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HIGHLY CONNECTED POINCARÉ COMPLEXES

Dedicated to Professor A. Komatu on his 70th birthday

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Introduction.

We are interested in the following problem proposed by Wall in [2] "Classify up to homotopy $(n-1)$-connected Poincaré complexes of dimension $2n+1$ and $2n+2$."

In this paper we shall discuss the case of dimension $2n+2$ under some additional conditions. Let K be a Poincaré complex which is $(n-1)$-connected and of dimension $2n+2$. If K has the same rational homology as the sphere, then the homology $H_*(K;Z)$ is as follows

$$H_0(K;Z)=Z=H_{2n+2}(K;Z)$$
$$H_n(K;Z)=G=H_{n+1}(K;Z)$$
$$H_i(K;Z)=0 \qquad \text{for other dimensions,}$$

where G denotes a finite abelian group. We denote by $P(n, n+1;G)$ the complex K such as above and call it a Poincaré complex of type $(n, n+1;G)$. Then our main results are

THEOREM A. *Let $n\geqq 3$ and $G\otimes Z_2=0$. Then $P(n, n+1;G)$ has the same homotopy type as the connected sum of $P(n, n+1;G_1)$ and $P(n, n+1;G_2)$ if G is a direct sum of G_1 and G_2.*

THEOREM B. *Under the same conditions as Theorem* A, *if $P(n, n+1;G)$ is S-reducible it's homotopy type is unique with respect to n and G.*

By applying these theorems to the case of manifolds we shall prove

THEOREM C. *Let M be a $(n-1)$-connected rational homology sphere which is a smooth manifold of dimension $2n+2$ with no 2-torsion. Then M is uniquely determined up to homotopy by homology for $n\equiv 0$, 1 mod 4.*

The case of $G\otimes Z_2\neq 0$ (essencially, G is a 2-group) is more complicated, therefore we shall discuss it in the subsequent paper.

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The plan of this paper is as follows. First, in § 1, we study the homotopy of Moore spaces and in § 2 characterize Poincaré complexes of type $(n, n+1; G)$. In § 3 we shall prove Theorem A and B, and in § 4 the proof of Theorem C shall be given. Throughout this paper we assume that groups $G, H, \cdots$ are finite abelian with no 2-torsion and $n \geqq 3$.

§ 1. Homotopy of Moore spaces.

We denote by M_G^n the Moore space of type (n, G) and by $\sharp$ the integer $2n+1$. We first note the following easy

LEMMA 1.1. *$\pi_i(M_G^n)$ is trivial for $i=n+1$, $n+2$.*

Now we define a homomorphism

$$\mu_H^G : \pi_\sharp(M_G^n \vee M_H^{n+1}) \longrightarrow \mathrm{Hom}(G, H)$$

by $$\mu_H^G(f)=\mu_f\cap : G=H^{n+1}(M_G^n; Z)=H^{n+1}(c(f); Z) \longrightarrow H_{n+1}(c(f); Z)=H_{n+1}(M_H^{n+1}; Z)=H,$$

where $c(f)$ denotes the mapping cone for a map $f: S^\sharp \to M_G^n \vee M_H^{n+1}$ and μ_f is the oriented generator of $H_{2n+2}(c(f); Z)$. Let h be a map $M_G^n \vee M_H^{n+1} \to M_{G'}^n \vee M_{H'}^{n+1}$. Clearly h is decomposed into the sum of four maps;

$$h_1: M_G^n \longrightarrow M_{G'}^n,\ h_2: M_G^n \longrightarrow M_{H'}^{n+1},\ h_3: M_H^{n+1} \longrightarrow M_{G'}^n \text{ and } h_4: M_H^{n+1} \to M_{H'}^{n+1}.$$

Then, from the commutative diagram

$$\begin{array}{ccc} H^{n+1}(c(f); Z) & \xrightarrow{\mu_f\cap} & H_{n+1}(c(f); Z) \\ h_1^* \uparrow & & \downarrow h_{4*} \\ H^{n+1}(c(hf); Z) & \xrightarrow[\mu_{hf}\cap]{} & H_{n+1}(c(hf); Z), \end{array}$$

we obtain

LEMMA 1.2. *(The naturality of μ_H^G) $\mu_{H'}^{G'}(hf)=h_{4*}\mu_H^G(f)h_1^*$.*

