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ON CRITERIA OF $\tilde{g}$-HYPERELLIPTICITY

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I. Introduction. Let S be a compact Riemann surface of genus $g\geqq 2$. S is called $\tilde{g}$-hyperelliptic provided that S is a two-sheeted covering of a surface of genus $\tilde{g}$. 0-hyperelliptic and 1-hyperelliptic are called hyperelliptic and elliptic-hyperelliptic, respectively. Let P be an arbitrary point of S. Let $\phi_1, \cdots, \phi_g$ be a basis of the space of abelian differentials of the first kind on S. Let k_i be the order of the zero of ϕ_i at P. Then we can choose $\phi_1, \cdots, \phi_g$ such that $0=k_1<k_2<\cdots<k_g\leqq 2g-2$. The sequence $G(P)=\{k_1+1, k_2+1, \cdots, k_g+1\}$ is called the Weierstrass gap sequence at P. P is called a Weierstrass point of S if $k_g\geqq g$. Denote $N(P)$ the sequence $\{1, 2, \cdots, 2g\}-G(P)$. If k is in $N(P)$, then there is a meromorphic function on S which is holomorphic except for a pole of order k at P.

It is well known that if $N(P)=\{2, 4, \cdots, 2g\}$ for some P, then S is hyperelliptic and vice versa. If S is elliptic-hyperelliptic and P is a fixed point of an elliptic-hyperelliptic involution, then $N(P)$ contains $\{4, 6, 8, \cdots 2g\}$ and no odd number less than $2g-3$ can be contained in $N(P)$. Moreover, if S is $\tilde{g}$--hyperelliptic, $g\geqq 4\tilde{g}-1$ and P is a fixed point of the $\tilde{g}$-hyperelliptic involution, then $l(P^{g-1})$ is equal to $(g+1)/2-\tilde{g}$ or $g/2-\tilde{g}$ [6]. Here, $l(P^{g-1})$ is the dimension of the space of meromorphic functions on S whose divisors are multiples of P^{1-g}. This is related directly with the vanishing property of the theta function at $K(P)$, the Riemann constant vector in the Jacobian variety.

In this paper we shall study some criteria of $\tilde{g}$-hyperellipticity in terms of a property of the Weierstrass gap sequence, which is also reflected with a vanishing property of the theta function. Accola [1] has treated a related problem in terms of the vanishing property at half periods of the Jacobian variety.

2. Statement of Theorems. We shall prove the following theorems.

THEOREM 1. *Let S be a compact Riemann surface of odd genus $g\geqq 11$. If $l(P^{g-1})=(g-1)/2$ for some point P on S, then S is elliptic-hyperelliptic.*

THEOREM 2. *Let S be a compact Riemann surface of even genus $g\geqq 14$. If $l(P^{g-1})=g/2-1$ for some point P on S, then S is elliptic-hyperelliptic.*

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THEOREM 3. *Let S be a compact Riemann surface of odd genus $g\geqq 17$. If $l(P^{g-1})=(g-3)/2$ for some point P on S, then S is 2-hyperelliptic.*

THEOREM 4. *Let S be a compact Riemann surface of even genus $g\geqq 20$. If $l(P^{g-1})=g/2-2$ for some point P on S, then S is 2-hyperelliptic.*

For $\tilde{g}\geqq 3$, we cannot obtain such a criterion as above. We, however, obtain another criterion by a similar approach to Accola's [1, p. 70]. A sequence $N=\{i_1, i_2, \cdots, i_g\}$, $0<i_1<i_2<\cdots<i_g=2g$, is called admissible provided that for i_m and i_n is N, i_m+i_n also in N unless $i_m+i_n>2g$. Then we have

THEOREM 5. *Let S be a compact Riemann surface of genus g. Let $\{i_1, \cdots, i_{\tilde{g}}\}$ be an admissible sequence, where $g\geqq 8\tilde{g}+3$. Suppose that $N(P)=\{j_n\}$, for some point P on S, where $j_n=2i_n$ for $1\leqq n\leqq\tilde{g}$, $j_n=2n+2\tilde{g}$ for $\tilde{g}+1\leqq n\leqq 5\tilde{g}+2$ and $j_g=2g$. Then S is $\tilde{g}$-hyperelliptic.*

3. Lemmas. To prove Theorems we shall prepare some lemmas.

LEMMA 1. *Give integers $t\geqq 2$, $\alpha\geqq 2$, $\beta\geqq 0$, $1\leqq s_1<s_2<\cdots<s_n\leqq\alpha$. Then the number of integers $k\alpha+s_j$ $(k=0, 1, 2, \cdots, j=1, \cdots, n)$, satisfying $1+\beta\leqq k\alpha+s_j\leqq t+\beta$, does not exceed $n(t+\alpha-n)/\alpha$. Equality occurs only if $n=\alpha$ or $n=t-m\alpha$, for some m.*

Proof. Let M be the number which we shall estimate. Put $t=m\alpha+t_1$, $0\leqq t_1<\alpha$. Then we have $M\leqq nm+\min(t_1, n)$. If $n\leqq t_1$, then we have $M\leqq nm+n=n(t-t_1)/\alpha+n\leqq n(t-n)/\alpha+n$. If $n>t_1$, then we have $M\leqq nm+t_1=n(t+\alpha-n)/\alpha+(n-\alpha)(n-t_1)/\alpha\leqq n(t+\alpha-n)/\alpha$.

