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COEFFICIENTS OF INVERSES OF UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSIONS

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§1. Introduction.

Let Σ' denote the family of univalent functions

$$g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

in $\tilde{\Delta}=\{z: 1<|z|<\infty\}$. For $0\leqq k<1$ let Σ'_k be the family of functions in Σ' that admit k-quasiconformal extensions to $\bar{\Delta}=\{z:|z|\leqq 1\}$. That is, each $g\in\Sigma'_k$ has a homeomorphic extension to $\bar{\Delta}$, that is absolutely continuous on a.e. horizontal and vertical line in $\bar{\Delta}$ and satisfies

$$|g_{\bar{z}}|\leqq k|g_z| \qquad \text{a.e. in } \bar{\Delta}.$$

If $k=0$, then g is an entire univalent function. Consequently, Σ'_0 contains only the identity function. As $k\to 1$, the families Σ'_k are dense in Σ', and we therefore define $\Sigma'_1=\Sigma'$. Since $\Sigma'_{k_1}\subset\Sigma'_{k_2}$ for $k_1<k_2$, the families Σ'_k interpolate in a monotonic fashion from the identity function to the family Σ'.

R. Kühnau [2] and O. Lehto [5] have obtained the sharp coefficient estimates

$$|b_1|\leqq k \quad \text{and} \quad |b_2|\leqq(2/3)k \tag{1}$$

for functions $g\in\Sigma'_k$. In this article we shall study the coefficients of their inverse functions.

That is, if G is the inverse of a function g in Σ'_k, i.e., $G=g^{-1}$, then G has an expansion

$$G(w)=w+\sum_{n=1}^{\infty}\frac{B_n}{w^n}$$

in some neighborhood of $w=\infty$. Since $B_1=-b_1$ and $B_2=-b_2$, the sharp estimates

$$|B_1|\leqq k \quad \text{and} \quad |B_2|\leqq(2/3)k \tag{2}$$

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are a consequence of (1).

For the class $\Sigma_1'=\Sigma'$, G. Springer [8] proved that $|B_3|\leqq 1$ and conjectured that

$$(3)\qquad |B_{2n-1}|\leqq\frac{(2n-2)!}{n!(n-1)!},\qquad n=3, 4, 5, \cdots.$$

Very recently, Y. Kubota [1] and the present author [7] have verified (3) for $2\leqq n\leqq 7$. Based on a technique of Lehto [5] we may therefore conclude that

$$(4)\qquad |B_{2n-1}|\leqq\frac{(2n-2)!}{n!(n-1)!}k$$

for $g\in\Sigma_k'$ and $2\leqq n\leqq 7$.

It is the purpose of this article to improve the estimates (4). We shall also obtain anestimate when $n=8$ and verify the conjecture (3) for $n=8$ as a special case.

§ 2. Results.

The following theorem contains the results of this article. Its proof will be given in Section 5.

PRINCIPAL THEOREM. *Let g belong to Σ_k', $0\leqq k\leqq 1$, and let*

$$G(w)=g^{-1}(w)=w+\sum_{n=1}^{\infty}\frac{B_n}{w^n}$$

be the expansion of its inverse function in a neighborhood of $w=\infty$. Then

$$|B_3|\leqq k-\frac{1}{2}k(1-k)\leqq k$$

$$|B_5|\leqq 2k-\frac{1}{3!}k(1-k)(10+7k)\leqq 2k$$

$$|B_7|\leqq 5k-\frac{1}{4!}k(1-k)(114+103k+49k^2)\leqq 5k$$

$$|B_9|\leqq 14k-\frac{1}{5!}k(1-k)(1656+1606k+1181k^2+451k^3)\leqq 14k$$

$$|B_{11}|\leqq 42k-\frac{1}{6!}k(1-k)(30120+29846k+26381k^2+17776k^3+6241k^4)\leqq 42k$$

$$|B_{13}|\leqq 132k-\frac{1}{7!}k(1-k)(664560+662796k+631632k^2+529887k^3+317892k^4+98841k^5)\leqq 132k$$

$$|B_{15}|\leqq 429k-\frac{3}{8!}k(1-k)(5764080+5759724k+5658280k^2+5247149k^3+4075349k^4+2274655k^5+666699k^6)\leqq 429k\,.$$

