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ON THE EXISTENCE OF A COMPLEX ALMOST CONTACT STRUCTURE

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§ 1. Introduction.

A complex manifold of complex dimension $2m+1$ is said to be a complex contact manifold if it admits an open covering $\{\tilde{O}_i\}$ such that on each $\tilde{O}_i$ there is a holomorphic 1-form γ_i with $\gamma_i\wedge(d\gamma_i)^m\neq 0$ and on $\tilde{O}_i\cap\tilde{O}_j\neq\phi$, $\gamma_i=\tilde{f}_{ij}\gamma_j$ for some non-vanishing holomorphic function $\tilde{f}_{ij}$. In general such a structure is not given by a global 1-form γ; in fact, this is the case for a complex manifold if and only if its first Chern class vanishes [7]. It is also shown in [7] that the structural group of the tangent bundle of a complex contact manifold is reducible to $(Sp(m)\otimes U(1))\times U(1)$. Standard examples of complex contact manifolds are the odd dimensional complex projective space PC^{2m+1}, the complex projective cotangent bundle of a complex manifold, etc.. (See [3], [7].)

On the other hand, some of complex contact manifolds are base spaces of principal fibre bundles with 1-dimensional fibres and real contact 3-structure. A typical example of this is a Hopf map $S^{4m+3}\longrightarrow PC^{2m+1}$. Generalizing this situation Ishihara and Konishi studied in [5] fiberings with 1-dimensional fibres of a manifold with real contact 3-structure and defined in the base space a new structure called a complex almost contact structure. In [2] an equivalent definition is given in terms of global tensor fields. The structural group of the tangent bundle of a complex almost contact manifold is also reducible to $(Sp(m)\otimes U(1))\times U(1)$. The notion of a complex almost contact structure is naturally weaker than that of a complex contact structure. In fact, a manifold with a complex contact structure admits a complex almost contact structure, and the converse is true if the complex almost contact structure is normal [5], [6].

Let M be a complex manifold of complex dimension $2m+1$. Let $\mathcal{O}=\{O_i\}$ be an open covering of M. We say, in this paper, M has a Ω-structure if the structural group of the tangent bundle of M is reducible to $(Sp(m)\otimes U(1))\times U(1)$, that is equivalent to the existence of a local 2-form Ω_i of type (2, 0) on each O_i such that $(\Omega_i)^m\neq 0$ and a non-vanishing function $f_{ij}\in U(1)$ such that $\Omega_i=f_{ij}\Omega_j$ on $O_i\cap O_j\neq\phi$.

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The existence of a real almost contact structure is discussed by Hatakeyama [4]. In this paper we generalize his result to the complex case and prove that the existence of a Ω-structure is a sufficient condition for the existence of a complex almost contact structure.

§ 2. Decomposition of $GL(n, \boldsymbol{C})$.

Let $H^{+}(n)$ denote the set of all positive definite hermitian (n, n)-matrices. Then $H^{+}(n)$ is naturally imbedded as a real analytic submanifold in the general linear group $\mathrm{GL}(n, \boldsymbol{C})$, and its analytic structure is defined by the components of its element. Let $U(n)$ denote the unitary group with usual analytic structure. Both $H^{+}(n)$ and $U(n)$ are considered to act on n-complex variables. Algebraically it is well known that any $\lambda \in \mathrm{GL}(n, \boldsymbol{C})$ is written in one and only one way as the product $\lambda=\eta\upsilon$ of $\eta \in H^{+}(n)$ and $\upsilon \in U(n)$, thus the components of η and υ of the decomposition of λ depend continuously on those of λ. In this section we check this decomposition is in fact analytic. First we prove

PROPOSITION 1. *Let Φ be a map from the product manifold $H^{+}(n) \times U(n)$ to the group* $\mathrm{GL}(n, \boldsymbol{C})$ *defined by*

$$\Phi(\eta, \upsilon)=\eta\upsilon, \quad \eta \in H^{+}(n), \quad \upsilon \in U(n).$$

Then Φ is an analytic map, and its differential map $d\Phi$ is everywhere an onto isomorphism.