LEMMA 2 (Jenkins [5]). *Let S be a compact Riemann surface of genus g and let P be a point on S. If h and k are in $N(P)$ and $(h, k)=1$* i.e. *h and k are coprime, then $g\leqq(h-1)(k-1)/2$.*

Since $N(P)$ is an admissible sequence, we have immediately the following:

LEMMA 3 (Hurwitz [4], Kusunoki [7]). *Let S and P be as in Lemma 2. Let k be the least number of $N(P)$,* i.e. *k is the first non-gap. Then $G(P)$ consists of*

$$\begin{Bmatrix} 1, & k+1, & \cdots, m_1k+1 \\ 2, & k+2, & \cdots, m_2k+2 \\ \cdots & & \cdots \\ k-1, & k+k-1, & \cdots, m_{k-1}k+k-1 \end{Bmatrix}$$

where $\sum_{j=1}^{k-1}(m_j+1)=g$ and $0<m_jk+j<2g$.

To prove the following four lemmas, we shall use Lemma 3 frequently but implicitly. Since all the proofs of these four lemmas are in similar ways, we shall only note the proof of Lemma 7. The proofs of the other lemmas are simpler than that.

LEMMA 4. *Let S and P be as in Theorem* 1. *Then $N(P)$ is either* $\{4, 6, 8, \cdots, 2g-4, 2g-3, 2g-2, 2g\}$ *or* $\{4, 6, 8, \cdots, 2g-4, 2g-2, 2g-1, 2g\}$.

LEMMA 5. *Let S and P be as in Theorem* 2. *Then $N(P)$ is either* $\{4, 6, 8, \cdots, 2g-4, 2g-3, 2g-2, 2g\}$ *or* $\{4, 6, 8, \cdots, 2g-4, 2g-2, 2g-1, 2g\}$.

LEMMA 6. *Let S and P be as in Theorem* 3. *Then $N(P)$ is one of the following*: $\{4, 8, 10, 12, \cdots, 2g-8, 2g-6, 2g-4, 2g-3, 2g-2, 2g-1, 2g\}$, $\{4, 8, 10, 12, \cdots 2g-8, 2g-6, 2g-5, 2g-4, 2g-2, 2g-1, 2g\}$, $\{4, 8, 10, 12, \cdots, 2g-8, 2g-7, 2g-6, 2g-4, 2g-3, 2g-2, 2g\}$, $\{6, 8, 10, 12, \cdots, 2g-8, 2g-6, 2g-4, 2g-3, 2g-2, 2g-1, 2g\}$, $\{6, 8, 10, 12, \cdots, 2g-8, 2g-6, 2g-5, 2g-4, 2g-2, 2g-1, 2g\}$, $\{6, 8, 10, 12, \cdots, 2g-8, 2g-6, 2g-5, 2g-4, 2g-3, 2g-2, 2g\}$, $\{6, 8, 10, 12, \cdots, 2g-8, 2g-7, 2g-6, 2g-4, 2g-2, 2g-1, 2g\}$.

LEMMA 7. *Let S and P be as in Theorem* 4. *Then $N(P)$ is as in the preceding lemma.*

Proof. Put $N(P)=\{k_j; j=1, 2, \cdots, 2r+6\}$. Here $r=g/2-3\geqq 7$, $k_i<k_j$, if $i<j$ and $k_{2r+6}=2g$. Since $l(P^{g-1})=r+1$, we have $k_r\leqq g-1<k_{r+1}$. Put $M=\{k_1, k_2, \cdots, k_r, k_r+k_1, k_r+k_2, \cdots, k_r+k_{r-1}, 2k_r, 2g\}$. It is obvious that M is included in $N(P)$. Since $^{\#}M=2r+1$, there are at most five k's between k_r and k_r+k_1. We shall consider the following 16 cases.

Case I) $k_1\leqq 3$. Case II) $k_1\geqq 4$ and $k_r+k_1=k_{r+1}$. Case III) $k_1=4, 5$ and $k_r+k_1=k_{r+2}$. Case IV) $k_1\geqq 6$ and $k_r+k_1=k_{r+2}$. Case V) $k_1=4$ and $k_r+k_1=k_{r+3}$. Case VI) $5\leqq k_1\leqq 7$ and $k_r+k_1=k_{r+3}$. Case VII) $k_1\geqq 8$ and $k_r+k_1=k_{r+3}$. Case VIII) $k_1=4, 5$ and $k_r+k_1=k_{r+4}$. Case IX) $6\leqq k_1\leqq 9$ and $k_r+k_1=k_{r+4}$. Case X) $k_1\geqq 10$ and $k_r+k_1=k_{r+4}$. Case XI) $4\leqq k_1\leqq 7$ and $k_r+k_1=k_{r+5}$. Case XII) $8\leqq k_1\leqq 10$ and $k_r+k_1=k_{r+5}$. Case XIII) $k_1\geqq 11$ and $k_r+k_1=k_{r+5}$. Case XIV) $4\leqq k_1\leqq 8$ and $k_r+k_1=k_{r+6}$. Case XV) $9\leqq k_1\leqq 12$ and $k_r+k_1=k_{r+6}$. Case XVI) $k_1\geqq 13$ and $k_r+k_1=k_{r+6}$.

