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ON A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION AND THE COSINE FUNCTION BY FACTORIZATION

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1. Introduction. Recently there have appeared quite many results in the theory of factorization by composition. However, so far as the present author knows, all of them are concerned with the possibility or the impossibility of factorization of certain functions. The factorization theory is still in the infancy so that almost all fundamental problems remain unsettled.

A meromorphic function $F(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be pseudo-prime if every factorization of the above form implies that $g(z)$ is a polynomial unless $f(z)$ is rational. If $F(z)$ is representable as $f_1(f_2 \cdots (f_n(z))\cdots)$ and $g_1(g_2 \cdots (g_n(z))\cdots)$ and if with suitable linear transformations λ_j, $j=1, \cdots, n-1$

$$f_1=g_1(\lambda_1),\ f_2=\lambda_1^{-1}(g_2(\lambda_2)),\ \cdots,\ f_n=\lambda_{n-1}^{-1}(g_n)$$

hold, then two factorizations are called to be equivalent.

It is well-known that e^z and $\cos z$ are both pseudo-prime and further they admit infinitely many non-equivalent left polynomial factors, that is, $e^z=w^n\circ e^{z/n}$ and $\cos z=P_n(\cos z/n)$ with a suitable polynomial P_n of degree n. This means that e^z and $\cos z$ occupy a quite special situation in the factorization theory. In this paper we shall discuss the inverse problem and prove the following characterization of the exponential function and the cosine function.

THEOREM. *Let $F(z)$ be an entire function, for which for every positive integer m there is a polynomial $P_m(z)$ of degree m such that*

$$F(z)=P_m(f_m(z))$$

for an entire function $f_m(z)$. Then

$$F(z)=A\cos\sqrt{H(z)}+B$$

with two constants A, B and an entire function $H(z)$, unless

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$$F(z)=Ae^{H(z)}+B.$$

Our proof is divided into several steps, since it needs a little bit complicated process.

2. We shall make use of the following famous theorem and its consequences repeatedly.

THEOREM A. (*The second fundamental theorem for an entire function.*) *Let $f(z)$ be an entire function. Then*

$$(q-1)m(r,f)<\sum_{1}^{q}\bar{N}(r,a_{\nu},f)+S(r),$$

where $a_{\nu}\neq\infty$ and

$$S(r)=O(\log r\, m(r,f))$$

except for a set of finite measure.

THEOREM B. *Let $f(z)$ be an entire function. Then*

$$\sum_{a\neq\infty}\Theta(a)\leqq 1,$$

where

$$1-\Theta(a)=\overline{\lim_{r\to\infty}}\frac{\bar{N}(r,a,f)}{m(r,f)}.$$

Let $\nu(a)$ denote the least order of almost all a-points of $f(z)$. Then

$$\sum_{a\neq\infty}\left(1-\frac{1}{\nu(a)}\right)\leqq 1.$$

3. Proof of Theorem. The first step. Let us start from $m=2$. Then with constants A_2, b and w_0

$$F(z)-b=A_2(f_2(z)-w_0)^2.$$

Hence $F(z)-b$ has only zeros of even positive integral order if they exist. In the first place we assume that $F(z)-b$ has only finitely many zeros. Then

$$F(z)-b=Q(z)^2e^{L(z)}$$

with a polynomial $Q(z)$ and an entire function $L(z)$. $F(z)$ is trivially transcendental by our assumption. Hence $L(z)$ is not a constant. Let m be an arbitrary large integer. Then

$$F(z)-b=A_m\prod_{j=1}^{m}(f_m(z)-w_j).$$

By Picard's theorem for entire functions we have only one representation

$$F(z)-b=A_m(f_m(z)-w_1)^m ,$$

$$f_m(z)-w_1=Q_m(z)e^{L_m(z)}$$

with a polynomial $Q_m(z)$ and an entire function $L_m(z)$. Thus

$$m \deg Q_m=2 \deg Q .$$

This is possible if and only if $\deg Q=\deg Q_m=0$. Therefore

$$F(z)=b+Ae^{L(z)} .$$

The second step. From now on we assume that $F-b$ admits infinitely many zeros of even integral order. Let us consider the case $m=3$. Then by Theorem B we have only two possibilities:

1) $F(z)-b=A_3(f_3(z)-w_1)(f_3(z)-w_2)^2,\ w_1\neq w_2$
2) $F(z)-b=A_3(f_3(z)-w_1)^3$.