In order to prove Proposition 1 we begin with the following

LEMMA 1. *Let η be an element of $H^{+}(n)$. Then the characteristic values of the linear map $ad(\eta)$: $\mathfrak{gl}(n, \boldsymbol{C}) \longrightarrow \mathfrak{gl}(n, \boldsymbol{C})$, defined by*

$$ad(\eta)L=\eta L \eta^{-1}, \quad L \in \mathfrak{gl}(n, \boldsymbol{C}),$$

are all positive numbers.

Proof. Since η belongs to $H^{+}(n)$, the characteristic values of η are all positive real numbers, which we shall denote by a_i $(i=1, \cdots, n)$. According to a classical result, there is a unitary matrix ν such that $\nu\eta\nu^{-1}$ is a diagonal matrix δ whose diagonal components are the a_i's. Since $ad(\eta)=ad(\nu)^{-1}ad(\delta)ad(\nu)$ the characteristic values of $ad(\eta)$ are equal to those of $ad(\delta)$ and hence, by direct computation, equal to the n^2 real numbers $a_i a_j^{-1}$ $(i, j=1, \cdots, n)$, which are all positive real numbers. q. e. d.

COROLLARY. *Let A be a skew hermitian (n, n)-matrix and η be an element of $H^{+}(n)$. If ηA is a hermitian matrix, then $A=0$.*

Proof. If ηA is a hermitian matrix, then

$$\eta A={}^t\overline{\eta A}={}^t\bar{A}^t\bar{\eta}=-A\eta,$$

that is $ad(\eta)A=-A$. Thus $A=0$ by Lemma 1. q. e. d.

Proof of Proposition 1. Φ is an analytic map because $H^+(n)$ and $U(n)$ are real analytic submanifolds of $\mathrm{GL}(n, \boldsymbol{C})$ and the multiplication of the elements of $\mathrm{GL}(n, \boldsymbol{C})$ is algebraic. In order to prove $d\Phi$ is everywhere an onto isomorphism we have only to prove that the differential map $d\Phi$ is univalent since the dimensions of $H^+(n)\times U(n)$ and $\mathrm{GL}(n, \boldsymbol{C})$ are both equal to $2n^2$. Let X be a tangent vector to $H^+(n)\times U(n)$ at a point (η, υ) which is tangent to a parameterized curve $(\eta+tB, \upsilon\exp tA)$, where A is skew hermitian and B is hermitian. Then $d\Phi(X)$ is given by

$$\begin{aligned} d\Phi(X)&=\lim_{t\to 0}\frac{(\eta+tB)(\upsilon\exp tA)-\eta\upsilon}{t} \\ &=\eta A\upsilon+B\upsilon. \end{aligned}$$

Assuming $d\Phi(X)=0$, we have $\eta A\upsilon+B\upsilon=0$ and therefore $\eta A=-B$, which means ηA is hermitian. Applying the preceding corollary, we have $A=0$ and so $B=0$, which shows $d\Phi$ is univalent. q. e. d.

Since $d\Phi$ is univalent, there exists an analytic inverse map of Φ (p. 80 [1]), which we denote by Ψ. Thus we have

PROPOSITION 2. *Any complex regular matrix λ can be written in one and only one way as the product $\lambda=\eta\upsilon$ of a positive definite hermitian matrix η and a unitary matrix υ. The map*

$$\Psi:\ \mathrm{GL}(n, \boldsymbol{C})\longrightarrow H^+(n)\times U(n)$$

defined by this decomposition gives an analytic homeomorphism of $\mathrm{GL}(n, \boldsymbol{C})$ *to* $H^+(n)\times U(n)$ *with respect to their usual analytic structures.*

Remark. It easily follows from Propositions 1 and 2 that any $\lambda\in\mathrm{GL}(n, \boldsymbol{C})$ can also be decomposed analytically in one and only one way as $\lambda=\upsilon'\eta'$, where $\upsilon'\in U(n)$ and $\eta'\in H^+(n)$.

§ 3. Existence Theorem.

Let $(M, F, \hat{g})$ denote a complex manifold of odd complex dimension $2m+1$ ($\geqq 3$) with complex structure F and hermitian metric $\hat{g}$ and let M be covered by a system of coordinate neighborhoods $\mathcal{O}=\{O_i\}$. We shall suppose M has a Ω-structure.

