CERTAIN INTEGRAL EQUALITY AND INEQUALITY FOR HYPERSURFACES OF $S^n(R)$

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§ 0. Introduction.

We have the following well known isoperimetric inequality for any simply connected domain $\Omega$ in the sphere $S^2(R)$ of radius $R$ with smooth boundary $\partial \Omega$:

Let $A=\text{area}(\Omega)$ and $L=\text{length}(\partial \Omega)$, then

$$L^2 \geq A\left(4\pi - \frac{1}{R^2}A\right)$$

and the equality is true if and only if $\Omega$ is a geodesic circular disk.

We can prove this inequality by a method of the integral geometry in which for any integer $k$ and positive real number $r$, the set of points $y$ of $S^2(R)$ such that the spherical circle with center at $y$ and of radius $r$ intersects $\partial \Omega$ at $k$ points are used effectively. In the present paper, the author will try to get analogous results to this fact in a higher dimensional sphere $S^n(R)$ by means of the same way.

In § 1, we state some preliminary facts. In § 2, we shall obtain an integral equality for oriented hypersurfaces (Theorem 1). Then, in § 3, we shall have an equality on the volumes of a convex domain and the $r$-neighborhood $\Omega_r$ of $\Omega$ (3.5)). Finally, in §§ 4 and 5, combining the results in §§ 2 and 3 and using the Fenchel-Borsuk’s theorem:

For any closed curve $C$ in a Euclidean space, $\int_C |k(s)| ds \geq 2\pi$, where $k(s)$ is the curvature of $C$ and $s$ denotes the arclength of $C$, we shall obtain a kind of isoperimetric inequality for a convex domain in $S^3(R)$ (Theorem 3).

§ 1. Preliminaries.

Let $S^n(R)$ be the standard $n$-sphere in $R^{n+1}$ of radius $R$ and with its centor at the origin, and $\Omega$ a domain in $S^n(R)$ with smooth boundary $\partial \Omega=M^{n-1}$ (= $M$). Let $\Phi=\{\xi | \xi \in T_x S^n(R), x \in M, |\xi|=1\}$ and $\pi: \Phi \rightarrow M$ be the projection of the sphere bundle $(\Phi, M, \pi)$. For a positive real number $r>0$, let $\phi_r$ be the mapping $\phi_r: \Phi \rightarrow S^n(R)$ with $\phi_r(\xi)=\exp_x r\xi$, where $\exp_x$ denotes the exponential mapping of $S^n(R)$ at $x=\pi(\xi)$.

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Let \( (x, e_1, \cdots, e_{n-1}, e_n, e_{n+1}) \) be a moving orthonormal frame of \( \mathbb{R}^{n+1} \) such that
\[
x \in M, \quad e_1, e_2, \cdots, e_{n-1} \in T_x M, \quad e_{n+1} = \frac{1}{R} x
\]
and the orientation of \( (e_1, e_2, \cdots, e_{n-1}, e_n, e_{n+1}) \) coincides with the canonical one of \( \mathbb{R}^{n+1} \). Then, we have

\[
\begin{align*}
\frac{dx}{dt} &= \sum_{\beta=1}^{n-1} \omega_\beta e_\beta, \\
de_\alpha &= \sum_{\beta=1}^{n-1} \omega_{\alpha\beta} e_\beta + \omega_{\alpha n} e_n - \frac{1}{R} \omega_\alpha e_{n+1}, \\
de_n &= - \sum_{\beta=1}^{n-1} \omega_{\beta n} e_\beta, \quad de_{n+1} = - \frac{1}{R} \sum_{\beta=1}^{n-1} \omega_{\beta} e_\beta,
\end{align*}
\]

where

\[
\omega_{\alpha\beta} = - \omega_{\beta\alpha}, \quad \omega_{\alpha n} = - \omega_{n\alpha} = \sum_{\beta=1}^{n-1} A_{\alpha\beta} \omega_\beta,
\]

are the components of the 2nd fundamental form of \( M \) in \( S^n(R) \) for the unit normal vector \( e_n \) with respect to \( (x, e_1, e_2, \cdots, e_{n-1}) \).

Setting \( y = \varphi_r(\xi), \xi \in \Phi, \pi(\xi) = x \) and \( \xi = \sum_{i=1}^n \xi_i e_i, \) we have easily

\[
y = R e_{n+1} \cos \frac{r}{R} \frac{\xi}{R} + \xi \sin \frac{r}{R}
\]

and

\[
dy = \cos \frac{r}{R} \sum_{\alpha=1}^{n-1} \omega_\alpha e_\alpha + R \sin \frac{r}{R} \sum_{i=1}^n D\xi_i e_i - \sin \frac{r}{R} \sum_{\alpha=1}^{n-1} \xi_\alpha \omega_\alpha e_{n+1}
\]

\[
= \sum_{\alpha=1}^{n-1} \omega_\alpha \left( \cos \frac{r}{R} e_\alpha - \xi_\alpha \sin \frac{r}{R} e_{n+1} \right) + R \sin \frac{r}{R} \sum_{i=1}^n D\xi_i e_i,
\]

where \( D \) denotes the covariant differentiation \( S^n(R) \) with respect to its Riemannian connection.

Now, when \( \cos \frac{r}{R} \neq 0 \) and \( \sin \frac{r}{R} \neq 0 \), noticing that \( \cos \frac{r}{R} e_\alpha - \xi_\alpha \sin \frac{r}{R} e_{n+1}, \alpha = 1, 2, \cdots, n-1, \) and \( \sum D\xi_i e_i \) are all orthogonal to \( e_{n+1} \cos \frac{r}{R} + \xi \sin \frac{r}{R} \), we obtain by a straightforward calculation
\[
\left( \cos \frac{r}{R} e_1 - \xi_1 \sin \frac{r}{R} e_{n+1} \right) \wedge \left( \cos \frac{r}{R} e_2 - \xi_2 \sin \frac{r}{R} e_{n+1} \right) \wedge \ldots \\
\wedge \left( \cos \frac{r}{R} e_{n-1} - \xi_{n-1} \sin \frac{r}{R} e_{n+1} \right) \wedge \sum_i D\xi_i e_i \wedge \left( \cos \frac{r}{R} e_{n+1} + \sin \frac{r}{R} \xi \right) \\
= \left( \cos \frac{r}{R} \right)^{n-2} D\xi_n (e_1 \wedge e_2 \wedge \ldots \wedge e_{n+1}).
\]

We denote the volume element of \( S^n(R) \) by \( dV_S \). Then from the above equalities we have

\[
(1.6) \quad \phi^* dV_S = R \sin \frac{r}{R} \left( \cos \frac{r}{R} \right)^{n-2} \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{n-1} \wedge D\xi_n,
\]
in which we may replace \( D\xi_n \) by \( d\xi_n \).