Now we prove

PROPOSITION 1.3. $\pi_\sharp(M_G^n \vee M_H^{n+1})=\pi_\sharp(M_G^n) \oplus \pi_\sharp(M_H^{n+1}) \oplus \mathrm{Hom}(G, H)$

Proof. The proof follows from the standard isomorphism

$$\pi_\sharp(M_G^n \vee M_H^{n+1})=\pi_\sharp(M_G^n) \oplus \pi_\sharp(M_H^{n+1}) \oplus \partial\pi_{\sharp+1}(M_G^n \times M_H^{n+1}, M_G^n \vee M_H^{n+1})$$

if we can show that the restriction μ_H^G on the third factor is an isomorphism. Thus, by using isomorphisms

$$\pi_{\sharp+1}(M_G^n \times M_H^{n+1}, M_G^n \vee M_H^{n+1})=\pi_{\sharp+1}(M_G^n \wedge M_H^{n+1})=\pi_{\sharp+1}(M_{G\otimes H}^{2n+1} \vee M_{G*H}^{2n+2}),$$

where $\wedge$ denotes the smash product, the proof can be reduced to the case of $G=Z_{p^i}$ and Z_{p^j}. Let α be the generator of

$$\pi_{\#+1}(M^n_G\times M^{n+1}_H,\ M^n_G\vee M^{n+1}_H)\cong H_{\#+1}(M^n_G\times M^{n+1}_H,\ M^n_G\vee M^{n+1}_H\,;Z)\,.$$

Then there exists a map $\varphi: c(f)\to M^n_G\times M^{n+1}_H$ ($f=\partial\alpha$) such that $\varphi_*: H_{2n+2}(c(f),Z)\to H_{2n+2}(M^n_G\times M^{n+1}_H\,;Z)$ is surjective and $\varphi|M^n_G\vee M^{n+1}_H=$identity. Consider the commutative diagram

$$\begin{array}{ccc}
Z_{p^i}=H^{n+1}(c(f)\,;Z) & \xrightarrow{\qquad\qquad} & H_{n+1}(c(f)\,;Z)=Z_{p^j} \\
\uparrow & \mu_f\frown & \downarrow \\
Z_{p^i}=H^{n+1}(M^n_G\times M^{n+1}_H\,;Z) & \xrightarrow[\varphi_*(\mu_f)\frown]{\qquad\qquad} & H_{n+1}(M^n_G\times M^{n+1}_H\,;Z)=Z_{p^j}\,.
\end{array}$$

Then the proof is obtained from $\varphi_*(\mu_f)\frown 1=p^{j-k}(1)$ ($k=\min(i,\ j)$).

Now we investigate the N-fold suspension

$$E^N:\ \pi_\#(M^n_i\vee M^{n+1}_i)\longrightarrow \pi_{\#+N}(M^{n+N}_i\vee M^{N+1+n}_i)\qquad (N\longrightarrow\infty)\,,$$

where M^n_i denotes M^n_G for $G=Z_{p^i}$. First, in the decomposition given by Proposition 1.3, we can easily obtain

$$E(\mathrm{Hom}\,(Z_{p^i},\ Z_{p^i}))=0\quad\text{and}\quad E^{-N}(0)\frown\pi_\#(M^{n+1}_i)=[\pi_{n+1}(M^{n+1}_i),\ \pi_{\#+1}(M^{n+1}_i)]\,,$$

where $[\ ,\ ]$ denotes the Whitehead product. Next, let $M^n_{i,\infty}$ be the reduced product for M^n_i. Using $\pi_{\#+1}(M^n_{i,\infty},\ M^n_i)=0$ and the homotopy exact sequence of the pair $(M^n_{i,\infty},\ M^n_i)$, we have

LEMMA 1.4. $E:\ \pi_\#(M^n_i)\longrightarrow\pi_{\#+1}(M^{n+1}_i)$ *is injective.*

