ON PARALLEL CONFORMAL CONNECTIONS

BY RADU ROSCA

Introduction. Riemannian manifolds endowed with a parallel conformal connection $\nabla_{p,c}$ have been defined by the present author in [1]. In this paper one studies in the first section a type of such manifolds for which the principal field $X$ associated with $\nabla_{p,c}$ is parallel. In this case $X$ is an infinitesimal homothety of the volume element of $M_c$ and is an invariant section of the canonical form in the set of 2-frames $O^2(M_c)$. If $M_c$ is of even dimension $2m$, then the connection $\nabla_{p,c}$ defines on $M_c$ a conformal symplectic form $\varphi$ and the dual field of the principal 1-form $\alpha$ ($\alpha$ is the dual form of $X$ with respect to the metric of $M_c$) with respect to $\varphi$ is a Killing field. Finally it is shown that $M_c$ is of constant scalar curvature and is Ricci flat in the direction of $X$. In the second section, making use of some notions introduced by K. Yano and S. Ishihara in [5] and by J. Klein in [7] one studies different properties of the tangent bundle manifold $TM_c$. Thus the complete lift $\varphi^c$ of $\varphi$, on $TM_c$ is a homogenous form of degree 1 and is also conformal symplectic. If $V$ is the canonical field on $TM_c$, then the Lie bracket $[V, X]$ is an infinitesimal automorphism of $\varphi^c$. Further some properties involving the canonical symplectic form $\Omega$ on $TM_c$ ($\Omega$ is a Finslerian form) and a second conformal symplectic form $\Theta$, which is homogenous of degree 2, are discussed. In the last section one considers a regular mechanical system (in the sense of J. Klein [8]), $H=\{M_c, T, \pi\}$ such that the kinetic energy $T$ is homogenous of degree 2 and the dynamical system $Z$ associated with $H$ is a spray on $M_c$.

1. $M_c$ manifold. Let $M$ be an $n$-dimensional $C^\infty$-Riemannian manifold and let $O(M)$ be the bundle of orthonormal frames of $M$. If $O=O(M)$ is such a frame, let $\{e_i\}$, $\{\omega^i\}$ and $\omega^i=\delta_k^i\omega^j$, $i, k, j=1, \ldots, n$, be the vectorial and dual basis and the connection forms associated with $O$ respectively. Then the line element $dp$ ($p\in M$), the connection equations and the structure equations (E. Cartan) are respectively

\begin{align}
(1.1) & \quad dp=\omega^i \otimes e_i, \\
(1.2) & \quad \nabla e_i=\omega^k \otimes e_k, \\
(1.3) & \quad d\omega^i=\Omega^i+\omega^k \wedge \omega_k^i,
\end{align}

Received December 8, 1976
2 RADU ROSCA

\( d \wedge \omega^i = \Omega^i + \omega^i \wedge \omega^j, \)

where \( \Omega^i \) and \( \Omega^i_j \) are the \textit{torsion} and the \textit{curvature 2-forms} respectively.

A connection \( \nabla \) such that

\( \omega^i = t^i \omega^j - t_j \omega^i; \quad t_i \in C^\infty(M) \)

has been called in [1], a \textit{parallel conformal connection}, and denoted by \( \nabla_{p.c} \). If \( T_p(M) \) is the tangent space at \( p \in M \) we shall call

\( X = \sum t_i e_i \in T_p(M) \)

the \textit{principal field} (p.f.) associated with \( \nabla_{c.p} \) and if \( \mathcal{J} \) is the canonical isomorphism (with respect the metric of \( M \)) the Pfaffian

\( \mathcal{J}X = \alpha = \sum t_i \omega^i \)

is the \textit{principal Pfaffian} (p.P.) associated with \( \nabla_{p.c} \).