In the following discussion we shall not write down that "this is a contradiction". But it will be almost clear in the context.

Put $N'(P)=N(P)\cap\{1, 2, \cdots, g-1\}$ and $G'(P)=G(P)\cap\{g, g+1, \cdots, 2g-1\}$.

Case I) If $k_1=2$, then S is hyperelliptic. Hence, $l(P^{g-1})=(g-2)/2=r+2$. Suppose $k_1=3$. If k is in $N(P)$ and $(3, k)=1$, then by Lemma 2, $k\geqq g+1$. Hence,

$$^{\#}N'(P)=r=\left[\frac{g-1}{k_1}\right]\leqq\frac{2r+5}{3} \quad \text{and} \quad r\leqq 5.$$

Here $[s]$ denotes the integer part of s.

Case II) In this case $k_1|k_j$ for $j\leqq r$. Therefore,

$$^{\#}N'(P)=r=\left[\frac{g-1}{k_1}\right]\leqq\frac{2r+5}{4} \quad \text{and} \quad r<3.$$

Case III) Suppose $k_1=5$. If k is in $N(P)$ and $(5, k)=1$, then by Lemma 2,

$k \geqq r+4 \geqq 11$. Hence,

$$^{\#}N'(P)=r \leqq 2\left[\frac{2r+5}{5}\right]-1 \quad \text{and} \quad r \leqq 5.$$

Suppose $k_1=4$ and $(4, k_2)=1$. Then by Lemma 2 we have $k_2 \geqq 15$. Substituting $n=2$, $\alpha=4$, $\beta=14$ and $t=g-15$ in Lemma 1, we have

$$^{\#}N'(P)=r \leqq \frac{2(g-15+4-2)}{4}+3=r-\frac{1}{2}.$$

Suppose $k_1=4$ and $(4, k_2) \geqq 2$. If $k_2 \geqq 10$, then

$$^{\#}N'(P)=r \leqq \left[\frac{2r+5}{2}\right]-3=r-1.$$

If $k_2=6$, then

$$^{\#}N'(P)=r=\left[\frac{2r+5}{2}\right]-1=r+1.$$

If $k_2=8$ and $k_3=10$, then $N(P)$ is one of the first three of the desired result. If $k_2=8$, $k_3=12$, then

$$^{\#}N'(P)=r \leqq \left[\frac{2r+5}{2}\right]-3=r-1.$$

If $k_2=8$ and k_3 is odd, then by Lemma 2, $k_3 \geqq 15$. But 12 is in $N(P)$.

Case IV) Substituting $n=2$, $\alpha=k_1$, $\beta=k_1-1$ and $t=g-k_1$ in Lemma 1, we have

$$^{\#}N'(P)=r \leqq \frac{2(g-2)}{k_1} \leqq \frac{2(2r+4)}{6} \quad \text{and} \quad r \leqq 4.$$

Case V) Substituting $n=1$, $\alpha=4$, $\beta=g-1$ and $t=g$ in Lemma 1, we have

$$^{\#}G'(P)=r+1 \leqq \frac{g+3}{4}=\frac{2r+9}{4} \quad \text{and} \quad r \leqq 2.$$

Case VI) Since $^{\#}(M \cup \{k_{r+1}, k_{r+2}\})=2r+3$, we shall consider the following five subcases. Case VI-1) $k_2=2k_1$, $k_4=3k_1$. Case VI-2) $k_2=2k_1$, $k_5=3k_1$. Case VI-3) $k_3=2k_1$, $k_5=3k_1$. Case VI-4) $k_3=2k_1$, $k_6=3k_1$. Case VI-5) $k_4=2k_1$, $k_7=3k_1$.

Case VI-1) Since $k_2=2k_1$, by use of Lemma 1 we have

$$^{\#}(N'(P)-\{k_1\})=r-1 \leqq \frac{3(g-k_1-3)}{k_1} \quad \text{and} \quad k_1=5.$$

Since $k_2=2k_1$, $k_4=3k_1$, we have

$$\begin{aligned} N(P)=\{&k_1, \cdots, k_6, k_1+k_4, k_1+k_5, k_1+k_6, 2k_1+k_4, 2k_1+k_5, \\ &2k_1+k_6, k_4+k_5, k_4+k_6, \cdots, k_4+k_r, k_5+k_r, k_6+k_r, \\ &\cdots, k_r+k_r, 2g\}. \end{aligned}$$

Hence, $k_5+k_6=7k_1$. If $k_1+k_3=k_5$, then $2k_3=k_1+k_6$. Therefore, $3k_3=7k_1$ and $k_1=5$. If $k_1+k_3=k_6$, then $2k_3=2k_1+k_4=5k_1$ and $k_1=5$.