For the investigation of $E:\ \pi_{\#+1}(M^{n+1}_i)\to\pi_{\#+2}(M^{n+2}_i)$ we define a homomorphism $h_n:\pi_{2n}(M^n_i)\to Z_{p^i}$ as follows. Let $c(f)=M^n_i\cup e^\#$ be the mapping cone for a map $f: S^{2n}\to M^n_i$ and let α, β, γ be generators of $H^n(c(f)\,;Z_{p^i})$, $H^{n+1}(c(f)\,;Z_{p^i})$ and $H^{2n+1}(c(f)\,;Z_{p^i})$ respectively. Then put $\mu f\frown(\alpha\cup\beta)=h_n(f)$.

LEMMA 1.5. (1) $h_n(E\pi_{2n-1}(M^{n-1}_i))=0$
(2) *if n is even, h_n is trivial*
(3) *if n is odd, h_n is surjective*

Proof. (1) follows from the definition of h_n and (2) is deduced from applying the Bockstein operator. For (3), consider the boundary homomorphism ∂: $\pi_{2n+1}(M^n_{i,\infty},\ M^n_i)=Z_{p^i}\to\pi_{2n}(M^n_i)$. We assert

$$h_n(\partial(1))=\text{a generator of } Z_{p^i}\,.$$

Clearly there exists a map $\psi:\ c(f)\to M^n_{i,\infty}$ such that $\psi|M^n_i=$identity and $\psi_*:\ H_{2n+1}(c(f)\,;Z)=Z\to H_{2n+1}(M^n_{i,\infty}\,;Z)$ is surjective. Then our assertion follows from the cohomologyring structure of $M^n_{i,\infty}$.

LEMMA 1.6. $E^2:\ \pi_\#(M^n_i)\to\pi_{\#+2}(M^{n+2}_i)$ *is injective.*

Proof. Consider the diagram

$$
\begin{array}{ccccccc}
 & & & & \pi_{\#}(M^n_\iota) & & \\
 & & & & \downarrow & & \\
\pi_{\#+2}(M^{n+1}_{\iota,\infty}) & \xrightarrow[j]{} & \pi_{\#+2}(M^{n+1}_{\iota,\infty},\ M^{n+1}_\iota) & \longrightarrow & \pi_{\#+1}(M^{n+1}_\iota) & \xrightarrow[i]{} & \pi_{\#+1}(M^{n+1}_{\iota,\infty}) \\
 & & \| & & \downarrow h_{n+1} & & \\
 & & Z_{p\iota} & & Z_{p\iota} & &
\end{array}
$$

If n is even, the proof follows from lemma 1.4 and (1) of lemma 1.5. If n is odd, j is surjective by (1) and (3) of lemma 1.5, and hence i is injective. Thus the proof is completed.

Thus, from combining lemmas, we have

PROPOSITION 1.7. *The kernel of E^N is the subgroup*

$$[\pi_{n+1}(M^{n+1}_\iota),\ \pi_{n+1}(M^{n+1}_\iota)] \oplus \mathrm{Hom}\,(Z_{p\iota},\ Z_{p\iota})\,.$$

Now let ι_{n+1} be the generator of $\pi_{n+1}(M^{n+1}_\iota)$ and define the map $\nu_r\colon M^n_\iota \vee M^{n+1}_\iota \to M^n_\iota \vee M^{n+1}_\iota$ by $\nu_r | M^{n+1}_\iota = \text{identity}$ and $\nu_r | M^n_\iota = \text{identity} + r\iota_{n+1} \circ id/S^n$. For the later, we note

LEMMA 1.8. *For $id \in \mathrm{Hom}\,(Z_{p\iota}, Z_{p\iota}) \subset \pi_{\#}(M^n_\iota \vee M^{n+1}_\iota)$ we have $\nu_{r*}(id) = r[\iota_{n+1}, \iota_{n+1}] + id$.*

Proof. Since $E^N(id)=0$, by Proposition 1.8, $\nu_{r*}(id)$ has a representation

$$\nu_{r*}(id) = x[\iota_{n+1},\ \iota_{n+1}] + y(id)$$

for some integers x and y. Then $y=1$ follows from the naturality of cup-product and $x=r$ is easily deduced from the cohomology ring structure of the mapping cone for id.