Then, when \( \cos \frac{r}{R} = 0 \), (1.4) and (1.5) turn into

\[
(1.4_0) \quad y = \varepsilon R\xi, \quad \varepsilon = \sin \frac{r}{R} = \pm 1
\]
and

\[
(1.5_0) \quad dy = \varepsilon \left\{ R \sum_{\alpha=1}^{n-1} D\xi_\alpha e_\alpha - \sum_{\alpha=1}^{n-1} \xi_\alpha \omega_\alpha e_{n+1} \right\}.
\]

For \( \xi \in \Phi \) with \( \xi_n \neq 0 \), substituting \( D\xi_n = -\frac{1}{\xi_n} \sum_\alpha \xi_\alpha D\xi_\alpha \) into the above equality, we get

\[
(1.5_0') \quad dy = \varepsilon \left\{ R \sum_\alpha D\xi_\alpha \left( e_\alpha - \frac{1}{\xi_n} \xi_\alpha e_n \right) - \sum_\alpha \xi_\alpha \omega_\alpha e_{n+1} \right\}.
\]
Noticing \( e_\alpha - \frac{1}{\xi_n} \xi_\alpha e_n, \ \alpha = 1, 2, \ldots, n-1, \) and \( e_{n+1} \) are all orthogonal to \( \xi \), we obtain

\[
\left( e_1 - \frac{1}{\xi_n} \xi_1 e_n \right) \wedge \ldots \wedge \left( e_{n-1} - \frac{1}{\xi_n} \xi_{n-1} e_n \right) \wedge e_{n+1} \wedge \xi \\
= -\frac{1}{\xi_n} e_1 \wedge \ldots \wedge e_n \wedge e_{n+1}.
\]

Hence we have in this case

\[
(1.6_0) \quad \phi^* dV_S = \varepsilon^{n+1} \frac{1}{\xi_n} R^{n-1} D\xi_1 \wedge \ldots \wedge D\xi_{n-1} \wedge \sum_\alpha \xi_\alpha \omega_\alpha, \quad \varepsilon = \sin \frac{r}{R}.
\]
The induced Riemannian metric on $M$ from $R^{n+1}$ is written as
\[ ds^2_M = \sum_{a=1}^{n+1} \omega_a \omega_a \]
and we define a natural Riemannian metric on $\Phi$ by
\[ ds^2_{\Phi} = \sum_{a=1}^{n} \omega_a \omega_a + \sum_{i=1}^{n} D\xi_i D\xi_i. \]
Then, their volume elements $dV_M$ and $dV_{\Phi}$ are clearly given by
\[ dV_M = \omega_1 \wedge \cdots \wedge \omega_{n-1} \]
and
\[ dV_{\Phi} = \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \sum_{i=1}^{n} (-1)^{n-1} \xi_i D\xi_1 \wedge \cdots \wedge \hat{D}\xi_i \wedge \cdots \wedge D\xi_n. \]
We can easily prove that the following is a differential $(n-1)$-form on $\Phi$:
\[ d\mu_{n-1} = \sum_{i=1}^{n} (-1)^{n-1} \xi_i D\xi_1 \wedge \cdots \wedge \hat{D}\xi_i \wedge \cdots \wedge D\xi_n, \]
whose restriction on the unit $(n-1)$-sphere $\pi^{-1}(x)$ of $T_x S^n(R)$, $x \in M$, is its volume element. We have
\[ dV_{\Phi} = dV_M \wedge d\mu_{n-1}. \]
We can also easily prove that
\[ d\mu_{n-1} = (-1)^{n-1} \frac{1}{\xi_j} D\xi_1 \wedge \cdots \wedge \hat{D}\xi_j \wedge \cdots \wedge D\xi_n \]
at $\xi \in \Phi$ with $\xi_j \neq 0$, by using $\sum_x \xi_i \xi_i = 1$ and $\sum_x \xi_i D\xi_i = 0$, and so especially
\[ dV_{\Phi} = \frac{1}{\xi_n} \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge D\xi_1 \wedge \cdots \wedge D\xi_n \]
at $\xi$ with $\xi_n \neq 0$.

§ 2. An integral equality for hypersurfaces $S^n(R)$.

In the following, we suppose that $0 < r < \pi R$. For any point $y \in S^n(R)$, we denote the $(n-1)$-sphere on $S^n(R)$ of geodesic radius $r (0 < r < \frac{\pi}{2} R)$ or $\pi R - r (\frac{\pi}{2} R \leq r < \pi R)$ with its centor at $y$ or $-y$, by $F^{n-1}_r(y)$. We can easily see that
\[ F^{n-1}_r(y) = \{ \exp_y v | v \in T_y S^n(R), |v| = r \} \]
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and

\[ \phi^{-1}(y) = \{ \text{tangent unit vectors } \xi \text{ at } x \in F_{r}^{n-1}(y) \cap M \text{ such that } y = \exp_s r \xi \}. \]

Now, when \( \cos \frac{r}{R} \neq 0 \), from (1.5) we have

\[
\begin{align*}
D\xi \alpha &= -\frac{1}{R} \cot \frac{r}{R} \omega_\alpha, \quad \alpha = 1, 2, \ldots, n-1, \\
D\xi_n &= 0, \\
\sum_{\alpha=1}^{n-1} \xi_\alpha \omega_\alpha &= 0 \quad \text{along } \phi^{-1}(y).
\end{align*}
\]

(2.1)

Hence, the induced Riemannian metric on \( \phi^{-1}(y) \) from \( \omega \) on \( \Phi \) can be written as

\[ ds^2 = \left(1 + \frac{1}{R^2} \cot^2 \frac{r}{R} \right) \sum_{\alpha=1}^{n-1} \omega_\alpha \omega_\alpha, \]

which implies the following equality

\[ \text{vol}(\phi^{-1}(y)) = \left(1 + \frac{1}{R^2} \cot^2 \frac{r}{R} \right)^{\frac{n-1}{2}} \text{vol}(F_{r}^{n-1}(y) \cap M). \]

(2.2)

On the other hand, we consider a differential \((n-2)\)-form in \( \Phi \) of the from as

\[ \Theta_{n-2} = \sum_{\alpha=1}^{n-1} (-1)^{\alpha-1} \lambda_\alpha D\xi_1 \wedge \cdots \wedge \hat{D}\hat{\xi}_\alpha \wedge \cdots \wedge D\xi_{n-1}, \]

where \( \lambda_\alpha \) will be determined so that

(2.3)

\[ dV_{\phi} = \phi^*(dV_s) \wedge \Theta_{n-2}. \]