Since Σ'_0 contains only the identity function, the theorem is trivial in this case. The case $k=1$ is of special interest:

COROLLARY. *If g belongs to Σ', then the coefficients of its inverse function satisfy*

$$|B_3|\leqq 1,\quad |B_5|\leqq 2,\quad |B_7|\leqq 5,\quad |B_9|\leqq 14,$$

$$|B_{11}|\leqq 42,\quad |B_{13}|\leqq 132,\quad \text{and}\quad |B_{15}|\leqq 429.$$

Equality in any of these occurs if and only if $g(z)=z+\dfrac{e^{i\alpha}}{z}$ *for some real* α.

In this special case the result for $|B_{15}|$ is new. One easily verifies that equality occurs (also in (3)) for the indicated functions. To see that these are the only extremal functions, we observe at the end of Section 5 that equality can occur only if $|b_1|=1$, which by the Area Theorem (see Section 3) can occur only for the indicated functions. For $0<k<1$ we do not assert that the estimates of the Theorem are sharp.

§ 3. The Principal Lemma.

The set $H(\tilde{\Delta})$ of all analytic functions on $\tilde{\Delta}$ with the topology of locally uniform convergence is a locally convex topological vector space. We denote its topological dual space by $H'(\tilde{\Delta})$. If $h(z, \zeta)$ is analytic in $\tilde{\Delta}\times\tilde{\Delta}$ and $L\in H'(\tilde{\Delta})$, we define

$$L^2(h(z, \zeta))=L(L(h(z, \zeta)))\quad\text{and}\quad |L|^2(h(z, \zeta))=L(\overline{L(h(z, \bar{\zeta}))})$$

where L is applied first to the function of z and then to the function of ζ. In this framework we state (cf. [3; 6, Theorem 14.15]):

Grunsky-Kühnau Inequalities. If $g\in\Sigma'_k$, $0\leqq k\leqq 1$, and $L\in H'(\tilde{\Delta})$, then

$$\left|L^2\left(\log\frac{g(z)-g(\zeta)}{z-\zeta}\right)\right|\leqq k|L|^2\left(\log\frac{1}{1-(z\bar{\zeta})^{-1}}\right).$$

These inequalities may be "exponentiated" in the following manner. If $\phi(w)=\sum_{n=0}^{\infty}c_n w^n$ is any entire function and $\phi^+(w)=\sum_{n=0}^{\infty}|c_n|w^n$, then [6, Theorem 11.16]

$$\left|L^2\left(\phi\circ\log\frac{g(z)-g(\zeta)}{z-\zeta}\right)\right|\leqq|L|^2(\phi^+\circ\log[1-(z\bar{\zeta})^{-1}]^{-k}).$$

In particular, if $\phi(w)=e^{-w}$, then

$$\left|L^2\left(\frac{z-\zeta}{g(z)-g(\zeta)}\right)\right|\leqq|L|^2([1-(z\bar{\zeta})^{-1}]^{-k}) \tag{5}$$

$$=\sum_{m=0}^{\infty}\frac{k(k+1)\cdots(k+m-1)}{m!}|L(z^{-m})|^2.$$

We now distinguish a special functional L. Fix $g\in\Sigma_k'$ and $n\geqq 1$. Denote $G(w)=g^{-1}(w)=w+\sum_{\nu=1}^{\infty}\frac{B_\nu}{w^\nu}$, and define L to be the functional that associates to $h\in H(\tilde{\Delta})$ the coefficient d_n in the expansion of $h\circ G(w)=\sum_{\nu=-\infty}^{\infty}\frac{d_\nu}{w^\nu}$ in a neighborhood of $w=\infty$. Then

$$L^2\Big(\frac{z-\zeta}{g(z)-g(\zeta)}\Big)=B_{2n-1}$$

and we have the following consequence of (5).