§2. Poincaré complexes of type $(n,\ n+1\,;G)$.

First we note

LEMMA 2.1. *$P(n,\ n+1\,;G)$ has the same homotopy type as the mapping cone for a map $f\colon S^{\#} \to M^n_G \vee M^{n+1}_G$.*

Remark: This is not true in the case of $G \otimes Z_2 \neq 0$.

Proof. Let X be a Poincaré complex of type $(n,\ n+1\,;G)$. Since $\pi_i(X)=0$ $(0 \leqq i \leqq n-1)$ and $\pi_n(X)=G$, we may regard M^n_G as a subcomplex of X. Then we have

$$\pi_{n+1}(X) \cong \pi_{n+1}(X,\ M^n_G) \cong H_{n+1}(X,\ M^n_G) \cong H_{n+1}(X) \cong G\,,$$

using lemma 1.1 and the homotopy-homology exact sequence of the pair $(X,\ M^n_G)$. Hence there is a map $g\colon M^{n+1}_G \to X$ such that

$$g_*\colon H_{n+1}(M^{n+1}_G\,;Z) \longrightarrow H_{n+1}(X\,;Z)$$

is an isomorphism. Then, since the map $id\vee g: M_G^n\vee M_G^{n+1}\to X$ induces an isomorphism of homology up to dimension $2n+1$ the proof is completed by the standard argument.

Thus, from the point of view of homotopy, we can replace a complex of type $(n, n+1; G)$ with $c(f)$.

LEMMA 2.2 *$c(f)$ is a Poincaré complex if and only if $f(\in\pi_\#(M_G^n\vee M_G^{n+1}))$ is contained in the subgroup*

$$\pi_\#(M_G^n)\oplus\pi_\#(M_G^{n+1})\oplus \mathrm{Aut}\, G\,.$$

Proof. The part "only if" follows from the definition of decomposition in Proposition 1.3. For the part "if" we must show that two homomorphisms

(1) $\mu_f\frown: H^{n+1}(c(f); Z)\longrightarrow H_{n+1}(c(f); Z)$

(2) $\mu_f\frown: H^{n+2}(c(f); Z)\longrightarrow H_n(c(f); Z)$

are both isomorphisms, where μ_f denotes the generator of $H_{2n+2}(c(f); Z)$.

Clearly (1) holds by the definition. Let Z_{pi}, Z_{pj} be two direct summands of G and let p_i (p_j) be the projection $G\to Z_{pi}$ (Z_{pj}). Since p_i, p_j naturally induce the maps

$$\hat{p}_i: M_G^n\longrightarrow M_i^n \quad\text{and}\quad \hat{p}_j: M_G^{n+1}\longrightarrow M_j^{n+1} \qquad (M_i^n=M_{Z_{pi}}^n)\,,$$

we have the map

$$\hat{p}_i\vee\hat{p}_j=p: M_G^n\vee M_G^{n+1}\longrightarrow M_i^n\vee M_j^{n+1}\,.$$

On the other hand, by lemma 1.2, we may suppose that f has a representation $f=\alpha\oplus\beta\oplus id$ (Proposition 1.3). Then we have

$$p_*(f)=\hat{p}_{i*}(\alpha)\oplus\hat{p}_{j*}(\beta)\oplus id \qquad \text{if}\quad Z_{pi}=Z_{pj} \tag{2.3}$$

$$=\hat{p}_{i*}(\alpha)\oplus\hat{p}_{j*}(\beta) \qquad \text{if}\quad Z_{pi}\neq Z_{pj}\,, \tag{2.4}$$

using lemma 1.2. Let $\hat{p}$ be the map: $c(f)\to c(pf)$ which is the natural extension of p and consider the commutative diagram

$$\begin{array}{ccc} G=H^{n+2}(c(f); Z) & \xrightarrow{\mu_f\frown} & H_n(c(f); Z)=G \\ \uparrow \hat{p}_* & & \downarrow \hat{p}_*=p_i \\ Z_{pj}=H^{n+2}(c(pf); Z) & \xrightarrow[\mu_{pf}\frown]{} & H_n(c(pf); Z)=Z_{pi}\,. \end{array}$$