By means of (1.6), where \( \xi_n \neq 0 \), the right-hand side of this equality becomes

\[ -R \sin \frac{r}{R} \left( \cos \frac{r}{R} \right)^{n-2} \frac{1}{\xi_n} \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \sum_{\alpha=1}^{n-1} \xi_\alpha D\xi_\alpha \]

\[ \wedge \sum_{\beta=1}^{n-1} (-1)^{\beta-1} \lambda_\beta D\xi_1 \wedge \cdots \wedge \hat{D}\hat{\xi}_\beta \wedge \cdots \wedge D\xi_{n-1} \]

\[ = -R \sin \frac{r}{R} \left( \cos \frac{r}{R} \right)^{n-2} \frac{1}{\xi_n} \sum_{\alpha=1}^{n-1} \xi_\alpha \lambda_\alpha \omega_1 \wedge \cdots \wedge \omega_{n-1}. \]

Comparing this with (1.13), we see that it is sufficient to take \( \lambda_\alpha \) as
Thus we define \( \Theta_{n-2} \) by
\[
(2.4) \quad \Theta_{n-2} := \frac{1}{R \sin \frac{r}{R}} \left( \frac{r}{R} \right)^{n-2} \cdot \frac{1}{1 - \xi_n \xi_n} \sum_{\alpha=1}^{n-1} (-1)^{\alpha-1} \xi_\alpha D \xi_\alpha \wedge \cdots \wedge \hat{\xi}_\alpha \wedge \cdots \wedge \hat{\xi}_{n-1},
\]
where \( \xi_n \xi_n \neq 1 \).

Let \( \gamma : \psi^{-1}_r(y) \to \Phi \) be the inclusion map. Then, by (2.1) we have
\[
(2.5) \quad \Theta_{n-2} = - \frac{1}{R \sin \frac{r}{R}} \left( \frac{r}{R} \right)^{n-1} \cdot \frac{1}{1 - \xi_n \xi_n} \sum_{\alpha=1}^{n-1} (-1)^{\alpha-1} \xi_\alpha \omega_\alpha \wedge \cdots \wedge \omega_{n-1},
\]
and especially
\[
(2.5') \quad \Theta_{n-2} = - \frac{1}{R \sin \frac{r}{R}} \left( \frac{r}{R} \right)^{n-2} \frac{1}{1 - \xi_n \xi_n} \omega_1 \wedge \omega_2 \cdots \wedge \omega_{n-2},
\]
where \( \xi_{n-1} \neq 0 \).

Next, we observe the volume element of \( F_r^{n-1}(y) \cap M \). On \( F_r^{n-1}(y) \cap M \), we obtain from (2.1)
\[
ds^2 = \sum_{\alpha=1}^{n-1} \omega_\alpha \omega_\alpha = \sum_{\alpha=1}^{n-2} \omega_\alpha \omega_\alpha + \left( - \frac{1}{\xi_{n-1}} \sum_{\alpha=1}^{n-1} \xi_\alpha \omega_\alpha \right)^2
\]
\[
= \sum_{\alpha=1}^{n-2} \left( \delta_{ab} + \frac{1}{\xi_{n-1} \xi_{n-1}} \xi_a \xi_b \right) \omega_a \omega_b
\]
and
\[
\det \left( \delta_{ab} + \frac{1}{\xi_{n-1} \xi_{n-1}} \xi_a \xi_b \right) = \frac{1}{\xi_{n-1} \xi_{n-1}} \sum_{\alpha=1}^{n-1} \xi_\alpha \xi_\alpha = \frac{1 - \xi_n \xi_n}{\xi_{n-1} \xi_{n-1}},
\]
where \( \xi_{n-1} \neq 0 \). Hence, the volume element of \( F_r^{n-1}(y) \cap M \) is given by
\[
(2.6) \quad dV_{F_r^{n-1}(y) \cap M} = \frac{\sqrt{1 - \xi_n \xi_n}}{\xi_{n-1}} \omega_1 \wedge \cdots \wedge \omega_{n-2},
\]
where \( \xi_{n-1} \neq 0 \). In general, we have

\[
(2.6') \quad dV_{F_{r}^{n-1}(y) \cap M} = (-1)^{n-1-\beta} \frac{1}{\xi_{\beta}} \sqrt{1 - \xi_{1}^{2}} \xi_{n} \omega_{1} \wedge \cdots \wedge \omega_{n-1}.
\]

where \( \xi_{\beta} \neq 0 \), and

\[
(2.6^*) \quad dV_{F_{r}^{n-1}(y) \cap M} = \frac{1}{\sqrt{1 - \xi_{1}^{2}}} \sum_{\alpha=2}^{n-1} (-1)^{n-1-\alpha} \xi_{\alpha}^{n} \omega_{1} \wedge \cdots \wedge \omega_{n-1}.
\]

Since we have

\[
(2.7) \quad dV_{\phi_{r}^{n-1}(y)} = \left(1 + \frac{1}{R^{2}} \cot^{2} \frac{r}{R}\right)^{n/2-1} dV_{F_{r}^{n-1}(y) \cap M},
\]

hence

\[
(2.8) \quad dV_{\phi_{r}^{n-1}(y)} = \left(1 + \frac{1}{R^{2}} \cot^{2} \frac{r}{R}\right)^{n/2-1} \frac{1}{\sqrt{1 - \xi_{1}^{2} \xi_{n}^{2}}} \sum_{\alpha=1}^{n-1} (-1)^{n-1-\alpha} \xi_{\alpha} \omega_{1} \wedge \cdots \wedge \omega_{n-1}.
\]

and

\[
(2.8^*) \quad dV_{\phi_{r}^{n-1}(y)} = \left(1 + \frac{1}{R^{2}} \cot^{2} \frac{r}{R}\right)^{n/2-1} \frac{1}{\sqrt{1 - \xi_{1}^{2} \xi_{n}^{2}}} \xi_{n-1} \omega_{n-2}.
\]

where \( \xi_{n-1} \neq 0 \). From (2.5) and (2.8), we obtain

\[
(2.9) \quad \xi_{n-1} = \frac{1}{R \sin \frac{r}{R} \left(\cos \frac{r}{R}\right)^{n-2} \left(1 + R^{2} \tan^{2} \frac{r}{R}\right)^{n/2-1}}.
\]

Finally, we consider the case \( \cos \frac{r}{R} = 0 \), i.e. \( r = \frac{\pi R}{2} \). From (1.5) we have

\[
(2.10) \quad \left\{ \begin{array}{l}
D_{\xi_{1}} = 0, \quad i=1, 2, \ldots, n, \\
\sum_{\alpha=1}^{n-1} \xi_{\alpha} \omega_{\alpha} = 0 \quad \text{along} \quad \phi_{r}^{n-1}(y).
\end{array} \right.
\]