Since Ω_i is of rank $2m$, if we set

$$D_i(p)=\{X; X\in T_p(O_i),\ \tau(X)\Omega_i=0\} \qquad \text{for any} \quad p\in O_i,$$

where $\tau(X)\Omega_i$ is defined by $(\tau(X)\Omega_i)(Y)=\Omega_i(X, Y)$ for any $Y\in T_p(O_i)$, then the correspondence $p\mapsto D_i(p)$ gives a distribution D_i on each O_i of complex dimension 1 and of class C^∞. The condition $\Omega_i=f_{ij}\Omega_j$ on $O_i\cap O_j\neq\phi$, where f_{ij} is a non-vanishing function, shows that $D_i=D_j$ on $O_i\cap O_j$, that is, D_i is globally extended to a distribution D on M of complex dimension 1 and of class C^∞.

On each 0_i we can choose a field of unitary frames $\{E_{i1}, \cdots, E_{i2m}, N_i\}$ of class C^∞ with respect to $\hat{g}$ such that the last vector N_i belongs to the distribution D. The transformation of the field of such unitary frames on $O_i\cap O_j\neq\phi$ is of the form

$$T_{ij}=\left(\begin{array}{cc|cc} \tau_{ij} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & 0 \\ 0, \cdots 0, & \tau'_{ij} & & \\ \hline & 0 & \bar{\tau}_{ij} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ & & 0, \cdots 0, & \bar{\tau}'_{ij} \end{array}\right)$$

where $\tau_{ij}\in U(2m)$, $\tau'_{ij}\in U(1)$ and $\bar{\tau}_{ij}$ denotes the complex conjugate of τ_{ij}.

The matrix W_i consisting of components of the form Ω_i relative to the field of unitary frames defined on 0_i has the form

$$W_i=\left(\begin{array}{cc|cc} \omega^i & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & 0 \\ 0, \cdots 0, & 0 & & \\ \hline & 0 & \bar{\omega}_i & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ & & 0, \cdots 0, & 0 \end{array}\right)$$

where ω_i is a regular $(2m, 2m)$-skew symmetric matrix. As a consequence of Proposition 2, ω_i is written as

$$\omega_i=\xi_i\bar{\sigma}_i \tag{3.1}$$

where $\xi_i\in H^+(2m)$ and $\sigma_i\in U(2m)$, and the components of ξ_i and σ_i depend analytically on those of ω_i. So if we set

(3. 2)
$$Z_\iota=\left(\begin{array}{c|c} 0 & \begin{array}{cc} \xi_\iota & \begin{array}{c}0\\ \vdots\\ 0\end{array} \\ 0,\cdots 0, & 1\end{array} \\ \hline \begin{array}{cc} \bar{\xi}_\iota & \begin{array}{c}0\\ \vdots\\ 0\end{array} \\ 0,\cdots 0, & 1\end{array} & 0 \end{array}\right),$$

$$\Sigma_i=\left(\begin{array}{c|c} 0 & \begin{array}{cc} \sigma_\iota & \begin{array}{c}0\\ \vdots\\ 0\end{array} \\ 0,\cdots 0, & 0\end{array} \\ \hline \begin{array}{cc} \bar{\sigma}_\iota & \begin{array}{c}0\\ \vdots\\ 0\end{array} \\ 0,\cdots 0, & 0\end{array} & 0 \end{array}\right),$$

then (3. 1) is equivalent to

(3. 1)′
$$W_\iota=Z_\iota \Sigma_\iota.$$

Since ω_i is a skew symmetric matrix and $\sigma_\iota \in U(2m)$, we have

$$\xi_i \bar{\sigma}_i=-{}^t\bar{\sigma}_\iota\, {}^t\xi_i=-\sigma_i^{-1}\, {}^t\xi_\iota$$

and thus

$$\xi_i=-\sigma_i^{-1}\, {}^t\xi_i \bar{\sigma}_i^{-1}=\sigma_i^{-1}\, {}^t\xi_\iota \sigma_i\cdot(-\sigma_i^{-1}\bar{\sigma}_i^{-1}).$$