We assert that

$$\mu_{pf}\frown Z_{pi}=0 \qquad \text{if}\quad Z_{pi}\neq Z_{pj}$$

$$=Z_{pi} \qquad \text{if}\quad Z_{pi}=Z_{pj}\,.$$

The case of $Z_{pi}\neq Z_{pj}$. By (2.4) there exists a map

$$q: c(pf) \longrightarrow c(\hat{p}_i\alpha)\vee c(\hat{p}_j\beta)$$

such that $q|M_i^n\vee M_j^{n+1}=\iota d$ and $q_*(\mu_{pf})=\mu_{\hat{p}_i\alpha}+\mu_{\hat{p}_j\beta}$. Since $\mu_{\hat{p}_i\alpha}$ and $\mu_{\hat{p}_j\beta}$ are both trivial we have that $\mu_{pf}\frown$ is also trivial.

The case of $Z_{p^\iota}=Z_{p^j}$. For our purpose it is sufficient to consider Z_p-coefficient instead of Z-coefficient. Then we can take generators $x(\in H^n(c(pf)\,;\,Z_p))$ and $y(\in H^{n+1}(c(pf)\,;Z_p))$ such that $\beta_\iota x$ and $\beta_j y$ both generators, where β_ι denotes the Bockstein operator. Thus, using Kronecker product and (2.3), we have

$$\langle x,\ \mu_{pf}\frown\beta_\iota y\rangle=\pm\langle x\cup\beta_\iota y,\ \mu_{pf}\rangle=\pm\langle\beta_\iota x\cup y,\ \mu_{pf}\rangle$$
$$=\pm\langle y,\ \mu_{pf}\frown\beta_\iota x\rangle=\pm 1\,.$$

These show our assertion, and therefore the proof of (2) is completed.

3. The proof of Theorem A and B.

First we replace a space of type $(n,\ n+1\,;G)$ with $c(f)$ by lemma 2.1. Let $G=G_1\oplus G_2$ and let $Z_{p^i}(x)$, $Z_{p^j}(y)$ be direct summands of G_1 and G_2 respectively. By the decomposition

$$\pi_\#(M_G^n\vee M_G^{n+1})=\pi_\#(M_G^n)\oplus\pi_\#(M_G^{n+1})\oplus \mathrm{Hom}\,(G,\ G)$$
$$=\pi_\#(M_{G_1}^n)\oplus\pi_\#(M_{G_2}^n)\oplus\pi_\#(M_{G_1}^{n+1})\oplus\pi_\#(M_{G_2}^{n+1})\oplus[G_1,\ G_2]\oplus \mathrm{Hom}\,(G,\ G)\,,$$

where we identify G_ι with $\pi_{n+1}(M_{G_i}^{n+1})$, we may suppose that f has the representation

$$f=\alpha_1+\alpha_2+\beta_1+\beta_2+\sum_{x,y}s[x,\ y]+\iota d\,.$$

For fixed $Z_{p^i}(x_0)$ and $Z_{p^j}(y_0)$, let p_0 be the map $M_G^n\to M_\iota^n$ induced by the projection $G_1\to Z_{p^\iota}(x_0)$ and let p_0^r be the composite map

$$M_{G_1}^n \underset{p_0}{\longrightarrow} M_\iota^n \longrightarrow M_\iota^n/S^n=S^{n+1} \underset{ry_0}{\longrightarrow} M_{G_2}^{n+1}\,.$$

Consider the map $F_r:\ M_G^n\vee M_G^{n+1}\to M_G^n\vee M_G^{n+1}$ defined by $F_r|M_{G_1}^{n+1}$=identity, $F_r|M_{G_2}^{n+1}$=identity, $F_r|M_{G_2}^n$=identity and $F_r|M_{G_1}^n$=identity$+p_0^r$. F_r is clearly a homotopy equivalence and we prove

