M_\rho \omega \quad \text{in } \Omega_\rho ,$$

and hence $h_\rho(0) \geqq -\varepsilon$. Since ε is arbitrary, we see that $h_\rho(0) \geqq 0$. Letting $\rho \to 1$, we obtain

(12) $$\widehat{f \circ \pi_2 \circ \phi} \geqq \widehat{f \circ \pi_2} \circ \phi$$

since $\lim_{\rho \to 1} u_\rho = \widehat{f \circ \pi_2}$. Using Lemma 1, we obtain $\widehat{f \circ \phi} \geqq \hat{f} \circ \phi$, as desired.

Proof of Theorem 2.

1. (a) implies (b). Suppose that (b) does not hold, then there exists $\alpha \in U$ such that $(\phi(z) - \alpha)/(1 - \bar{\alpha}\phi(z))$ is not a Blaschke product. Let S denote its singular part, then S is represented as

(13) $$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right),$$

where μ is a positive singular measure on ∂U (see [3, p. 24]). Let $f(z) = -G_2(z, \pi_2(\alpha))$, then $f \in S(R_2)$ and

(14) $$(f \circ \pi_2)(z) = \sum_{T \in \Gamma} \log \left| \frac{T(z) - \alpha}{1 - \bar{\alpha} T(z)} \right|$$

by Myrberg's theorem (see [8, p. 522]). Therefore

(15) $$\begin{aligned} (f \circ \phi \circ \pi_1)(z) &= (f \circ \pi_2 \circ \phi)(z) \\ &= \sum_{T \in \Gamma} \log \left| \frac{(T \circ \phi)(z) - \alpha}{1 - \bar{\alpha}(T\phi)(z)} \right| \\ &= \log \left| \frac{\phi(z) - \alpha}{1 - \bar{\alpha}\phi(z)} \right| + \sum_{\substack{T \in \Gamma \\ T \neq id.}} \log \left| \frac{(T \circ \phi)(z) - \alpha}{1 - \bar{\alpha}(T \circ \phi)(z)} \right| \\ &\leqq \log |S(z)| \\ &= -\int_0^{2\pi} \mathrm{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) , \end{aligned}$$

and hence

$$f\circ\phi\circ\pi_1 \leqq -P[\mu]<0\,, \tag{16}$$

where $P[\mu]$ denotes the Poisson integral of μ. Using Lemma 1, we have $\widehat{f\circ\phi}<\hat{f}\circ\phi$ from which we see that (a) does not hold.

2. (b) implies (c). The following lemma is well known, for the proof, for example see [6, p. 335].

LEMMA 2. *Let B be a Blaschke product, then*

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\log|B(re^{i\theta})|\,d\theta=0\,. \tag{17}$$

Remark. (17) means

$$\widehat{\log|B|}\equiv 0 \tag{18}$$

in our language, since the left-hand side of (17) is the value at 0 of the least harmonic majorant of $\log|B|$.

Let $f(z)=-G_2(z, t)$ for arbitrarily fixed $t\in R_2$. Then, by Myrberg's theorem,

$$(f\circ\pi_2)(z)=\sum_{T\in\Gamma}\log\left|\frac{T(z)-\alpha}{1-\bar{\alpha}T(z)}\right|, \tag{19}$$

where α is a point in U with $\pi_2(\alpha)=t$. Therefore we see

$$\begin{aligned}(f\circ\phi\circ\pi_1)(z)&=(f\circ\pi_2\circ\psi)(z)\\ &=\sum_{T\in\Gamma}\log\left|\frac{(T\circ\psi)(z)-\alpha}{1-\bar{\alpha}(T\circ\psi)(z)}\right|\\ &=\sum_{T\in\Gamma}\log|B_T(z)|\\ &=\log|B(z)|,\end{aligned} \tag{20}$$

where $B_T=(T\circ\psi-\alpha)/(1-\bar{\alpha}T\circ\psi)$, which is a Blaschke product by (b) for every $T\in\Gamma$, and $B=\prod_{T\in\Gamma}B_T$. Using Lemma 2, we obtain

$$\widehat{f\circ\phi\circ\pi_1}=\widehat{\log|B|}\equiv 0\,. \tag{21}$$

Then, by Lemma 1, we see that $\widehat{f\circ\phi}\equiv 0$, which means that ϕ is of type Bl_1, as desired.

3. (c) implies (a). Any superharmonic function s on a Riemann surface $R\notin O_G$ which is represented as

$$s(z)=\int_R G(z, t)\,d\mu(t)\,, \tag{22}$$

where G is the Green's function of R and μ is a nonnegative measure on R, is called a (Green's) potential. By Riesz's theorem (Satz 4.6 and Folgesatz 4.6 of

[2, pp. 41-42]), a nonnegative superharmonic function is a potential if and only if its greatest harmonic minorant is 0. Let $f \in S(R_2)$, then $\hat{f}-f$ is a potential on R_2, i.e.

$$\hat{f}(w)-f(w)=\int_{R_2} G_2(w, \zeta)dm(\zeta), \tag{23}$$

where m is a nonnegative measure on R_2. By (c) $G_2(\phi(z), t)$ is a potential on R_1 for any $t \in R_2$, i.e.

$$G_2(\phi(z), t)=\int_{R_1} G_1(z, \tau)d\nu_t(\tau), \tag{24}$$

where ν_t is a nonnegative measure on R_1 for every $t \in R_2$. It is known that ν_t is the sum of the point masses at points s such that $\phi(s)=t$ counting with multiplicity (see [4, p. 440]). Therefore we can easily see that for any compact $K \subset R_1$, $\nu_t(K)$ is upper semi-continuous as a function of t. Define a nonnegative measure ν on R_2 by

$$\nu(K)=\int_{R_2} \nu_t(K)dm(t), \tag{25}$$

for any compact $K \subset R_1$. From (23), (24) and (25) we obtain

$$(\hat{f}\circ\phi)(z)-(f\circ\phi)(z)=\int_{R_1} G_1(z, \tau)d\nu(\tau), \tag{26}$$

which means that $\hat{f}\circ\phi-f\circ\phi$ is a potential on R_1, and hence its greatest harmonic minorant is 0 by Riesz's theorem cited above (cf. [4, pp. 449-451]). Therefore we obtain $\hat{f}\circ\phi=\widehat{f\circ\phi}$, as desired.

REFERENCES

[1] AHLFORS, L. V. AND L. SARIO, Riemann surfaces, Princeton university press, 1960.

[2] CONSTANTINESCU, C. AND A. CORNEA, Ideale Ränder Riemannscher Flächen, Springer-Verlag, 1963.

[3] DUREN, P. L. Theory of H^p spaces, Academic Press, New York, 1970.

[4] HEINS, M. On the Lindelöf principle, Ann. of Math. 61 (1955), 440-473.

[5] RUDIN, W. Analytic functions of class H_p, Trans. Amer. Math. Soc. 78 (1955), 46-66.

[6] RUDIN, W. Real and complex analysis, MacGraw-Hill, New York, 1974.

[7] RYFF, J. V. Subordinate H^p functions, Duke Math. J. 33 (1966), 347-354.

[8] TSUJI, M. Potential theory in modern function theory, Maruzen, Tokyo, 1959.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.