We take a differential \((n-2)\)-form in \( \Phi \) of the form

\[
\Psi_{n-2} = \sum_{\alpha=1}^{n-1} (-1)^{n-1} \lambda_{\alpha} \omega_{1} \wedge \cdots \wedge \omega_{n-1} \wedge \omega_{n+1} \wedge \cdots \wedge \omega_{n},
\]

where \( \lambda_{\alpha} \) will be determined so that

\[
(2.3^*) \quad dV_{\phi} = \phi_{\Phi}^{*}(dV_{S}) \wedge \Psi_{n-2}.
\]
By means of (1.6), at $\xi$ with $\xi_n \neq 0$, the right-hand side of this equality becomes

$$\frac{1}{\xi_n} R^{n-1} D^{\xi_1} \wedge \cdots \wedge D^{\xi_{n-1}} \wedge \sum_{a=1}^{n-1} \xi_a \omega_a \wedge \sum_{\beta=1}^{n-1} (-1)^{\beta-1} \lambda_\beta \omega_1 \wedge \cdots \wedge \omega_{n-1}$$

$$= (-1)^{n-1} R^{n-1} \frac{1}{\xi_n} \sum_{a=1}^{n-1} \xi_a \lambda_a \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge D^{\xi_1} \wedge \cdots \wedge D^{\xi_{n-1}}$$

Comparing this equality with (1.13), we see that it is sufficient to take $\lambda_\alpha$ as

$$\lambda_\alpha = \frac{(-1)^{n-1} \xi_a}{R^{n-1} (1 - \xi_n \xi_n)}$$

when $\xi_n \xi_n < 1$. Thus, we define $\Psi_{n-2}$ by

$$(2.10) \quad \Psi_{n-2} := - \frac{1}{R^{n-1} (1 - \xi_n \xi_n)} \sum_{a=1}^{n-1} (-1)^{n-1-a} \xi_a \omega_1 \wedge \cdots \wedge \omega_{a-1} \wedge \omega_{a+1} \wedge \cdots \wedge \omega_{n-1},$$

where $\xi_n \xi_n < 1$. In this case, from (2.10) we see that

$$(2.7) \quad d V_{\phi^{-1}(y)} = d V_{\phi^{-1}(y) \cap M}$$

through an isometry. We can also use the formula (2.6) and obtain

$$(2.9) \quad \epsilon_n \Psi_{n-2} = - \frac{1}{R^{n-1} \sqrt{1 - \xi_n \xi_n}} d V_{\phi^{-1}(y)}.$$

Making use of these formulas and noticing that these hold good for oriented hypersurfaces in $S^n(R)$ in general, we obtain the following

**Theorem 1.** Let $M^{n-1} = M \subset S^n(R)$ be a smooth oriented hypersurface and $0 < r < \pi R$. Then, we have the following integral equality:

$$(2.11) \quad \int_{S^{n}(R)} \left( \int_{\phi^{-1}(y)} \frac{1}{\sqrt{1 - \xi_n \xi_n}} d V_{\phi^{-1}(y) \cap M} \right) d V_s$$

$$= \left( R \sin \frac{r}{R} \right)^{n-1} c_{n-1} \text{vol}(M),$$

where $c_{n-1}$ is the volume of the unit $(n-1)$-sphere.

**Proof.** From (1.10), we see that

$$\int_{\phi} d V_{\phi} = \text{vol}(\phi) = c_{n-1} \cdot \text{vol}(M).$$
We prove the case \( r \neq \pi R/2 \). By means of (2.3), (2.8) and (2.7), the left-hand side of the above equality can be also computed as follows:

\[
\int_\varnothing dV_\varnothing = \int_{S^{n}(R)} \frac{dV_S}{R \sin \frac{r}{R} \left| \cos \frac{r}{R} \right|^{n-2} \left( 1 + R^2 \tan^2 \frac{r}{R} \right)^{n/2-1}} \int_{\tilde{\varnothing}_r^{-1}(y)} \phi_r^{-1}.
\]

\[
\frac{1}{\sqrt{1 - \frac{e_n e_n}{\tilde{e}_n \tilde{e}_n}}} dV_{\tilde{\varnothing}_r^{-1}(y)}
\]

\[
= \frac{1}{(R \sin \frac{r}{R})^{n-1}} \int_{S^{n}(R)} \left( \int_{\tilde{\varnothing}_r^{-1}(y) \cap M} \frac{1}{\sqrt{1 - \frac{e_n e_n}{\tilde{e}_n \tilde{e}_n}}} dV_{\tilde{\varnothing}_r^{-1}(y) \cap M} \right) dV_S,
\]

from which we obtain immediately (2.11).

For the case \( r = \pi R/2 \), we can prove it analogously by (2.3) and (2.9).

q.e.d.

§ 3. An integral equality for a convex domain in \( S^n(R) \).

We say a domain \( \varnothing \) in \( S^n(R) \) convex if \( \varnothing \) contains no pair of points \( y \) and \(-y\) of \( S^n(R) \) and for any two points \( p \) and \( q \) of \( \varnothing \) it contains the minimum geodesic segment of \( S^n(R) \) joining \( p \) and \( q \).

If \( \varnothing \subset S^n(R) \) is convex, it must be contained in a half \( n \)-sphere of \( S^n(R) \). We see this fact easily by considering a contacting great \((n-1)\)-sphere of \( S^n(R) \) to \( \partial \varnothing \). Hence we have

\[
V = \text{vol} \left( \varnothing \right) \leq \frac{e_n}{2} R^n.
\]

In the following, we suppose that \( \varnothing \) is convex and has smooth boundary \( M = \partial \varnothing \). For \( r > 0 \), we set

\[
\varnothing_r = \{ x \mid x \in S^n(R), \text{dis}_{S^n(R)}(x, \varnothing) < r \}, \quad V_r = \text{vol} \left( \varnothing_r \right).
\]

In this case, \( M \) must be diffeomorphic to an \((n-1)\)-sphere. Furthermore, we suppose \( 0 < r \leq \pi R/2 \). Using the notation in §§ 1, 2 and taking notice of that for the orthonormal frame \((x, e_1, \cdots, e_n, e_{n+1})\), \( x \in M \), \( e_n \) directs inward of \( \varnothing \) at \( x \) and \( e_{n+1} = (1/R)x \), we see that \( \varnothing_r - \varnothing \) is the set of points \( y \) written as

\[
y = R \left( e_{n+1} \cos \frac{t}{R} - e_n \sin \frac{t}{R} \right), \quad 0 \leq t < r.
\]
Hence, we have

\[ dy = \sum_{\alpha=1}^{n-1} \left\{ e_\alpha \cos \frac{t}{R} + R \left( \sum_{\beta=1}^{n-1} A_{\alpha\beta} \right) \sin \frac{t}{R} \right\} \omega_\alpha \\
- \left( e_{n+1} \sin \frac{t}{R} + e_n \cos \frac{t}{R} \right) dt. \]