(1) $F_{r*}(\alpha_2)=\alpha_2\,,\qquad F_{r*}(\beta_\iota)=\beta_\iota\qquad(\iota=1,\ 2)$

(2) $F_{r*}([x,\ y])=[x,\ y]$

(3) $F_{r*}(\alpha_1)=\alpha_1+p_{0*}^r(\alpha_1)$

(4) $F_{r*}(\iota d)=\iota d+r[x_0,\ y_0]\,.$

For, (1) and (2) are obvious by the definition of F_r and (3) follows from $E\pi_{2n}(M_{G_1}^{n-1})=\pi_{2n+1}(M_{G_1}^n)$. Since it is easy to obtain

$$F_{r*}(\iota d)=\iota d+\sum_{x,y}a[x,\ y]$$

we must determine a for each x, y. Now consider the commutative diagram

$$\begin{array}{ccc} M^n_{G_1}\vee M^{n+1}_{G_1}\vee M^n_{G_2}\vee M^{n+1}_{G_2} & \xrightarrow{F_r} & M^n_{G_1}\vee M^{n+1}_{G_1}\vee M^n_{G_2}\vee M^{n+1}_{G_2} \\ p^n_x\vee p^{n+1}_y\vee p^n_x\vee p^{n+1}_y \downarrow & & \downarrow p^n_x\vee p^{n+1}_x\vee p^n_y\vee p^{n+1}_y \\ M^n_\imath\vee M^{n+1}_\imath\vee M^n_\jmath\vee M^{n+1}_\jmath & \xrightarrow[G_r]{} & M^n_\imath\vee M^{n+1}_\imath\vee M^n_\jmath\vee M^{n+1}_\jmath=X_{x,y}, \end{array}$$

where $G_r=\imath d\vee\imath d\vee\imath d\vee\imath d$ $((x, y)\neq(x_0, y_0))$, p^n_x is the map $M^n_G\to M^n_\imath$ induced by the projection $G\to Z_{p^\imath}(x)$, and

$$G_r=(\imath d+ry_0\circ M^n_\imath/S^n)\vee\imath d\vee\imath d\vee\imath d \qquad ((x, y)=(x_0, y_0)).$$

Then we have

$$G_{r*}(\imath d)=\imath d+a[x, y].$$

Let α_x, β_x be generators for $H^{n+1}(M^n_\imath;Z_{p^k})$ and $H^{n+1}(M^{n+1}_\imath;Z_{p^k})$ (k=mim$(\imath, j)$) respectively and we denote by $\hat{X}_{x,y}$ the mapping cone for $\imath d\in\pi_\#(X_{x,y})$. In the cohomology ring $H^*(\hat{X}_{x,y};Z_{p^k})$, we have

$$\alpha_x\cup\beta_x=a \text{ generator and } \beta_x\cup\beta_y=0.$$

On the other hand, in the cohomology ring $H^*(c(G_r(\imath d))$, we have $\beta_x\cup\beta_y=a(1)$. Hence the proof of (4) follows from

$$\begin{aligned} a(1)=G_{r*}(\beta_x)\cup G_{r*}(\beta_y)=\beta_x\cup\beta_y=0 \qquad &((x, y)\neq(x_0, y_0)) \\ =\beta_x\cup(r\alpha_x+\beta_y)=r(1) \qquad &((x, y)=(x_0, y_0)). \end{aligned}$$

Thus the proof of Theorem A is completed by using iteratedly F_r for various r.