On the other hand it is easily checked

$$\sigma_i^{-1}\, {}^t\xi_\iota \sigma_i \in H^+(2m), \quad -\sigma_i^{-1}\bar{\sigma}_i^{-1} \in U(2m).$$

Therefore by the uniqueness of the decomposition we have

$$\xi_i=\sigma_i^{-1}\, {}^t\xi_\iota \sigma_\iota, \quad \text{i. e.,} \quad \sigma_i \xi_i={}^t\xi_\iota \sigma_\iota$$

and

(3. 3)
$$\sigma_\iota \bar{\sigma}_i=-I_{2m}.$$

Where I_k denote the unit (k, k)-matrix. The condition (3. 3) is equivalent to

$$\Sigma_i^2 = \left(\begin{array}{cccc|cccc} & -I_{2m} & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & & & 0 & \\ 0, & \cdots 0, & & 0 & & & & \\ \hline & & & & & -I_{2m} & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ & 0 & & & 0, & \cdots 0, & & 0 \end{array}\right) \tag{3.3$'$}$$

Next, the relation $\Omega_i = f_{ij}\Omega_j$ on $O_i \cap O_j \neq \phi$ is represented as

$$W_i = {}^tT_{ij}\,{}^*f_{ij}\,W_j\,T_{ij},$$

where

$$^*f_{ij} = \left(\begin{array}{c|c} f_{ij}I_{2m+1} & 0 \\ \hline 0 & \bar{f}_{ij}I_{2m+1} \end{array}\right)$$

or equivalently as

$$\omega_i = {}^t\tau_{ij}f_{ij}\omega_j\tau_{ij}$$

relative to the field of unitary frames defined on O_i. If we substitute (3.1) to this relation, we have

$$\xi_i\bar{\sigma}_i = {}^t\tau_{ij}f_{ij}\xi_j\bar{\sigma}_j\tau_{ij} = {}^t\tau_{ij}\xi_j\bar{\tau}_{ij}\bar{\tau}_{ij}^{-1}f_{ij}\bar{\sigma}_j\tau_{ij}.$$

Since ${}^t\tau_{ij}\xi_j\bar{\tau}_{ij} \in H^+(2m)$ and $\bar{\tau}_{ij}^{-1}f_{ij}\bar{\sigma}_j\tau_{ij} \in U(2m)$, by the uniqueness of the decomposition of this type, we have

$$\xi_i = {}^t\tau_{ij}\xi_j\bar{\tau}_{ij}, \quad \sigma_i = \tau_{ij}^{-1}\bar{f}_{ij}\sigma_j\bar{\tau}_{ij}, \tag{3.4}$$

or

$$Z_i = {}^tT_{ij}Z_jT_{ij}, \quad \Sigma_i = T_{ij}^{-1}\,{}^*\bar{f}_{ij}\Sigma_jT_{ij}. \tag{3.4$'$}$$

Thus, there is a global (0, 2)-tensor field g of class C^∞ on M defined by the set $\{Z_i\,; O_i \in \mathcal{O}\}$. (See [8].) Since each ξ_i is positive definite, g is regarded as a positive definite hermitian metric on M. Also there is a local (1, 1)-tensor field G_i of class C^∞ and of rank $2m$ defined by Σ_i on each O_i in such a way that the condition $G_i = \bar{f}_{ij}G_j$ is satisfied on $O_i \cap O_j \neq \phi$.

Now, we shall define on each O_i vector fields U_i and V_i, 1-forms u_i and v_i and (1, 1)-tensor field H_i respectively by

$$
\begin{aligned}
&U_i=N_\iota,\\
&V_i=-FN_\iota,\\
&u_i=\iota(U_\iota)g,\\
&v_i=\iota(V_\iota)g,\\
&H_i=-FG_i.
\end{aligned}
\tag{3.5}
$$