Case VI-2) Since $k_2=2k_1$, we have $k_1=5$. Put

$$M'=\{k_1, \cdots, k_7, k_2+k_2, k_2+k_3, \cdots, k_2+k_r, k_3+k_r, \cdots, k_r+k_r, 2g\}.$$

Then M' is included in $N(P)$ and $^{\#}M'=2r+5$. If $2k_3=k_2+k_4$ and $k_3+k_4=k_2+k_5=5k_1$, then $3k_3=7k_1$ and $k_1=5$. If $2k_3=k_2+k_4$ and $k_3+k_4\neq k_2+k_5$, then neither k_3+k_4 nor k_3+k_7 is in M' and $^{\#}N(P)\geqq 2r+7$. If $2k_3=k_2+k_5$, then $2k_3=5k_1$ and $k_1=5$. If $2k_3\neq k_2+k_4$ and $2k_3\neq 5k_1$, then neither $2k_3$ nor k_3+k_6 is in M' and $^{\#}N(P)\geqq 2r+7$.

Case VI-3) Applying Lemma 1, we have

$$^{\#}(N'(P)-\{k_1, \cdots, k_4\})=r-4\leqq \frac{3(g-3-2k_1)}{k_1} \quad \text{and} \quad k_1=5.$$

If $2k_3=k_7$, then $k_6=2k_1+k_2$ and $2k_2=3k_1$. Since $r\geqq 7$, if $2k_3>k_7$ and $2k_2>3k_1$, then $k_4=k_2+k_1$, $k_6=2k_2$, $k_7=2k_1+k_2$, $k_8=4k_1$, $k_9=2k_2+k_1$, $k_{10}=3k_1+k_2$ and $3k_2>3k_1+k_2$. Since $5k_1=3k_2$, $k_9+k_1\geqq k_{13}$. By use of Lemma 1, we have

$$^{\#}(N'(P)-\{k_1, \cdots, k_8\})=2r-2\geqq \frac{4(4r-10+5-4)}{5}$$

$$=\frac{4(4r-9)}{5} \quad \text{and} \quad r\leqq 5.$$

Case VI-4) Since $r\geqq 7$, we have $k_7=k_1+k_4$, $k_8=k_1+k_5$, $k_9=4k_1$, $k_{10}=k_1+k_7$. If $k_1+k_2=k_4$ and $2k_2=k_5$, then $k_2+k_5<k_2+k_6=k_1+k_7=k_{10}$. Hence, $k_2+k_5=k_9=4k_1$ and $3k_2=4k_1$, that is $k_1=6$, $k_2=8$, $k_5=16$. This implies

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-3=r-1.$$

If $k_1+k_2=k_4$ and $2k_2=k_6=3k_1$, then $k_2+k_5<k_2+k_6=k_{10}$. Hence, $k_2+k_5=k_9=k_2+k_4$. If $k_1+k_2=k_5$ and $2k_2=k_6$, then $k_2+k_4<k_2+k_5=k_9$. Hence, $k_2+k_4=k_1+k_5$ and $k_5=2k_2$. If $k_1+k_2=k_5$ and $2k_2=k_7$, then $k_2+k_4<k_2+k_5=k_{10}$. In the case of $k_2+k_4=k_8$, we have $k_4=2k_1$. In the case of $k_2+k_4=k_9$, since $k_4=2k_2-k_1$, we have $3k_2=5k_1$. Hence,

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-3=r-1.$$

Case VI-5) Since $r\geqq 7$, we have $k_5=k_1+k_2$, $k_6=k_1+k_3$, $k_8=2k_1+k_2$ and $k_9=2k_1+k_3$. If $2k_2=k_6$, then $k_2+k_3=k_7=3k_1$. Hence, $3k_2=4k_1$. Therefore, $N(P)$ is one of the last four of the desired result. If $2k_2=k_7$, then $k_7<k_2+k_3<2k_1+k_2=k_8$.

Case VII) Substituting $n=3$, $\alpha=k_1$, $\beta=k_1-1$ and $t=g-k_1$ in Lemma 1, we have

$$^{\#}N'(P)=r\leqq \frac{3(g-3)}{k_1}=\frac{3(2r+3)}{k_1} \quad \text{and} \quad r\leqq 4.$$

Case VIII) Substituting $n=k_1-4$, $\alpha=k_1$, $\beta=g-1$ and $t=g$ in Lemma 1, we have

$$^{\#}G'(P)=r+1\leqq\frac{(k_1-4)(g+4)}{k_1}\leqq\frac{2r+10}{5}\quad\text{and}\quad r\leqq 1\,.$$

Case IX) We shall divide this case into the following four subcases. Case IX-1) $2k_1=k_3, 3k_1=k_7$. Case IX-2) $2k_1=k_4$, $3k_1=k_7$. Case IX-3) $2k_1=k_4$, $3k_1=k_8$. Case IX-4) $2k_1=k_5$, $3k_1=k_9$.