Especially we have

COROLLARY 3.1. *Let $G=\sum_p\sum_\imath\sum Z_{p^\imath}$ be the direct-sum decomposition of G. Then $P(n, n+1;G)$ has the same homotopy type as the connected sum of $P(n, n+1;Z_{p^\imath})$s.*

Next we consider the proof of Theorem B. Let $G=\sum_p\sum_\imath\sum Z_{p^\imath}$ and let x be the generator of a $Z_{p^\imath}$-component. We denote by $M^n_\imath(x)$ the Moore space corresponding to the $Z_{p^\imath}$-component generated by x. By Corollary 3.1 we may assume that $P(n, n+1;G)$ has a decomposition

$$P(n, n+1;G)=(\bigvee_x M(x))\bigcup_f e^{2n+2}, \qquad f=\bigoplus_x\sigma_x \qquad (\sigma_x=f_x+f'_x+\imath d),$$

where $M(x)$ is the space $M^n_\imath(x)\vee M^{n+1}_\imath(x)$ and $\sigma_x\in\pi_\#(M(x))$. If $P(n, n+1;G)$ is S-reducible we can know from Proposition 1.7 that

$$f_x=0 \text{ and } f'_x\in[\pi_{n+1}(M^{n+1}_\imath(x)), \pi_{n+1}(M^{n+1}_\imath(x))]$$

Then, by applying the map F_r, the proof is completed.

§ **4.** π-manifolds.

We describe a closed smooth manifold as a manifold of type $(n, n+1; G)$ if it's underlying Poincaré complex is of type $(n, n+1; G)$.

If M is a π-manifold of type $(n, n+1; G)$, M is S-reducible and hence it's homotopy type is unique with respect to n and G by Theorem B. Conversely we prove

PROPOSITION 4.1. *If K is a S-reducible Poincaré complex of type $(n, n+1; G)$, then K has the homotopy type of a π-manifold.*

Proof. Consider the product manifold $S^n\times S^{n+2}$ and let ι be the generator of $\pi_n(S^n\times S^{n+2})$. Since $S^n\times S^{n+2}$ is a π-manifold, a new π-manifold K_m is obtained from killing the class $m\iota$ (Theorem 2 of [1]). Clearly K_m is a π-manifold of type $(n, n+1; Z_m)$ and hence it's homotopy type is unique. Then the proof is completed by Theorem B and Corollary 3.1.

Next, for the proof of Theorem C, we prove

PROPOSITION 4.2. *Let $n\equiv 0,1$ mod 4. Then manifolds of type $(n, n+1; G)$ are all π-manifolds.*

Proof. Let M be a manifold of type $(n, n+1; G)$ and let ν_M be the stable normal bundle for M. By lemma 2.1 we may suppose

$$M=(M_G^n\vee M_G^{n+1})\cup e^{2n+2} \qquad \text{(up to homotopy)}$$

Let P be the natural map $M\rightarrow S^{2n+2}=M/M_G^n\vee M_G^{n+1}$. Then, from Puppe's sequence, we obtain two isomorphisms

$$P^*: Z=[S^{2n+2}, BO]_0 \longrightarrow [M, BO]_0 \qquad (n\equiv 1 \bmod 4)$$

$$P^*: Z_2=[S^{2n+2}, BO]_0 \longrightarrow [M, BO]_0 \qquad (n\equiv 0 \bmod 4).$$

Thus, there exists a bundle ξ over S^{2n+2} with $P^*(\xi)=\nu_M$. Since the Thom space $T(\nu_M)$ is S-reducible and P is of degree 1, $T(\xi)$ is also reducible, hence we have $J(\xi)=0$. If $n\equiv 1$ mod 4, $J(\xi)=0$ is equivalent to $\xi=0$. Therefore we have $\nu_M=p^*(\xi)=0$. If $n\equiv 1$ mod 4, ξ is determined by it's Pontrijagin class. Using Hirzeburch formula for ν_M and Index $(M)=0$, we can know that the top Pontrijagin class of ν_M is zero. Thus we get $\xi=0$, i.e. $\nu_M=0$.

Now Theorem C is clear from Proposition 4.2. Finally we note

PROPOSITION 4.3. *Let M be an almost parallerizable manifold of type $(n, n+1; G)$. Then M is a π-manifold and hence it's homotopy type is unique with respect to n and G.*

Proof. Let ν_M be the stable normal bundle for M. Since the restriction $\nu_M|M_G^n\vee M_G^{n+1}$ is trivial, the proof follows from the same argument as the proof of Proposition 4.2.

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