Then from (3. 2), (3. 3)′, and (3. 5) the following relations are easily varified:

$$
\begin{aligned}
&FU_i=-V_\iota, \quad FV_i=U_\iota,\\
&g(U_\iota, U_\iota)=g(V_\iota, V_\iota)=1, \quad g(U_\iota, V_\iota)=0,\\
&G_iF=-FG_i=H_\iota, \quad H_\iota F=-FH_i=-G_i,\\
&G_iU_i=G_iV_i=H_\iota U_i=H_\iota V_\iota=0,\\
&v_i\circ G_i=v_i\circ G_i=u_i\circ H_i=v_i\circ H_i=0,\\
&G_i^2=H_\iota^2=-I+u_\iota\otimes U_i+v_\iota\otimes V_\iota,\\
&H_\iota G_i=-G_iH_i=F+u_i\otimes V_i-v_i\otimes U_\iota.
\end{aligned}
\tag{3.6}
$$

Furthermore, taking account of $\tau_{\iota j}\in U(2m)$, $\tau'_{\iota j}\in U(1)$ and $f_{\iota j}\in U(1)$ on $O_\iota\cap O_j\neq\phi$, and by (3. 4), we obtain

$$
\begin{cases} u_i=au_j+bv_j,\\ v_i=-bu_j+av_j,\end{cases} \qquad \begin{cases} G_i=cG_j+dH_j,\\ H_i=-dG_j+cH_j.\end{cases}
\tag{3.7}
$$

where a, b, c, and d are real valued functions satisfying $a^2+b^2=1$, and $c^2+d^2=1$.

The preceding argument shows that the set $\{(O_\iota, u_\iota, v_\iota, G_i, H_\iota): O_i\in\mathcal{O}\}$ defines a complex almost contact structure on M. Thus we have

THEOREM. *Let $(M, F, \hat{g})$ denote a complex manifold of odd complex dimension $2m+1$ ($\geqq 3$) with complex structure F and hermitian metric $\hat{g}$. If M has a Ω-structure, then there are a hermitian metric g and a complex almost contact structure on M.*

Remark. The significance of the Theorem proved in the preceding way is that if we suppose the original hermitian metric $\hat{g}$ and 2-forms Ω_ι's are of class C^ω, then we get a hermitian metric g and (1, 1)-tensor fields G_i's of class C^ω.

It might be convenient to have a metric $\tilde{g}$ satisfying $\tilde{g}(G_iX, Y)=-\tilde{g}(X, G_iY)$ and $\tilde{g}(H_\iota X, Y)=-\tilde{g}(X, H_\iota Y)$ on each O_ι. We get such a metric from g in the Theorem in the following way.

First we define $\tilde{g}_\iota$ on each O_ι by

$$
\tilde{g}_\iota(X, Y)=g(X, Y)+\frac{1}{2}\{g(G_\iota X, G_\iota Y)+g(H_\iota X, H_\iota Y)\}.
\tag{3.8}
$$

Then by (3.7) we see $\tilde{g}_\iota(X, Y)=\tilde{g}_j(X, Y)$ on $O_\iota \cap O_j \neq \phi$. This shows that $\tilde{g}_\iota$ is extended to a global metric, which we denote by $\tilde{g}$. It is easily checked that $\tilde{g}$ satisfies the required conditions on each O_ι, namely, $\tilde{g}(G_\iota X, Y)=-\tilde{g}(X, G_\iota Y)$ and $\tilde{g}(H_\iota X, Y)=-\tilde{g}(X, H_\iota Y)$ as well as $\tilde{g}(FX, Y)=-\tilde{g}(X, FY)$. However changing metrics of this type does not affect the definitions of U_ι, V_ι, u_ι, and v_i. In fact, this is easily checked by (3.5) and (3.6).

Remark. By slightly modifying the argument in this section we have the following:

Instead of a Ω-structure, if there is on each O_ι a 2-form Ω_ι of type (2, 0) satisfying $(\Omega_\iota)^n \neq 0$ $(n \leqq m)$ and $\Omega_i = f_{\iota j}\Omega_j$, $f_{\iota j} \in U(1)$ on $O_\iota \cap O_j \neq \phi$, then there is a hermitian metric g on M such that the non-zero eigenvalues of each Ω_ι with respect to g are only ι and $-\iota$. (See (3.3)'.)

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