So by 1.3 and 1.5 we readily get

\( d \wedge \omega^i = \Omega^i + \alpha \wedge \omega^i. \)

Assume now that \( X \) is \textit{parallel}, that is,

\( \nabla X = 0. \)

By using 1.2 and 1.5 we obtain from 1.9

\( dt_i = t_i \alpha - t^2 \omega^i; \quad t^2 = \|X\|^2. \)

Taking account of 1.7 one finds instantly

\( t^i = \text{const.} \)

Next exterior differentiation of (1.10) gives

\( \Omega^i = 0 \)

and so by an easy argument follows

\( d \wedge \alpha = 0. \)

Hence if the \textit{p.f.} \( X \) is parallel then the connection \( \nabla_{p.c} \) is necessarily \textit{torsionless} and the \textit{p.P.} \( \alpha \) is \textit{closed}. In the following the manifolds under consideration will be of even dimension \( (n=2m) \) and structured by \( \nabla_{p.c} \) connection with parallel principal field. Such manifolds will be denoted by \( M_k \).

We have shown in [1] that if \( M \) is of even dimension \( (n=2m) \) then the connection \( \nabla_{p.c} \) defines on \( M \) a \textit{conformal symplectic structure} \( CSp(m; R) \). Thus if we consider the almost symplectic form
PARALLEL CONFORMAL CONNECTIONS

(1.13) \[ \varphi = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{m-1} \wedge \omega^m, \]
then by

(1.14) \[ d \wedge \omega^i = \alpha \wedge \omega^i, \]
one gets at once

(1.15) \[ d \wedge \varphi = 2\alpha \wedge \varphi. \]

Thus we see that \(2\alpha\) is the co-vector of Lee of the structure \(\text{CSp}(m; \mathbb{R})\). Let now \(\mu_\varphi: Z \rightarrow -\varphi\) be the isomorphism defined by \(\varphi\). An easy calculation gives

(1.16) \[ \mu_\varphi^{-1}(\alpha) = X_a = -t_2 e_1 + t_1 e_2 + \cdots + t_{2m-1} e_{2m} \]
and \(X_a\) will be called the associated field of \(X\). Taking the star operator * of \(\alpha\) one has

(1.17) \[ \star \alpha = \sum (-1)^{i-1} \omega^i \wedge \cdots \wedge \omega^{2m} \]
(the “roof” indicates the missing terms).

Making use of 1.10 and 1.13 one finds from (1.17) \(\delta \alpha = \text{div} X = -t^2 = \text{const.}\) and so \(X\) is an infinitesimal homothety of the volume element of \(M\).

Put now \(\mu_\varphi(X) = \alpha_a\) and call \(\alpha_a\) the associated 1-form of \(\alpha\). Denoting by \(\tilde{\omega}\) the symplectic adjoint operator \([2]\) one has

(1.18) \[ \pi \alpha_a = \tilde{\omega} \alpha = \frac{\alpha}{(m-1)!} \lambda^{m-1} \varphi. \]

Making use of (1.12) and (1.15) we readily see that \(\delta \alpha_a = 0\), that is \(\text{div} X_a = 0\).

On the other hand \(Z(Z') \in T_p(M)\) being any vector field, we derive from (1.2), (1.5) and (1.10)

(1.19) \[ \nabla_Z e_i = t_i Z - Z' X; \quad \nabla_Z: \text{covariant derivative.} \]

Now with the aid of (1.17) and since \(\langle X, X_a \rangle = 0\), one finds by a straightforward calculation

(1.20) \[ \langle \nabla_Z X_a, Z' \rangle + \langle \nabla_{Z'} X_a, Z \rangle = 0, \]
where \(Z\) and \(Z'\) are arbitrary vector fields. The above relation proves that \(X_a\) is a Killing vector field. So the equation

(1.21) \[ \mathcal{L} X_a = 0; \quad \mathcal{L}_Z = i_Z d + d \wedge i_Z : \text{Lie derivative} \]

for \(X_a\) is a Killing vector field. The above relation proves that \(X_a\) is a Killing vector field. So the equation