PRINCIPAL LEMMA. *If $g\in\Sigma_k'$, $0\leqq k\leqq 1$, and $G(w)=g^{-1}(w)=w+\sum_{\nu=1}^{\infty}\frac{B_\nu}{w^\nu}$ in a neighborhood of $w=\infty$, then*

$$(6)\qquad |B_{2n-1}|\leqq\frac{k(k+1)\cdots(k+n-1)}{n\,!}+\sum_{m=1}^{n-2}\frac{k(k+1)\cdots(k+m-1)}{m\,!}|B_n^{(-m)}|^2$$

for $n\geqq 1$, where $[G(w)]^{-m}=\frac{1}{w^m}+\sum_{\nu=m+2}^{\infty}\frac{B_\nu^{(-m)}}{w^\nu}$ in a neighborhood of $w=\infty$. The sum in (6) *is omitted for $n=1$ and $n=2$.*

The inequality (6) is our main tool. We shall also use the following Area Theorem of Kühnau [3] and Lehto [4].

AREA THEOREM. *If $g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$ belongs to Σ_k', $0\leqq k\leqq 1$, then $\sum_{n=1}^{\infty}n|b_n|^2\leqq k^2$. In particular, $|b_1|\leqq k$ with equality if and only if $g(z)=z+\frac{e^{i\alpha}k}{z}$ for some real α.*

The Area Theorem will be used in the following form.

COROLLARY. *If $g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$ belongs to Σ_k', $0\leqq k\leqq 1$, and r, s, t are real, then*

$$(7)\qquad (r|b_6|+s|b_4|+t|b_2|)^2\leqq\Big(\frac{r^2}{6}+\frac{s^2}{4}+\frac{t^2}{2}\Big)(k^2-|b_1|^2)$$

Proof. We have

$$0\leqq\frac{2}{3}\Big(r|b_4|-\frac{3}{2}s|b_6|\Big)^2+\frac{1}{3}(r|b_2|-3t|b_6|)^2+\frac{1}{2}(s|b_2|-2t|b_4|)^2$$

$$=\Big(\frac{r^2}{6}+\frac{s^2}{4}+\frac{t^2}{2}\Big)(6|b_6|^2+4|b_4|^2+2|b_2|^2)-(r|b_6|+s|b_4|+t|b_2|)^2$$

$$\leqq\Big(\frac{r^2}{6}+\frac{s^2}{4}+\frac{t^2}{2}\Big)(k^2-|b_1|^2)-(r|b_6|+s|b_4|+t|b_2|)^2$$

with the help of the Area Theorem.

§ 4. Coefficient Relations.

Several coefficient relations will be needed. If $g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$ and $G(w)=g^{-1}(w)=w+\sum_{n=1}^{\infty}\frac{B_n}{w^n}$ in a neighborhood of ∞, then

$$B_1=-b_1 \qquad B_4=-b_4-3b_1b_2$$

$$B_2=-b_2 \qquad B_5=-b_5-4b_1b_3-2b_2^2-2b_1^3$$

$$B_3=-b_3-b_1^2 \qquad B_6=-b_6-5b_1b_4-5b_2b_3-10b_1^2b_2\,.$$

Furthermore, if $[G(w)]^{-m}=\frac{1}{w^m}+\sum_{\nu=m+2}^{\infty}\frac{B_\nu^{(-m)}}{w^\nu}$ in a neighborhood of $w=\infty$, for $m\geqq 1$, then