Case IX-1) Since $k_{11}=2k_3$, we have

$$\begin{aligned}N(P)&=\{k_1,\cdots,k_{10}, 2k_3, k_3+k_4,\cdots,k_3+k_r, k_4+k_r,\cdots,2k_r, 2g\}\\&=\{k_1,\cdots,k_{10}, 2k_3, k_3+k_4, 2k_4, k_4+k_5,\cdots,k_4+k_r, k_5+k_r,\cdots,\\&\quad 2k_r, 2g\}\,.\end{aligned}$$

Therefore, $k_3+k_5=2k_4$, $k_3+k_6=k_4+k_5$, $k_3+k_7=k_4+k_6$, $k_3=2k_1$ and $k_7=3k_1$. Hence, $4k_4=9k_1$, $2k_5=5k_1$, $4k_6=11k_1$ and $k_1=8$, $k_4=18$, $k_5=20$, $k_6=22$. Thus we have

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-5=r-2\,.$$

Case IX-2) Since $k_5=k_1+k_2$, $k_6=k_1+k_3$ and $k_{11}=4k_1$, one of k_8, k_9 and k_{10}, denoted by k', is neither $2k_1+k_2$ nor $2k_1+k_3$. Then we have

$$\begin{aligned}N(P)=\{&k_1,\cdots,k_{11}, k_4+k_5, k_4+k_6,\cdots,k_4+k_r, k_5+k_r,\\&k_6+k_r,\cdots,2k_r, 2g\}\cup\{k_1+k', k'+k_r-k_1\}\end{aligned}$$

and $k_5<2k_2<k_2+k_3<2k_3<k_3+k_4<2k_4=k_{11}$.

i) Suppose $2k_2=k_6$ and $k_2+k_3=k_7$. Then we have $4k_1=3k_2$ and $k_1=6$, $k_2=8$, $k_3=10$. If $k_2+k'=k_3+3k_1$, then $k'=20=2k_1+k_2$. If $k_2+k'=5k_1$, then $k'=22=2k_1+k_3$.

ii) Suppose $2k_2=k_6$, $k_2+k_3=k_8$ and $2k_3=k_9$. Since $k_8<k_2+k_4$, we have $k_9=k_2+k_4$ and $k_2=2(k_3-k_1)$. Therefore, $3k_2=4k_1$, $3k_3=5k_1$ and $k_2+k_3=3k_1=k_7$.

iii) Suppose $2k_2=k_7$, $k_2+k_3=k_8$, $k_2+k_4=2k_3=k_9$ and $k_3+k_4=k_{10}$. Then we have $2k_2=3k_1$, $4k_3=7k_1$ and $k_1=8$, $k_2=12$, $k_3=14$. Therefore,

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-5=r-3\,.$$

Case IX-3) Since $r\geqq 7$, $k_9=k_5+k_1$, $k_{10}=k_6+k_1$ and $k_{11}=k_7+k_1$.

i) Suppose $k_5=k_2+k_1$, $k_6=k_3+k_1$. Then $k_5<2k_2<k_2+k_3<2k_3\leqq k_2+k_4=k_9$. If $2k_2=k_6$, $k_2+k_3=k_7$, $2k_3=k_8=3k_1$, then $4k_2=5k_1$ and

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-3=r-1\,.$$

If $2k_2=k_6$, $k_2+k_3=k_8$, then $3k_2=4k_1$, $3k_3=5k_1$ and $k_1=6$, $k_2=8$, $k_3=10$, $k_7=17$. Thus every $k\geqq 23$ is in $N(P)$. Therefore,

$$^{\#}N(P)=\left[\frac{2g}{2}\right]-2+\left[\frac{2g-21}{2}\right]+1\geqq g+8\,.$$

If $2k_2=k_7$, $k_2+k_3=k_8$ and $2k_3=2k_1+k_2$, then $3k_3=5k_1$ and $3k_2=4k_1$. Thus $2k_2=k_1+k_3<k_7$.

ii) Suppose $k_5=k_1+k_2$ and $k_7=k_1+k_3$. Then $k_2+k_3=k_8$. If $k_7=2k_2$, then $3k_2=4k_1$ and $3k_3=5k_1$. Hence, $k_1=6$, $k_2=8$, $k_3=10$, $k_6=15$. Thus every $k\geqq 21$ is in $N(P)$. If $k_9=2k_3$, then $k_7=2k_2$ which reduces to the above case. If $k_6=2k_2$ and $k_9\neq 2k_3$, then $k_{10}=2k_3$ and $k_1=8$, $k_2=10$, $k_3=14$. Hence,

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-4=r-2\,.$$

iii) Suppose $k_6=k_1+k_2$ and $k_7=k_1+k_3$. If $2k_2=k_7$, $k_2+k_3=k_8$, then $3k_2=4k_1$ and $3k_3=5k_1$. Therefore, $k_1=6$, $k_2=8$, $k_3=10$, $k_5=13$ and $k_{10}<k_2+k_5<k_{11}$. If $2k_2=k_7$ and $k_2+k_3=k_9$, then $2k_3=k_2+k_5$ and $k_9=k_1+k_5<k_2+k_5<k_2+k_6=k_{11}$. Hence, $k_2+k_5=k_1+k_6=2k_1+k_2$ and $k_5=2k_1$. If $2k_2=k_8$ and $k_2+k_3=k_9$, then $2k_3=k_{10}$. Therefore, $2k_2=3k_1$, $4k_3=7k_1$ and $k_1=8$, $k_2=12$, $k_3=14$, $k_5=18$. Hence,