(1.22) \[ \mathcal{L}_X X = 0. \]
Denote like usual by $R^i_{jkl}$ the Riemann curvature tensor, that is, $\Omega^i = (1/2)R^i_{jkl}\omega^k \wedge \omega^l$. By (1.4), (1.5) and (1.10) one finds

$$R^i_{jkl} = t^i - t^i_l - t^i_k,$$

(1.23)

$$R^i_{j0} = -t^i_k t^k,$$

$$R^i_{jij} = 0; k \neq i \neq j \neq l.$$

From the above expressions we derive the components of the Ricci tensor as follows

$$R_{ii} = (n-2)(t^i - t^i_i),$$

$$R_{ik} = -(n-2)t^i_k t^k.$$

(1.24)

From 1.24 and taking account of (1.10) one quickly finds that the scalar curvature of $M$ is constant, that is

$$R = (n-1)(n-2)t^2.$$

(1.25)

Next denote by $\text{Ric}(X)$ the Ricci curvature in the direction $X$.

In consequence of (1.24) and (1.7) a short calculation gives

$$\text{Ric}(X) = 0.$$

(1.26)

Hence the manifold $M$ is Ricci flat in the direction $X$.

On the other hand referring to (1.5) and (1.10) one finds that both $\omega^i$ and $\omega^i_k$ are invariant by $X$; that is $\mathcal{L}_X \omega^i = 0, \mathcal{L}_X \omega^i_k = 0$.

Therefore one may say that $X$ is an invariant section for the canonical form $\omega^i \otimes \epsilon_i + \sum \omega^i_k \otimes \epsilon_k$ of the set of 2-frames $\mathcal{O}^2(M)$ (frames of second order).

Finally coming back to the structure $\text{CSp}(m; R)$ defined by (1.14) we have

$$t^i \varphi = -\mu_a(X) = -\alpha_a = -t^i \omega^i + t^i \omega^a - \ldots + t_{2m-1} \omega^a_{2m}.$$

(1.26)

By (1.20) and (1.14) a short computation gives

$$d \wedge \alpha_a = \alpha \wedge \alpha_a + 2t^i \varphi.$$

(1.27)

Since $t^i$ is constant, this equation proves as is known [3] that $X$ is a conformal symplectic infinitesimal transformation of $\varphi$. From the preceding discussion we may state the

**Theorem.** Let $M_c$ be a Riemannian manifold of even dimension $2m$ structured by a $\nabla_{p, c}$ connection with principal field $X$ and let $X_a$ and $\gamma$ be the associated field of $X$ and the volume element of $M_c$ respectively. Then:

(i) the connection $\nabla_{p, c}$ defines on $M_c$ a conformal symplectic structure $\text{CSp}(m; R) = (\varphi, 2\alpha)$ having (up to a constant factor) the dual form of $X$ as covector of Lee,
(ii) the field $X$ has the following properties: it is an infinitesimal homothety of $\eta$, it is an invariant section of the canonical form of the set of 2-frames $O^2(M_c)$, it is a conformal symplectic infinitesimal transformation of $\text{CSp}(m; R)$;

(iii) the field $X_a$ has the following properties: it is an infinitesimal automorphism of $\eta$, it is a Killing field;

(iv) $M_c$ is of constant scalar curvature and is Ricci flat in the direction of $X$.

2. Tangent bundle manifold $TM_c$. $M_c$, being of constant scalar curvature, is as is known ($n \geq 3$) endowed with a conformal flat structure. Therefore referring to (1.14) we may get

\[ \alpha = -df/f ; f \in C^\infty(M_c) \]

and call $f$ the integrating factor associated with $\nabla_{\text{p.c.}}$. Denote by $TM_c$ the tangent bundle manifold having $M_c$ as basis and by $V(v^i)$ the canonical field (the field of Liouville) on $TM_c$. Thus we may consider the set $B^* = \{ \omega^i, dv^i \}$ as a co-vectorial basis of $TM_c$.

Denote (like usual) by $d_{\text{v}}$ and $\iota_\omega$ the vertical differentiation and the vertical derivation operators respectively taken with respect to $B^*$ ($d_{\text{v}}$ is an antiderivation of degree 1 of $\Lambda(TM)$ and $\iota_\omega$ is a derivation of degree 0 of $\Lambda(TM)$ [4]).