$$B_3^{(-1)}=-B_1$$

$$B_4^{(-1)}=-B_2$$

$$B_5^{(-1)}=-B_3+B_1^2$$

$$B_6^{(-1)}=-B_4+2B_1B_2$$

$$B_7^{(-1)}=-B_5+2B_1B_3+B_2^2-B_1^3$$

$$B_8^{(-1)}=-B_6+2B_1B_4+2B_2B_3-3B_1^2B_2$$

$$B_4^{(-2)}=-2B_1$$

$$B_5^{(-2)}=-2B_2$$

$$B_6^{(-2)}=-2B_3+3B_1^2$$

$$B_7^{(-2)}=-2B_4+6B_1B_2$$

$$B_8^{(-2)}=-2B_5+6B_1B_3+3B_2^2-4B_1^3$$

$$B_5^{(-3)}=-3B_1$$

$$B_6^{(-3)}=-3B_2$$

$$B_7^{(-3)}=-3B_3+6B_1^2$$

$$B_8^{(-3)}=-3B_4+12B_1B_2$$

$$B_6^{(-4)}=-4B_1$$

$$B_7^{(-4)}=-4B_2$$

$$B_8^{(-4)}=-4B_3+10B_1^2$$

$$B_7^{(-5)}=-5B_1$$

$$B_8^{(-5)}=-5B_2$$

$$B_8^{(-6)}=-6B_1$$

§ 5. Estimates.

The Principal Lemma will be applied for $1\leqq n\leqq 8$. It will be convenient to set $\beta=|b_1|=|B_1|$. Then $0\leqq\beta\leqq k$ for the family Σ_k' by the Area Theorem.

$n=1$. In this case (6) is identical to the first half of (2).
$n=2$. In this case (6) gives $|B_3|\leqq 1/2\,k(k+1)=k-1/2\,k(1-k)$ directly.
$n=3$. Since $|B_3^{(-1)}|^2=\beta^2\leqq k^2$, the estimate (6) leads to

$$|B_5|\leqq\frac{k(k+1)(k+2)}{3!}+k^3=2k-\frac{1}{3!}k(1-k)(10+7k)\,.$$

$n=4$. We use $|B_4^{(-2)}|^2=4\beta^2$ and apply (7) to $|B_4^{(-1)}|^2=|b_2|^2\leqq(1/2)(k^2-\beta^2)$. With these relations the inequality (6) reduces to

$$|B_7|\leqq\frac{k}{4!}[(6+11k+18k^2+k^3)+12(3+4k)\beta^2]\,.$$

Since the coefficient of β^2 is positive, we may estimate β^2 by k^2 and rewrite the resulting bound in the form stated in the Principal Theorem.

$n=5$. We substitute the bounds

$$|B_5^{(-3)}|^2=9\beta^2\,,\qquad |B_5^{(-2)}|^2=4|b_2|^2\leqq 2(k^2-\beta^2)\,,$$

$$|B_5^{(-1)}|^2\leqq[|B_3|+\beta^2]^2\leqq[(1/2)\,k(k+1)+\beta^2]^2$$

into (6) to obtain

$$|B_9|\leqq\frac{k}{5!}[(24+50k+185k^2+190k^3+31k^4)+60(4+9k+5k^2)\beta^2+120\beta^4]\,.$$

Since the coefficients of β^2 and β^4 are positive, we may replace β by k and rewrite the resulting bound in the form of the Principal Theorem.

$n=6$. Making use of (7), we substitute the bounds

$$|B_6^{(-4)}|^2=16\beta^2\,,\qquad |B_6^{(-3)}|^2=9|b_2|^2\leqq\frac{9}{2}(k^2-\beta^2)\,,$$

$$|B_6^{(-2)}|^2\leqq[2|B_3|+3\beta^2]^2\leqq[k(k+1)+3\beta^2]^2\,,$$

$$|B_6^{(-1)}|^2=|b_4+5b_1b_2|^2\leqq\left(\frac{1}{4}+\frac{25}{2}\beta^2\right)(k^2-\beta^2)$$

into (6) and obtain the estimate

$$|B_{11}|\leqq\frac{k}{6!}[(120+274k+1845k^2+2785k^3+1635k^4+361k^5)$$
$$+60(27+97k+261k^2+44k^3)\beta^2-360(16-9k)\beta^4]\,.$$

Since the coefficient of β^4 is negative, the estimate

(8) $$-\beta^4\leqq k^4-2k^2\beta^2$$

leads to

$$|B_{11}|\leqq\frac{k}{6!}[(120+274k+1845k^2+2785k^3+7395k^4-2879k^5)$$
$$+60(27+97k+69k^2+152k^3)\beta^2]\,.$$

The coefficient of β^2 is positive, and so we may estimate β^2 by k^2 and rearrange the result into the form given in the Principal Theorem.