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-4=r-2\,.$$

Case IX-4) Since $k_7=k_1+k_3\leqq 2k_2<k_2+k_3<2k_3<k_3+k_4<k_{11}$, we have $k_1+k_3=2k_2$, $k_2+k_3=k_1+k_4$ and $2k_3=3k_1$. Hence, $k_1=8$, $k_2=10$, $k_3=12$, $k_4=14$ and

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-3=r-1\,.$$

Case X) Substituting $n=4$, $\alpha=k_1$, $\beta=k_1-1$ and $t=g-k_1$ in Lemma 1, we have

$$^{\#}N'(P)=r\leqq\frac{4(g-4)}{k_1}\leqq\frac{8r+8}{10}\quad\text{and}\quad r\leqq 4\,.$$

Case XI) Substituting $n=k_1-5$, $\alpha=k_1$, $\beta=g-1$ and $t=g$ in Lemma 1, we have

$$^{\#}G'(P)=r+1\leqq\frac{(k_1-5)(g+5)}{k_1}=\frac{2(2r+11)}{7}\quad\text{and}\quad r\leqq 5\,.$$

Case XII) In this case we shall consider the following two subcases. Case XII-1) $2k_1=k_5$. Case XII-2) $2k_1=k_6$.

Case XII-1) Since $^{\#}\{k_1,\cdots,k_r,\cdots,k_{r+4},k_r+k_1,k_r+k_2,\cdots,2k_r,2g\}=2r+5$, $k_{j+5}-k_j=k_1$ for every j, $5\leqq j\leqq g-5$. Therefore, $k_{10}=3k_1$, $k_{11}=k_1+k_6$, $k_{12}=k_1+k_7$, $k_{13}=k_1+k_8$.

i) Suppose $k_6=k_1+k_2$, $k_7=k_1+k_3$ and $k_8=k_1+k_4$. Then we have $k_7\leqq 2k_2<k_2+k_3<2k_3<k_3+k_4<2k_4<k_{13}$. If $2k_2=k_7$, $k_2+k_3=k_8$ and $2k_4=k_{11}$, then $5k_2=6k_1$, $5k_3=7k_1$ and $5k_4=8k_1$. Therefore, $k_1=10$, $k_2=12$, $k_3=14$, $k_4=16$ and $k_9=28$; that is

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-5=r-3\,.$$

If $2k_2=k_7$, $k_2+k_3=k_8$ and $2k_4=k_{12}$, then $4k_2=5k_1$, $2k_3=3k_1$ and $4k_4=7k_1$. Hence,

$k_1=8$, $k_2=10$, $k_3=12$, $k_4=14$ and $k_9=23$. Therefore, every $k \geqq 31$ is in $N(P)$. If $2k_3 \neq k_2+k_4$ or if $2k_3=k_2+k_4=k_9$, then $2k_2=k_7$ and $k_2+k_3=k_8$. If $2k_3=k_2+k_4=k_{10}$, then $k_3+k_4=k_{11}$ and $2k_4=k_{12}$. Hence, every $k \geqq 31$ is in $N(P)$.

ii) Suppose $k_6=k_1+k_2$, $k_7=k_1+k_3$ and $k_9=k_1+k_4$. Then we have $k_9<k_2+k_4<k_3+k_4<k_{12}$ and $k_9 \leqq 2k_3<k_3+k_4$. If $2k_3=k_2+k_4=k_{10}$ and $k_3+k_4=k_{11}$, then $4k_2=5k_1$, $2k_3=3k_1$ and $4k_4=7k_1$. Hence, $k_1=8, \cdots, k_4=14$, $k_8=21$ and every $k \geqq 29$ is in $N(P)$. If $2k_3=k_9$, $k_2+k_4=k_{10}$ and $k_3+k_4=k_{11}$, then $5k_2=6k_1$, $5k_3=7k_1$ and $5k_4=9k_1$. Hence,

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-4=r-2\,.$$

iii) Suppose $k_6=k_1+k_2$, $k_8=k_1+k_3$ and $k_9=k_1+k_4$. Then we have $k_8<k_2+k_3<k_2+k_4<k_{11}$ and $k_{10}<k_3+k_4<2k_4<k_{14}$. If $2k_3=k_2+k_4$, then $4k_2=5k_1$, $2k_3=3k_1$ and $4k_4=7k_1$. Hence, $k_1=8, \cdots, k_4=14$, $k_7=19$ and every $k \geqq 27$ is in $N(P)$. If $2k_3>k_2+k_4$, then $2k_3=2k_1+k_2$. Hence, $5k_2=6k_1$, $5k_3=8k_1$ and $5k_4=9k_1$. Therefore,