Put

\[ \lambda = \int f v \in C^\infty(TM_c), \]

where

\[ v = \frac{1}{2} \sum_i (v^i)^2. \]

One has

\[ \iota_\omega d_{\text{v}} = \int f \sum_i v^i \omega^i = \iota_\omega \in \wedge^1(TM_c) \]

and by (2.1) and (1.14) we get

\[ d \wedge d_{\text{v}} = \int f \sum_i d v^i \wedge \omega^i = \Omega. \]

Clearly $\Omega$ is an exact symplectic form which will be called the canonical symplectic form on $TM_c$.

In addition we shall call $l$ and $\lambda$ the Liouville function and the Liouville form respectively on $TM_c$.

If $\iota: \Lambda^1(M) \to C^\infty(TM)$ is the operator of K. Yano and S. Ishihara [3] one has (with respect to $B^*$)

\[ \iota \alpha = \sum_i t_i v^i \]

and so
If \( \partial \) denotes the Pfaffian derivative with respect \( \omega \), then according to [5], complete lift \( \alpha^c \) of \( \alpha \) is defined by

\[(2.7) \quad d\omega(t\alpha)=\alpha,\]

With the help of (1.10) one finds

\[(2.9) \quad \alpha^c=(t\alpha)\alpha-(t^2/f)\lambda+\beta,\]

where

\[(2.10) \quad \beta=\sum t_i d\nu^i.\]

One obtains

\[(2.11) \quad d\nu \alpha^c=0; \quad i_{\nu} \alpha^c=\frac{\alpha}{\nu} i_{\nu} \beta\]

and so by (2.7) and remarking that \( \alpha^c=d(t\alpha) \), one checks \( (d\wedge d_x+d_\nu d\wedge)(t\alpha)=0 \). On the other hand since

\[(2.12) \quad i_{\nu}(t\alpha)=0\]

one checks \( [i_{\nu}, d]=d_{\nu} \).

The complete lift \( X^c \) of \( X \) is as is known

\[(2.13) \quad X^c=(t^1)\lambda=X+(t\alpha)X^\nu-t^2V\]

where \( X^\nu=(0\安娜\安娜) \) and \( V \) are the vertical lift of \( X \) and the canonical field respectively. Referring to (2.4) and (2.5) we find at once

\[(2.14) \quad i_{\nu}\Omega=\lambda, \quad i_{X^\nu}\Omega=f\alpha, \quad i_{X}\Omega=-f\beta.\]

On the other hand taking account of (2.5), exterior differentiation of (2.10) gives

\[(2.15) \quad d\wedge \beta=\alpha \wedge \beta+t^2/f\Omega.\]

Now making use of (2.15) we derive from (2.14) the following equations

\[(2.16) \quad \mathcal{L}_\nu\Omega=\Omega, \quad \mathcal{L}_{X^\nu}\Omega=0, \quad \mathcal{L}_X\Omega=-t^2\Omega.\]

These equations assert that \( \Omega \) is homogenous of rank 1 [7] and that \( X^\nu \) and \( X \) are an infinitesimal automorphism and an infinitesimal homothety of \( \Omega \) respectively.

Further by (2.13) and (2.14) we get

\[(2.17) \quad i_{X^c}\Omega=(t\alpha)f\alpha-f\beta-t^2\lambda.\]
and therefore

\begin{equation}
\mathcal{L}_X \omega = \alpha^c \wedge \mathfrak{f} \alpha - 2t^2 \omega.
\end{equation}

But \( \alpha^c \) being exact (as \( \alpha \)) we quickly obtain

\begin{equation}
d \wedge (\mathcal{L}_X \omega) = 0
\end{equation}

and this proves that \( \omega \) is a \textit{relatively invariant} 2-form of \( X^c[6] \).

Next making use of the vertical derivation operator \( \iota_\nu \) one finds \( \iota_\nu \Omega = 0 \), and so by virtue of the definition given in [7] one may say that \( \Omega \) is a \textit{Finslerian form}.