$n=7$. In this case we use (7), the bounds already obtained for $|B_3|$ and $|B_5|$, and the estimates $\beta\leqq k$, $\beta^3\leqq k\beta^2$, $\beta^6\leqq k^2\beta^4$ to arrive at

$$|B_7^{(-5)}|^2=25\beta^2, \qquad |B_7^{(-4)}|^2=16|b_2|^2\leqq 8(k^2-\beta^2),$$

$$|B_7^{(-3)}|^2\leqq[3|B_3|+6\beta^2]^2\leqq\left[\frac{3}{2}k(k+1)+6\beta^2\right]^2,$$

$$|B_7^{(-2)}|^2=|2b_4+12b_1b_2|^2\leqq(1+72\beta^2)(k^2-\beta^2),$$

$$\begin{aligned}|B_7^{(-1)}|^2&\leqq[|B_5|+2|B_3|\beta+|b_2|^2+\beta^3]^2\\&\leqq(|B_5|^2+4k|B_3||B_5|)+(4|B_3|^2+2k|B_5|)\beta^2+(4|B_3|+k^2)\beta^4\\&\quad+(|B_5|+2k|B_3|+k\beta^2)(k^2-\beta^2)+\frac{1}{4}(k^2-\beta^2)^2\\&\leqq\frac{1}{36}(4k^2+48k^3+160k^4+240k^5+133k^6)+\frac{1}{6}(-2k-2k^2+11k^3+20k^4)\beta^2\\&\quad+\frac{1}{4}(1+4k+12k^2)\beta^4.\end{aligned}$$

Substitution of these bounds into (6) leads to

$$\begin{aligned}|B_{13}|\leqq\frac{k}{7!}[&(720+1764k+18564k^2+41685k^3+49665k^4+44751k^5+20511k^6)\\&+210(60+286k+1343k^2+1238k^3+157k^4)\beta^2-1260(95+68k-36k^2)\beta^4].\end{aligned}$$

Since the coefficient of β^4 is negative, the estimate (8) implies

$$\begin{aligned}|B_{13}|\leqq\frac{k}{7!}[&(720+1764k+18564k^2+41685k^3+169365k^4+130431k^5-24849k^6)\\&+210(60+286k+203k^2+422k^3+589k^4)\beta^2].\end{aligned}$$

Finally, since the resulting coefficient of β^2 is positive, we may replace β^2 by k^2 and rewrite the bound in the form given in the Principal Theorem.

$n=8$. Just as in the previous case we use (7), the bounds for $|B_3|$ and $|B_5|$, and the estimates $\beta\leqq k$ and $\beta^3\leqq k\beta^2$ to obtain

$$|B_8^{(-6)}|^2=36\beta^2, \qquad |B_8^{(-5)}|^2=25|b_2|^2\leqq\frac{25}{2}(k^2-\beta^2),$$

$$|B_8^{(-4)}|^2\leqq[4|B_3|+10\beta^2]^2\leqq[2k(k+1)+10\beta^2]^2,$$

$$|B_8^{(-3)}|^3=|3b_4+21b_1b_2|^2\leqq\left(\frac{9}{4}+\frac{441}{2}\beta^2\right)(k^2-\beta^2),$$

$$\begin{aligned}|B_8^{(-2)}|^2&\leqq[2|B_5|+6|B_3|\beta+3|b_2|^2+4\beta^3]^2\\&\leqq(4|B_5|^2+24k|B_3||B_5|)+(36|B_3|^2+16k|B_5|)\beta^2+48|B_3|\beta^4\end{aligned}$$