If we choose especially \( e_1, \ldots, e_{n-1} \) in the principal directions of \( M \) at \( x \), then we can put \( A_{\alpha\beta} = k_{\alpha} \delta_{\alpha\beta} \). Denoting the normal exponential map of \( M \) in \( S^n(R) \) by \( \exp^1 \), we induce a volume element of the normal bundle \( NM \) from \( dV_s \) through \( \exp^1 \). From the above computation, we have

\[(\exp^1)_{(x,\cdot)}^* dV_i = - \prod_{\alpha=1}^{n-1} \left( \cos \frac{t}{R} + k_\alpha R \sin \frac{t}{R} \right) \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge dt,

hence

\[ V_r - V = \int_0^r \int_M \left\{ \left( \cos \frac{t}{R} \right)^{n-1} + \sum_{m=1}^{n-1} R^m \sigma_m(k_1, \ldots, k_{n-1}) \right. \\
\left. \left( \cos \frac{t}{R} \right)^{n-m-1} \left( \sin \frac{t}{R} \right)^m \right\} dV_M dt,

where \( \sigma_m(u_1, \ldots, u_{n-1}) \) denotes the fundamental symmetric polynomial of order \( m \) in \( u_1, u_2, \ldots, u_{n-1} \). Thus, we have

\[(3.4) \quad V_r = V + \int_0^r \left( \cos \frac{t}{R} \right)^{n-1} dt \int_M dV_M \\
+ \sum_{m=1}^{n-1} R^m \int_0^r \left( \cos \frac{t}{R} \right)^{n-m-1} \left( \sin \frac{t}{R} \right)^m dt \int_M \sigma_m(k_1, \ldots, k_{n-1}) dV_M.

For the 2nd fundamental form \( II = \sum_{\alpha, \beta} A_{\alpha\beta} \omega_\alpha \omega_\beta \), we set

\[ \det (I_{n-1} + uA) = 1 + \sum_{m=1}^{n-1} \binom{n-1}{m} u^m P_m(A), \]

where \( A = (A_{\alpha\beta}) \). Especially, we have

\[ P_1(A) = \frac{1}{n-1} \sum_{\alpha=1}^{n-1} k_\alpha = H \quad (\text{mean curvature}). \]
CERTAIN INTEGRAL EQUALITY

Using these $P_m(A)$, we can rewrite (3.4) as

\[(3.5) \quad V_r = V + \int_0^r \left( \cos \frac{t}{R} \right)^{n-1} dt \cdot \int_M dV_M \]
\[+ \sum_{m=1}^{n-1} \binom{n-1}{m} R^m \int_0^r \left( \cos \frac{t}{R} \right)^{n-m-1} \left( \sin \frac{t}{R} \right)^m dt \cdot \int_M P_m(A) dV_M.\]

Now, we compute the right-hand side of (3.4) in more exact form for the case $n=3$. Since we have

\[\int_0^r \cos^2 \frac{t}{R} dt = \frac{1}{2} \left( R \cos \frac{r}{R} \sin \frac{r}{R} + r \right), \quad \int_0^r \cos \frac{t}{R} \sin \frac{t}{R} dt = \frac{R}{2} \sin^2 \frac{r}{R}, \]
\[\int_0^r \sin^2 \frac{t}{R} dt = \frac{1}{2} \left( -R \cos \frac{r}{R} \sin \frac{r}{R} + r \right),\]

(3.4) becomes in this case

\[V_r = V + \frac{1}{2} \left( r + R \cos \frac{r}{R} \sin \frac{r}{R} \right) \int_M dV_M + R^2 \sin^2 \frac{r}{R} \int_M H dV_M \]
\[+ \frac{1}{2} R^2 \left( -R \cos \frac{r}{R} \sin \frac{r}{R} \right) \int_M k_1 k_2 dV_M.\]

On the other hand, denoting the Gaussian curvature of $M$ by $K$, we have easily

\[K = k_1 k_2 + \frac{1}{R^2}.\]

Hence, by means of the Gauss-Bonnet theorem we obtain

\[\int_M k_1 k_2 dV_M = \int_M K dV_M - \frac{1}{R^2} \int_M dV_M = 2\pi \chi(M) - \frac{1}{R^2} \int_M dV_M \]
\[= 4\pi - \frac{1}{R^2} \int_M dV_M,\]

since $M$ is homeomorphic to $S^2$. Substituting this into the above equality, we obtain

\[(3.6) \quad V_r = V + R \cos \frac{r}{R} \sin \frac{r}{R} \int_M dV_M + 2\pi R^2 \left( -R \cos \frac{r}{R} \sin \frac{r}{R} \right) \]
\[+ R^2 \sin^2 \frac{r}{R} \int_M H dV_M.\]
§ 4. An isoperimetric inequality for a convex domain in $S^3(R)$.

First of all, we investigate the integral in (2.11):

\begin{equation}
\int_{F^{n-1}_r(y) \cap M} \frac{1}{\sqrt{1 - \xi_n^2}} dV_{F^{n-1}_r(y) \cap M}.
\end{equation}

For any point $x \in F^{n-1}_r(y) \cap M$ and a frame $(x, e_1, e_2, \ldots, e_n, e_{n+1})$ as in § 1, we have

\begin{equation}
\xi = \frac{1}{R \sin \frac{r}{R}} (y - x \cos \frac{r}{R}),
\end{equation}

\begin{equation}
\xi_i = \langle \xi, e_i \rangle = \frac{y_i}{R \sin \frac{r}{R}}, \quad y_i = \langle y, e_i \rangle, \quad i = 1, 2, \ldots, n
\end{equation}

and

\begin{equation}
y_{n+1} = \langle y, e_{n+1} \rangle = R \cos \frac{r}{R}.
\end{equation}

Along $F^{n-1}_r(y) \cap M$, $\langle y, x \rangle = R^2 \cos \frac{r}{R}$ implies

\begin{equation}
\langle y, \sum_a \omega_a e_a \rangle = \sum_a \omega_a = 0.
\end{equation}

On the other hand, restricting the moving frame $(x, e_1, e_2, \ldots, e_{n+1}), \ x \in F^{n-1}_r(y) \cap M$, to the one such that $e_1, \ldots, e_{n-2} \in T_x(F^{n-1}_r(y) \cap M)$, we have

\begin{equation}
\omega_{n-1} = 0,
\end{equation}

and hence

\begin{equation}
y_1 = y_2 = \cdots = y_{n-2} = 0
\end{equation}

and

\begin{equation}
y = y_{n-1} e_{n-1} + y_n e_n + R \cos \frac{r}{R} e_{n+1}.
\end{equation}