According to [5] the complete lift \( \phi^c \) of \( \phi \) (with respect to \( B^a \)) is expressed by

\begin{equation}
\phi^c - dv^1 \wedge \omega^\nu + \cdots + dv^{2m-1} \wedge \omega^{2m} + \omega^1 \wedge dv^2 + \cdots + \omega^{2m-1} \wedge dv^{3m}.
\end{equation}

By virtue of (1.14) a short calculation gives

\begin{equation}
d \wedge \phi^c = \alpha \wedge \phi^c
\end{equation}

and so \( \phi^c \) defines on \( TM_c \) a conformal symplectic structure \( CS\{2m; R \} \).

From (2.20) we obtain

\begin{equation}
i_\nu \phi^c = -v^1 \omega^1 + v^1 \omega^2 - \cdots - v^{2m} \omega^{2m-1} + v^{2m} \omega^{3m}.
\end{equation}

Thus

\begin{equation}
\mathcal{L}_\nu \phi^c = \phi^c
\end{equation}

that is, \( \phi^c \) is \textit{homogenous of degree 1}.

If we put

\begin{equation}
i_\nu \phi = -t_1 dv^1 + t_1 dv^2 - \cdots + t_{2m} dv^{2m} = - \beta_a = - \mu v \phi (X).
\end{equation}

we obtain

\begin{equation}
i_\nu \beta_a = \alpha_a
\end{equation}

and one checks \( (i_\nu d_\nu + d_\nu i_\nu) \beta_a = i_\nu \beta_a \).

Exterior differentiation of (2.24) gives

\begin{equation}
d \wedge \beta_a = \alpha \wedge \beta_a + t^i \phi^c
\end{equation}

and from (2.24) and (2.25) we find

\begin{equation}
\mathcal{L}_\nu \beta_a = \beta_a,
\end{equation}

that is, \( \beta_a \) is homogenous of degree 1.

From (2.20) we also have
\( i_X \varphi^c = i_X \varphi = -\alpha_a \).

Now by (2.23), (2.27) and (2.28) we infer

\( i_{V \cdot X} \varphi^c = \mathcal{L}_{V \cdot X} \varphi^c = -i_X \mathcal{L}_{V} \varphi^c = 0. \)

Clearly \( i_{V \cdot X} \alpha = 0 \), and so referring to 2.21 we finally may write

\( \mathcal{L}_{i_{V \cdot X}} \varphi^c = 0 \)

that is, the Lie bracket \([ V, X ]\) is an infinitesimal automorphism of \( \varphi^c \).

Consider now the almost symplectic form

\( \Theta = (c\alpha)(\alpha \land \lambda + \Omega) \in \land^2(TM_c). \)

By 2.4 and 2.5 exterior differentiation of \( \Theta \) gives

\( d \land \Theta = \left( \frac{\alpha^c}{\iota\alpha} - \alpha \right) \land \Theta \)

and so \( \Theta \) defines a second conformal structure on \( TM_c \) having \( \frac{\alpha^c}{\iota\alpha} - \alpha \) as co-vector of Lee.

One has

\( d_e \Theta = \alpha \land \Omega, \quad i_v \Theta = 0 \)

and with the help of (2.11) and (2.32), one checks \( d_e \Theta = [i_v, d_e] \Theta \).

Now making use of 2.16 we derive from 2.31 and 2.32

\( \mathcal{L}_V \Theta = 2\Theta, \quad \mathcal{L}_X \Theta = -t^2 \Theta, \quad \mathcal{L}_{X \cdot V} \Theta = \frac{t^2}{\iota\alpha} \Theta. \)

Hence \( \Theta \) is homogenous of degree 2, \( X \) is an infinitesimal homothety of \( \Theta \) and \( X^\Psi \) is an infinitesimal conformal transformation of \( \Theta \).