$$+16\beta^6+(6|B_5|+18k|B_3|+12k\beta^2)(k^2-\beta^2)+\frac{9}{4}(k^2-\beta^2)^2$$

$$\leqq\frac{1}{36}(16k^2+264k^3+1021k^4+1464k^5+700k^6)$$

$$+\frac{1}{6}(-12k-13k^2+132k^3+166k^4)\beta^2+\frac{1}{4}(9+48k+96k^2)\beta^4+16\beta^6,$$

$$|B_8^{(-1)}|^2=|b_6+7b_1b_4+(-7B_3+14b_1^2)b_2|^2$$

$$\leqq\left[\frac{1}{6}+\frac{49}{4}\beta^2+\frac{1}{2}\left(\frac{7}{2}k(k+1)+14\beta^2\right)^2\right][k^2-\beta^2].$$

Substitution of these bounds into (6) leads to

$$|B_{15}|\leqq\frac{k}{8!}[(5040+13068k+200172k^2+573489k^3+1359120k^4+2089122k^5$$

$$+1516788k^6+398721k^7)+168(620+3928k+27145k^2+45230k^3$$

$$+28025k^4+3732k^5)\beta^2-1680(1431+2551k-1326k^2-388k^3)\beta^4$$

$$-80640(45-4k)\beta^6].$$

Since the coefficients of β^4 and β^6 are negative, we may use the estimates (8) and

$$-\beta^6\leqq(k^4-2k^2\beta^2)\beta^2\leqq k^4\beta^2+2k^2(k^4-2k^2\beta^2)=2k^6-3k^4\beta^2$$

to obtain

$$|B_{15}|\leqq\frac{k}{8!}[(5040+13068k+200172k^2+573489k^3+3763200k^4+6374802k^5 \tag{9}$$

$$+6546708k^6-898239k^7)+168(620+3928k-1475k^2-5790k^3$$

$$-10255k^4+17252k^5)\beta^2].$$

One easily shows that the coefficient of $168\beta^2$ is positive. For example, if we denote this polynomial by $p(k)$ and if $0\leqq k\leqq 1$, then

$$(1+k)p(k)=k^2(62-131k^2)^2+k(44-83k^2)^2+1391k(1-k)+q(k)>0$$

where $q(k)=620+1221k+39k^3+199k^4+108k^5+91k^6>0$. Cosnequently, we may replace β^2 by k^2 in (9) and rewrite the resulting bound in the form given in the Principal Theorem.

In each case we used the estimate $\beta\leqq k$. Therefore equality can occur only if $|b_1|=k$. For $k=1$ this occurs only for the functions indicated in the Corollary to the Principal Theorem.

REFERENCES

[1] Y. KUBOTA, "Coefficients of meromorphic univalent functions," Kōdai Math. Sem. Rep. **28** (1977), 253-261.

[2] R. KUHNAU, "Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung," Math. Nachr. **40** (1969), 1-11.

[3] R. KÜHNAU, "Verzerrungssätze und Koeffizientenbedingungen von Grunskyschen Typ für quasikonforme Abbildungen," Math. Nachr. **48** (1971), 77-105.

[4] O. LEHTO, "Schlicht functions with a quasiconformal extension," Ann. Acad. Sci. Fenn. Ser. AI ♯**500** (1971), 10 pp.

[5] O. LEHTO, "Quasiconformal mappings and singular integrals," Symposia Mathematica, Instituto Nazionale di Alta Mathematica **18** (1976), 429-453.

[6] G. SCHOBER, "Univalent functions-selected topics," Lecture Notes in Math. **478** (1975), Springer-Verlag, 200 pp.

[7] G. SCHOBER, "Coefficients of inverses of meromorphic univalent functions," Proc. Amer. Math. Soc. **67** (1977), 111-116.

[8] G. SPRINGER, "The coefficient problem for schlicht mappings of the exterior of the unit circle," Trans. Amer. Math. Soc. **70** (1951), 421-450.

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