Using these relations, $dy=0$ implies

\begin{equation}
dy_{n-1} = y_n \omega_{n-1}, \quad dy_n = -y_{n-1} \omega_{n-1},
\end{equation}

\begin{equation}
y_{n-1} \omega_{a_{n-1}} + y_n \omega_a = \cos \frac{r}{R} \cdot \omega_a, \quad a = 1, \ldots, n-2.
\end{equation}

From (4.5) and the structure equation we obtain
hence we can put

\begin{equation}
\omega_{a,n-1} = \sum_{b=1}^{n-2} B_{ab} \omega_b, \quad B_{ab} = B_{ba}.
\end{equation}

$B_{ab}$ are the components of the 2nd fundamental form of $F^a_{\tau^{-1}(y)} \cap M$ with respect to the normal unit vector $e_{n-1}$. By (4.4) and (4.7), we can put

\begin{equation}
y_{n-1} = R \sin \frac{r}{R} \cos \theta, \quad y_{n} = R \sin \frac{r}{R} \sin \theta.
\end{equation}

Substituting these into (4.7) and (4.8), we get

\begin{equation}
\omega_{n-1,n} = -d\theta,
\end{equation}

\begin{equation}
\cos \theta \cdot B_{ab} + \sin \theta \cdot A_{ab} = \frac{1}{R} \cot \frac{r}{R} \cdot \delta_{ab},
\end{equation}

with $a, b = 1, 2, \ldots, n-2$.

From the equality:

\begin{equation}
\omega_{n-1,n} = \sum_{a=1}^{n-2} A_{n-1,a} \omega_a \quad \text{along } F^a_{\tau^{-1}(y)} \cap M
\end{equation}

and (4.11), we can put

\begin{equation}
A_{n-1,a} = -\nabla_{e_a} \theta,
\end{equation}

where $\nabla$ denotes the covariant derivation of $F^a_{\tau^{-1}(y)} \cap M$.

Now, we suppose $n = 3$ in the following. Then, $F^3_{\tau(y)} \cap M$ is composed of curves in general. Setting $\omega_1 = ds$, (4.11) and (4.13) imply

\begin{equation}
\theta = -\int A_{12} ds + \text{const}.
\end{equation}

We have also

\begin{equation}
\frac{dV_{F^3_{\tau(y)} \cap M}}{\sqrt{1-e_3^2}} = \frac{ds}{\sqrt{1-\sin^2 \theta}} = \frac{ds}{\cos \theta} = -\frac{d\theta}{A_{12} \cos \theta}.
\end{equation}

In this case, (4.12) becomes

\begin{equation}
B_{11} \cos \theta + A_{11} \sin \theta = \frac{1}{R} \cot \frac{r}{R},
\end{equation}

from which we get

\begin{equation}
\cos \theta = \frac{1}{A_{11}^2 + B_{11}^2} \left( \frac{B_{11}}{R} \cot \frac{r}{R} + A_{11} \sqrt{A_{11}^2 + B_{11}^2 - \frac{1}{R^2} \cot^2 \frac{r}{R}} \right).
\end{equation}
Along the curve $F^2_2(y) \cap M$, we have

$$\frac{de_1}{ds} = B_{11}e_2 + A_{11}e_3 - \frac{1}{R}e_4$$

and hence its curvature as a curve in $\mathbb{R}^4$ is

(4.17) $$k(s) = \left| \frac{de_1}{ds} \right| = \sqrt{B_{11}^2 + A_{11}^2 - \frac{1}{R^2}}.$$ 

Using $k(s)$, the right-hand side of the above expression of $\cos \theta$ can be written as

(4.18) $$\cos \theta = \frac{1}{A_{11}^2 + B_{11}^2} \cdot \frac{1}{R \sin \frac{r}{R}} \left\{ B_{11} \cos \frac{r}{R} + A_{11} \sqrt{k^2 R^2 \sin^2 \frac{r}{R} - 1} \right\}.$$ 

Then, we have the following theorem which will be proved in § 5.

**Theorem 2.** Let $\Omega \subset S^3(R)$ be convex and for $\partial \Omega = M$ its normal curvature $A$ with respect to the inner normal unit vector satisfy $A_0 \leq A \leq A_1$. Then, supp. posing $0 < r < \pi R/2$, for $\cos \theta$ given by (4.18) there exists a constant $C_0$ depending only on $A_0$, $A_1$ and $r$ such that $1/\cos \theta \geq C_0 k(s)$.

Now, for a domain $\Omega$ in $S^3(R)$, let $r_1(r_e)$ be the supremum (infimum) of radius of 3-disk included in (containing) $\Omega$. Then, we have

**Theorem 3.** Let $\Omega$ be a convex domain of $S^3(R)$ with a smooth boundary $\partial \Omega = M$. Let $H$ be the mean curvature of $M$. Then, for a fixed number $r$ ($r_1(r_e) \leq r \leq r_n$) we have

(4.19) $$R \cos \frac{r}{R} \sin \frac{r}{R} \left( \frac{2R}{C_0} \tan \frac{r}{R} - 1 \right) \text{area}(M) \geq \text{vol}(\Omega) + 2 \pi R^2 \left( r - R \cos \frac{r}{R} \sin \frac{r}{R} \right) + R^2 \sin^2 \frac{r}{R} \int_M H dV_M.$$ 

**Proof.** Since $\Omega$ is convex, we have easily $r_e \leq \pi R/2$. Therefore, we can utilize Theorem 2 for the domain $\Omega$. For a general point $y \in S^3(R)$, let $n(y)$ be the number of the components of $F^2_2(y) \cap M$. Let $C$ be one of them. For the curvature $k(s)$ of $C$ as a curve in $\mathbb{R}^4$, we have $\int_0^L k(s) ds \geq 2 \pi$ by the Fenchel-Borsuk theorem, where $L = \text{length}(C)$. It is clear that the set of general points $y$ is open and dense in $S^3(R)$, and the function $n(y)$ is lower semi-continuous. Therefore, the set of $y$ with $n(y) = m$ is measurable with respect to the 3-dimensional measure of $S^3(R)$ for any integer $m$. Setting

$$F_m := \text{vol}\{ y \mid y \in S^3(R), n(y) = m \},$$
we obtain from Theorem 1 with $n=3$

$$R^2 \sin^2 \frac{r}{R} \cdot 4\pi \cdot \text{area}(M) = \int_{S^3(R)} \left( \int_{F^2_y \cap M} \frac{dV_{F^2_y \cap M}}{\sqrt{1 - \xi_3 \xi_3}} \right) \ dV_S(y)$$

$$\geq C_0 \int_{S^3(R)} \left( \int_{F^2_y \cap M} k_{F^2_y \cap M}(s) \ ds \right) \ dV_S(y)$$

$$\geq 2\pi C_0 \int_{S^3(R)} n(y) \ dV_S(y) = 2\pi C_0 (F_1 + 2F_2 + 3F_3 + \cdots),$$

i.e.