$$^{\#}N'(P) \leqq \left[\frac{2r+5}{2}\right]-4=r-2\,.$$

iv) Suppose $k_7=k_1+k_2$, $k_8=k_1+k_3$, $k_9=k_1+k_4$. Then we have $k_7<2k_2<k_2+k_3<k_2+k_4<k_3+k_4<2k_4<k_{14}$. If $2k_2=k_8$, $k_2+k_3=k_9$, $k_2+k_4=k_{10}$, then $4k_2=5k_1$, $2k_3=3k_1$ and $4k_4=7k_1$. Hence, $k_1=8, \cdots, k_4=14$, $k_6=17$ and every $k \geqq 25$ is in $N(P)$. If $k_2+k_3=k_9$, $k_3+k_4=k_{12}$ and $2k_4=k_{13}$, then $4k_2=5k_1$, $2k_3=3k_1$ and $4k_4=7k_1$. Hence, $k_1=8, \cdots, k_4=14$ and $k_6=17$. If $k_2+k_3=k_{10}$, $k_3+k_4=k_{12}$ and $2k_4=k_{13}$, then $5k_2=7k_1$, $5k_3=8k_1$ and $5k_4=9k_1$. Hence,

$$^{\#}N'(P) \leqq \left[\frac{2r+5}{2}\right]-4=r-2\,.$$

Case XII-2) Since $r \geqq 7$, $k_{12}-k_7=k_1$ and $k_{12}=2k_1+k_2=k_2+k_6$. Hence, $k_7<2k_2<k_2+k_3<k_2+k_4<k_2+k_5<k_2+k_6=k_{12}$. Therefore, $2k_2=k_1+k_3$, $k_2+k_3=k_1+k_4$, $k_2+k_4=k_1+k_5$ and $k_2+k_5=3k_1$. Hence, $k_1=10$, $k_2=12$, $k_3=14$, $k_4=16$ and $k_5=18$; that is

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-4=r-2\,.$$

Case XIII) Substituting $n=5$, $\alpha=k_1$, $\beta=k_1-1$ and $t=g-k_1$ in Lemma 1, we have

$$^{\#}N'(P)=r \leqq \frac{5(g-5)}{k_1} \leqq \frac{10r+5}{11} \quad \text{and} \quad r \leqq 5\,.$$

Case XIV) Substituting $n=k_1-6$, $\alpha=k_1$, $\beta=g-1$ and $t=g$ in Lemma 1, we have

$$^{\#}G'(P)=r+1 \leqq \frac{(k_1-6)(g+6)}{k_1} \leqq \frac{r+6}{2} \quad \text{and} \quad r \leqq 5\,.$$

Case XV) Since $N(P)=\{k_1, \cdots, k_r, k_{r+1}, \cdots, k_{r+5}, k_r+k_1, \cdots, 2k_r, 2g\}$, we have $k_7=2k_1$, $k_{13}=3k_1$ and $k_{14}=2k_1+k_2$. Therefore, $k_8<2k_2<k_2+k_3<k_2+k_4<k_2+k_5<k_2+k_6<k_2+k_7=k_{14}$. Hence, $k_2+k_3=k_1+k_4$, $k_2+k_4=k_1+k_5$, $k_2+k_5=k_1+k_6$ and $k_2+k_6=3k_1$. Thus $k_1=12, \cdots, k_6=22$ and

$$^{\#}N'(P)=\left[\frac{2r+5}{2}\right]-5=r-3\,.$$

Case XVI) Substituting $n=6$, $\alpha=k_1$ $\beta=k_1-1$ and $t=g-k_1$, we have

$$^{\#}N'(P)=r\leqq\frac{6(g-6)}{k_1}\leqq\frac{12r}{13}\quad\text{and}\quad r\leqq 0.$$

This completes the proof.

The following lemma is an analogy of a theorem of Castelnuovo (cf. Accola [1]).

LEMMA 8. *Under the same hypothesis as in Theorem* 5, *the linear series* $|P^{2r}|$ *is composite for every* r *such that* $3\tilde{g}+1\geqq r\geqq 2\tilde{g}+2$.

Proof. This proof is also an analogy of that of Accola. Suppose that $|P^{2r}|$ is simple. Since $\gamma_{5\tilde{g}+2}=12\tilde{g}+4\geqq 4r$, $|P^{2r}|$ and $|P^{4r}|$ are of dimension $r-\tilde{g}$ and $2r-\tilde{g}$, respectively.

Since $|P^{2r}|$ has no fixed point and is simple, there is a birational map of S onto a curve C in $\boldsymbol{P}^{r-g}(C)$ of degree $2r$. Fix a hyperplane H such that the points of the hyperplane section, say $\{P_1, P_2, \cdots, P_{2r}\}$, are in general position in H (cf. [2]).

Let $\{P_1, P_2, \cdots, P_r\}$ be a subset of $\{P_1, P_2, \cdots, P_{2r}\}$ satisfying $l(P^{2r}P_{r+1}P_{r+2}\cdots P_{2r})=l(P^{2r})$. Then any quadric through $\{P_1, P_2, \cdots, P_r\}$ passes through the remaining r points $\{P_{r+1}, P_{r+2}, \cdots, P_{2r}\}$.

Since through any $2(r-\tilde{g}-1)$ points of $\{P_1, P_2, \cdots P_{2r}\}$ we can find two hyperplanes (that is a quadric) containing no further P's we have $2r-2\tilde{g}-1\leqq r$, which is a contradiction.