We may formulate the preceding results as follows:

**Theorem.** Let \( TM_c \) be the tangent bundle manifold having as basis the
manifold \( M_c \) of section 1. Let \( V, \lambda, \Omega \) and \( \iota \) be the canonical field on \( TM_c \), the
Liouville form, the symplectic canonical form and the operator which assigns to
1-forms on \( M_c \) functions on \( TM_c \) respectively. Then,

(i) \( \Omega \) is a Finslerian form, \( X \) is an infinitesimal homothety of \( \Omega \), the vertical
lift \( X^V \) of \( X \) is an infinitesimal automorphism of \( \Omega \), and \( \Omega \) is a relatively in-
vARIANT form of the complete lift \( X^c \) of \( X \);

(ii) the complete lift \( \varphi^c \) of the conformal symplectic form \( \varphi \) on \( M_c \) is a con-
formal symplectic form on \( TM_c \) and the Lie bracket \([ V, X ]\) is an infinitesimal
automorphism of \( \varphi^c \);

(iii) the form \( \Theta = (c\alpha)(\alpha \land \lambda + \Omega) \) is homogenous of degree 2 and defines a second
conformal symplectic structure on \( TM_c \) having \( \frac{\alpha^c}{\iota\alpha} - \alpha \) as co-vector of Lee and \( X \)
is an infinitesimal homothety of \( \Theta \).
Note. Let \( S_{x_a}(M_c) \) be the cross section determined on \( TM_c \) by the associated vector field \( X_a \) of \( X \). In consequence of (1.22) (that is the Lie derivate of \( X \) with respect to \( X_a \) vanishes) and of the theorem stated in [5] one may say that \( X^c \) is tangent to the cross-section \( S_{x_a}(M_c) \).

3. Regular mechanical system \( \mathcal{M} = \{M_c, T, \pi\} \) on \( mTM_c \). Consider now on \( TM_c \) the mechanical system \( \mathcal{M} = \{M_c, T, \pi\} \) [8] such that the kinetic energy \( T \) and semi-basic 1-form \( \pi \) be defined respectively by

\[
T = l
\]

and

\[
\pi = l\alpha.
\]

Referring to (2.6), one has

\[
d \wedge dT = \Omega
\]

and so according to J. Klein’s definition, equation 3.3 proves that the system \( \mathcal{M} \) is regular. (it has as fundamental form the symplectic canonical form of \( TM_c \)).

On the other hand a short calculation gives

\[
V(T) = 2T,
\]

Hence \( T \) is homogenous of degree 2.

If \( Z \) is the dynamical system associated with \( \mathcal{M} \) it is as is known [4] well defined by

\[
\iota_Z \Omega = d(T - V(T)) + \pi.
\]

Since \( T \) is homogenous of degree 2, the following theorem of A. Lichnerowicz [9] holds: the form

\[
\Omega - (dT - \pi) \wedge dt \in \Lambda^2(TM_c \times R)
\]

is an integral relation of invariance for \( Z + \frac{\partial}{\partial t} \).

Further one has

\[
d_\pi \Pi = \lambda \wedge \alpha, \quad \iota_\lambda \Pi = 0
\]

and

\[
d \wedge \Pi = \frac{dv}{v} \wedge \Pi,
\]

and so the equation \( d_\lambda \Pi = [\iota_\lambda, d \wedge] \Pi \) is verified.

By 3.6 and 3.7 a short computation gives
Hence $II$ is homogenous of degree 2 as the kinetic energy $T$. This fact proves according to a known Proposition [3] that the dynamical system $Z$ is a spray on $M_c$.

Thus we have the

**Theorem.** Let $TM_c$ be the tangent bundle manifold discussed in section 2. Consider on $TM_c$ the mechanical system $\mathcal{M} = (M_c, T, \pi)$ whose kinetic energy is the Liouville function $l$ on $TM_c$, and whose semi-basic 1-form is the product by $l$ of the principal 1-form on $M_c$. Then:

(i) $\mathcal{M}$ is regular and has as fundamental form the canonical symplectic form on $TM_c$.

(ii) the kinetic energy $T$ is homogenous of degree 2 and the dynamical system associated with $\mathcal{M}$ is a spray on $M_c$.

**References**