$$\frac{2R^2}{C_0} \cdot \sin^2 \frac{r}{R} \cdot \text{area}(M) \geq F_1 + 2F_2 + 3F_3 + \cdots \tag{4.20}$$

On the other hand, we see easily that

$$\Omega_r := \{ x \mid x \in S^3(R), \ \text{dis}_{S^3(R)}(x, \ Omega) < r \}$$

$$= \{ y \mid n(y) > 0, \ y \in S^3(R) \} \ \text{(except a set of measure 0)}$$

and

$$V_r = \text{vol}(\Omega_r) = F_1 + 2F_2 + 3F_3 + \cdots \tag{4.21}$$

From (4.20) and (4.21), we have

$$\frac{2R^2}{C_0} \cdot \sin^2 \frac{r}{R} \cdot \text{area}(M) - V_r = F_1 + 2F_2 + 3F_3 + \cdots \geq 0,$$

and furthermore using (3.6) we obtain

$$\frac{2R^2}{C_0} \cdot \sin^2 \frac{r}{R} \cdot \text{area}(M) \geq V_r = \text{vol}(\Omega) + R \cos \frac{r}{R} \sin \frac{r}{R} \cdot \text{area}(M)$$

$$+ 2\pi R^2 \left( r - R \cos \frac{r}{R} \sin \frac{r}{R} \right) + R^2 \sin^2 \frac{r}{R} \int_M H dV_M,$$

which is equivalent to (4.19). \hspace{1cm} \text{q. e. d.}

§ 5. Proof of Theorem 2.

According to the formula (4.18), we have

$$k(s) \cos \theta = \sqrt{A_{11}^2 + B_{11}^2} + \frac{1}{R^2} \cdot \frac{1}{A_{11}^2 + B_{11}^2}$$

$$\times \left\{ B_{11} \frac{1}{R} \ cot \frac{r}{R} + A_{11} \sqrt{A_{11}^2 + B_{11}^2} - \frac{1}{R^2} \cot^2 \frac{r}{R} \right\}$$
and setting $A_{11}=A$, $B_{11}=B=u$ for simplicity we consider the following function of $u$

\[
(5.1) \quad f(u) := \sqrt{A^2 + \frac{1}{R^2} + u^2} \left\{ \frac{u}{R} \cot \frac{r}{R} + A \sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2} \right\}.
\]

Since $\Omega$ is convex, $A_{11} \geq 0$ everywhere on $M=\partial \Omega$. We shall try to find an upper bound of $f(u)$ for $u \geq 0$.

First of all, we write the right-hand side of (5.1) as

\[
f(u) = \sqrt{A^2 + \frac{1}{R^2} + u^2} \left\{ \frac{1}{R} \cot \frac{r}{R} \sqrt{A^2 + u^2} + \frac{A \sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2}}{\sqrt{A^2 + u^2}} \right\}.
\]

We can easily see that the function $\frac{\sqrt{A^2 + \frac{1}{R^2} + u^2}}{\sqrt{A^2 + u^2}}$ is decreasing, $\frac{u}{\sqrt{A^2 + u^2}}$ and $\frac{\sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2}}{\sqrt{A^2 + u^2}}$ are increasing for $u \geq 0$. Hence we have

\[
1 < \frac{\sqrt{A^2 + \frac{1}{R^2} + u^2}}{\sqrt{A^2 + u^2}} \leq \frac{\sqrt{A^2 + \frac{1}{R^2}}}{A}, \quad 0 \leq \frac{u}{\sqrt{A^2 + u^2}} < 1
\]

and

\[
\frac{\sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2}}{A} \leq \frac{\sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2}}{\sqrt{A^2 + u^2}} < 1 \quad \text{for} \quad u \geq 0.
\]

Thus, we obtain

\[
(5.2) \quad f(u) < \frac{\sqrt{A^2 + \frac{1}{R^2}}}{A} \left( \frac{1}{R} \cot \frac{r}{R} + A \right) := h(A).
\]

The function $h(A)$ of $A$ has the properties as follows:

\[
\lim_{A \to 0} h(A) = \lim_{A \to \infty} h(A) = +\infty
\]

and

\[
\frac{h'(A)}{h(A)} = \frac{1}{A \left( A + \frac{1}{R} \cot \frac{r}{R} \right) \left( A^2 + \frac{1}{R^2} \right) \left( A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} \right)},
\]
hence

\[ h'(A) < 0 \quad \text{for} \quad 0 < A < \frac{1}{R} \left( \cot \frac{r}{R} \right)^{1/3}, \]

\[ h'(A) > 0 \quad \text{for} \quad \frac{1}{R} \left( \cot \frac{r}{R} \right)^{1/3} < A. \]

Let us suppose from the convexity of \( \Omega \) that

\[ (0 < A_0 \leq A \leq A_1). \tag{5.3} \]

Setting

\[ \max(h(A_0), h(A_1)) = \frac{1}{C}, \]

we have

\[ f(u) < h(A) \leq \frac{1}{C}, \tag{5.4} \]

that is

\[ \frac{1}{\cos \theta} > Ck(s). \tag{5.5} \]

In the following, we shall show that \( C \) in (5.4) can be replaced with a more sharper constant \( C_0 \). Setting

\[
\begin{cases}
  f_1(u) := & \frac{u}{A^2 + u^2} \sqrt{A^2 + \frac{1}{R^2} + u^2}, \\
  f_2(u) := & \frac{1}{A^2 + u^2} \sqrt{A^2 + \frac{1}{R^2} + u^2} \sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2},
\end{cases}
\]

we write \( f(u) \) as

\[ f(u) = \frac{1}{R} \cot \frac{r}{R} \cdot f_1(u) + A f_2(u). \tag{5.7} \]

First of all, since we have

\[
f'_1(u) = \frac{1}{(A^2 + u^2)^{3/2}} \left[ A^2 \left( A^2 + \frac{1}{R^2} \right) + \left( A^2 - \frac{1}{R^2} \right) u^2 \right],
\]

we obtain easily the following:
i) when $A < \frac{1}{R}$, $f_1(u) \leq f_1\left(\frac{1}{2AR} \sqrt{1 + R^2A^2} = \frac{1 + R^2A^2}{2AR}\right)$ for $u \geq 0$;

ii) when $A \geq \frac{1}{R}$, $f_1(u)$ is monotone increasing and so

$$f_1(u) < \lim_{u \to +\infty} f_1(u) = 1$$ for $u \geq 0$.