4. Proof of Theorems.

It is evident that $8\tilde{g}+3=11$ for $\tilde{g}=1$ and the conclusions of Lemmas 4 and 5 satisfy the hypothesis of Theorem 5. Hence, the proofs of Theorems 1 and 2 are reduced to the proof of Theorem 5.

Proof of Theorems 3 *and* 4. We shall consider the following two cases.

Case I) $N(P)$ contains $\{4, 8, 10, 12, 14, 16, 18, 20\}$ and $G(P)$ contains $\{1, 2, 3, 5, 6, 7, 9, 11, 13, 15, 17, 19\}$.

Case II) $N(P)$ contains $\{6, 8, 10, 12, 14, 16, 18\}$ and $G(P)$ contains $\{1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17\}$.

By virtue of Lemmas 6 and 7 and $g\geqq 17$, it is sufficient to consider the above two cases. In this proof we shall consider meromorphic functions on S with pole only at P. For simplicity's sake, we call the order of the pole of such a meromorphic function "the order of the function".

Case I) Let x be a function of order 4 and let y be a function of order 10. Then x^2 is of order 8, x^3 is of order 12, xy is of order 14, x^4 is of order 16,

x^2y is of order 18 and both x^5 and y^2 are of order 20. Therefore, we have an algebraic equation

$$y^2+Q_1(x)y+Q_2(x)=0,$$

where $Q_1(x)$ is a polynomial of degree less than three and $Q_2(x)$ is a polynomial of degree 5. This represents an algebraic plane curve $\tilde{S}$ of genus at most two. Hence, S is an n-sheeted covering of $\tilde{S}$ for some $n\geqq 2$. Let P' be a point on $\tilde{S}$ which corresponds to $x=\infty$. Since $\deg Q_1\leqq 2$ and $\deg Q_2=5$, x is a meromorphic function on $\tilde{S}$ with pole at P' of order 2 and holomorphic elsewhere. Let π be the projection map of S into $\tilde{S}$. Then $\pi^{-1}(P')=\{P\}$. Thus $n=2$. If the genus of $\tilde{S}$ is zero or one, then there is a function of order 6 on S. Therefore, $\tilde{S}$ is of genus two.

Case II) Suppose that x is a function of order 6, y is a function of order 8 and that z is a function of order 10. Then x^2 is of order 12, xy is of order 14, both xz and y^2 are of order 16 and both x^3 and yz are of order 18. Considering xz and y^2, we have

$$y^2=Q_1(x)z+Q_2(x)y+Q_3(x),$$

where $\deg Q_1=1$, $\deg Q_2\leqq 1$ and $\deg Q_3\leqq 2$. Considering x^3, yz and xz, we have

$$yz=Q_4(x)z+Q_5(x)y+Q_6(x),$$

where $\deg Q_4\leqq 1$, $\deg Q_5\leqq 1$ and $\deg Q_6=3$. Eliminating z from the above two equations, we have

$$(y-Q_4(x))(y^2-Q_2(x)y-Q_3(x))=Q_1(x)(Q_5(x)y+Q_6(x)).$$

That is

$$y^3+Q_7(x)y^2+Q_8(x)y+Q_9(x)=0,$$

where $\deg Q_7\leqq 1$, $\deg Q_8\leqq 2$ and $\deg Q_9=4$. This represents an algebraic plane curve $\tilde{S}$ of degree 4. By the formula [3, p. 201], the genus of $\tilde{S}$ is less than or equal to 3. We know that the set of points corresponding to $x=\infty$ consists of only one point. Denote it by P'. Since $\tilde{S}$ is a three-sheeted covering of the x-sphere, the polar divisor of x on $\tilde{S}$ is P'^3, As in the preceding case, we conclude that S is a two-sheeted covering of $\tilde{S}$. Since $x, y, z\ (=(y^2-Q_2(x)y-Q_3(x))/Q_1(x))$ are meromorphic functions on $\tilde{S}$ whose polar divisors are P'^3, P'^4 and P'^5. respectively, the genus of $\tilde{S}$ is two.

Proof of Theorem 5. By Lemma 8, $|P^{6\tilde{g}+2}|$ is composite. Hence, there is a Riemann surface $\tilde{S}$ such that S is an n-sheeted covering of $\tilde{S}$ for some $n\geqq 2$ and there is a complete linear series $g^{2\tilde{g}+1}_{(6\tilde{g}+2)/n}$ on $\tilde{S}$. If $n\geqq 3$, then $2\tilde{g}+1\leqq(6\tilde{g}+2)/n \leqq(6\tilde{g}+2)/3$. This is a contradiction. Therefore, S is a two-sheeted covering of $\tilde{S}$. Hence, $|P^{3\tilde{g}+1}|$ is a complete linear series on $\tilde{S}$. Observing that there exists a function of order $2k$ on S if and only if there exists a function of order k on $\tilde{S}$ [6, p. 392], we know that ${}^{\#}G(P)$ on $\tilde{S}$ is $\tilde{g}$. Therefore, $\tilde{S}$ is of genus $\tilde{g}$.

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