Second, we have

$$f_2(u) = \frac{u \sqrt{A^2 + u^2}}{(A^2 + u^2)^{1/2}} \sqrt{A^2 + \frac{1}{R^2} + u^2} \sqrt{A^2 - \frac{1}{R^2} \cot^2 \frac{r}{R} + u^2}$$

$$\times \left\{ -\frac{A^2}{R^2} \left(1 - \cot^2 \frac{r}{R}\right) + \frac{2}{R^4} \cot^2 \frac{r}{R} - \frac{1}{R^2} \left(1 - \cot^2 \frac{r}{R}\right) u^2 \right\}.$$ 

Hence we have the following:

a) Case $0 < r \leq \frac{\pi}{4} R$, $f_2(u)$ is monotone increasing and so

$$f_2(u) < \lim_{u \to +\infty} f_2(u) = 1$$ for $u \geq 0$,

and

b) Case $\frac{\pi}{4} R < r \leq \frac{\pi}{2} R$, from the equation

$$-\frac{A^2}{R^2} \left(1 - \cot^2 \frac{r}{R}\right) + \frac{2}{R^4} \cot^2 \frac{r}{R} = 0,$$

we obtain $A = \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}}$, and so

i) when $0 < A < \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}}$, $f_3(u) \leq f_3\left(\frac{1}{R} \sqrt{\frac{2 - A^2R^4(\tan^2 \frac{r}{R} - 1)}{\tan^2 \frac{r}{R} - 1}}\right)$

$$= \frac{1}{\sin \frac{2r}{R}}$$ for $u \geq 0$, 

$$\times \left\{ -\frac{A^2}{R^2} \left(1 - \cot^2 \frac{r}{R}\right) + \frac{2}{R^4} \cot^2 \frac{r}{R} - \frac{1}{R^2} \left(1 - \cot^2 \frac{r}{R}\right) u^2 \right\}.$$ 

Hence we have the following:
ii) when $A \geq \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}}$, $f_2(u)$ is monotone decreasing and so

$$f_2(u) \leq f_2(0) = \frac{\sqrt{A^2 R^2 + 1}}{A^2 R^2} \sqrt{A^2 R^2 - \cot^2 \frac{r}{R}}$$

for $u \geq 0$.

On the other hand, we compare the separating values $\frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}}$ and $\frac{1}{R}$ for $A$ with respect to $f_2(u)$ and $f_1(u)$ respectively. We see easily that

$$\frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}} \begin{cases} > \frac{1}{R} & \text{for } \frac{\pi}{4} R < r < \frac{\pi}{3} R, \\ = \frac{1}{R} & \text{for } r = \frac{\pi}{3} R, \\ < \frac{1}{R} & \text{for } \frac{\pi}{3} R < r \leq \frac{\pi}{2} R. \end{cases}$$

From the above arguments, we define the following functions $h_i(A), i = 1, 2, 3$, as follows:

1) Case $0 < r \leq \frac{\pi}{4} R$,

$$h_1(A) := \begin{cases} \frac{1 + A^2 R^2}{2 A R^2} \cot \frac{r}{R} + A & \text{for } 0 < A < \frac{1}{R}, \\ \frac{1}{R} \cot \frac{r}{R} + A & \text{for } A \geq \frac{1}{R}. \end{cases}$$

2) Case $\frac{\pi}{4} R < r \leq \frac{\pi}{3} R$,

$$h_2(A) := \begin{cases} \frac{1 + A^2 R^2}{2 A R^2} \cot \frac{r}{R} + \frac{A}{\sin \frac{2r}{R}} & \text{for } 0 < A < \frac{1}{R}, \\ \frac{1}{R} \cot \frac{r}{R} + \frac{A}{\sin \frac{2r}{R}} & \text{for } \frac{1}{R} \leq A < \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}}. \end{cases}$$
\[
\frac{r}{R} \cot \frac{1}{R} + \frac{\sqrt{A^2 R^2 + 1} \sqrt{A^2 R^2 - \cot^2 \frac{r}{R}}}{AR^2}
\]
for \( \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}} \leq A ; \)

3) Case \( \frac{\pi}{3} R < r \leq \frac{\pi}{2} R , \)

\[
h_x(A) := \begin{cases} 
\frac{1 + A^2 R^2}{2AR^2} \cot \frac{r}{R} + \frac{A}{\sin 2 \frac{r}{R}} & \text{for } 0 < A < \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}}, \\
\frac{1 + A^2 R^2}{2AR^2} \cot \frac{r}{R} + \frac{\sqrt{A^2 R^2 + 1} \sqrt{A^2 R^2 - \cot^2 \frac{r}{R}}}{AR^2} & \text{for } \frac{\sqrt{2}}{R \sqrt{\tan^2 \frac{r}{R} - 1}} \leq A < \frac{1}{R} , \\
\frac{1}{R} \cot \frac{r}{R} + \frac{\sqrt{A^2 R^2 + 1} \sqrt{A^2 R^2 - \cot^2 \frac{r}{R}}}{AR^2} & \text{for } \frac{1}{R} \leq A . 
\end{cases}
\]

For each cases, we obtain from (5.7)

(5.8) \( f(u) \leq h_i(A) \) for \( u \geq 0 . \)

Furthermore, we can prove easily that

1) Case \( 0 < r \leq \frac{\pi}{4} R , \) \( h_1(A) \) takes its minimum value at

\[
A_1^* = \frac{1}{R \sqrt{2 \tan \frac{r}{R} + 1}} < \frac{1}{R}
\]

and it is monotone decreasing in \([0, A_1^*]\) and increasing in \([A_1^*, \infty)\); 

2) Case \( \frac{\pi}{4} R < r \leq \frac{\pi}{3} R , \) \( h_2(A) \) takes its minimum value at

\[
A_2^* = \frac{1}{R \sqrt{1 + \sec^2 \frac{r}{R}}} < \frac{1}{R}
\]

and it is monotone decreasing in \([0, A_2^*]\) and increasing in \([A_2^*, \infty)\);
3) Case $\frac{\pi}{3} R < r \leq \frac{\pi}{4} R$, $h_3(A)$ has the same property as $h_2(A)$.

Thus, making use of these functions $h_i(A), i=1, 2, 3$, for these three cases, we set

$$\max \{ h_i(A_0), h_i(A_1) \} = \frac{1}{C_0}.$$ 

Then we have

$$f(u) < h_i(A) \leq \frac{1}{C_0} \quad \text{for} \quad A_0 \leq A \leq A_1.$$ 

It is clear that this $C_0$ is more sharper than $C$ for our purpose.

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