In this step we consider the case 2). Then by Theorem B we have

$$F(z)-b=A_4(f_4(z)-w_1{}^*)^4 ,$$

since $F(z)-b$ has only zeros of order $6p$. We inductively assume that $F(z)-b$ has only zeros of order $3\cdot 2^{p-1}$. Then we consider

$$F(z)-b=A_{2^p}\prod_{j=1}^{s}(f_{2^p}(z)-\alpha_j)^{l_j} ,$$

$$\sum_{j=1}^{s} l_j=2^p .$$

If there are two odd integers among $\{l_j\}$, say l_1 and l_2, then $f_{2^p}(z)-\alpha_1$, $f_{2^p}(z)-\alpha_2$ have only zeros of order at least 2^{p-1}. This is impossible by Theorem B. Hence we may put $l_j=2m_j$. We may put $m_1\leqq m_2\leqq\cdots\leqq m_s$. Let n_j be the least order of zeros of $f_{2^p}-\alpha_j$. Then $2n_jm_j\geqq 3\cdot 2^{p-1}$ for all j. If $s\geqq 3$, $m_s\leqq 2^{p-1}-2$. Hence $n_s>3\cdot 2^{p-1}/2m_s\geqq 3\cdot 2^{p-1}/(2^p-4)>1$. Therefore $n_s\geqq 2$. In this case $3m_1\leqq m_1+\cdots+m_s=2^{p-1}$. Thus $n_1\geqq 3\cdot 2^{p-1}/2m_1\geqq 3^2\cdot 2^{p-1}/2^p=9/2$. Hence $n_1\geqq 5$. This is impossible by Theorem B. If $s=2$, then similarly $n_2\geqq 3\cdot 2^{p-1}/(2^p-2)>1$, which shows that $n_2\geqq 2$. Further $m_1\leqq 2^{p-2}$. Hence $n_1\geqq 3\cdot 2^{p-1}/2^{p-1}=3$. This is again absurd by Theorem B. Thus we have only one representation

$$F(z)-b=A_{2^p}(f_{2^p}(z)-\alpha_j)^{2^p} .$$

This implies that $F(z)-b$ has only zeros of order $3\cdot 2^p\cdot l$, where l is a positive integer. Thus $F(z)-b$ has only zeros of arbitrary high order. This is absurd. Thus the case 2) is now rejected.

The third step. We now discuss the case 1). Then $f_3-w_1=T^2$. Hence

$$T^3-(w_2-w_1)T=\left(\frac{A_2}{A_3}\right)^{1/2}(f_2-w_0) .$$

There are two constants C_1 and C_2 such that

$$T^3-(w_2-w_1)T+C_j=(T-\alpha_{1j})(T-\alpha_{2j})^2,\ j=1,\ 2.$$

Then

$$C_1=\sqrt{\frac{4}{27}(w_2-w_1)^3}\ ,\qquad C_2=-C_1$$

$$\alpha_{11}=-2\sqrt{\frac{1}{3}(w_2-w_1)}\ ,\qquad \alpha_{12}=-\alpha_{11}$$

$$\alpha_{21}=\sqrt{\frac{1}{3}(w_2-w_1)}\ ,\qquad \alpha_{22}=-\alpha_{21}.$$

Hence

$$\left(\frac{A_2}{A_3}\right)^{1/2}\left(f_2-w_0+\left(\frac{A_3}{A_2}\right)^{1/2}C_j\right)=(T-\alpha_{1j})(T-\alpha_{2j})^2$$

and on putting

$$x_j=w_0-\left(\frac{A_3}{A_2}\right)^{1/2}C_j$$

we have

$$(1)\qquad \left(\frac{A_2}{A_3}\right)^{1/2}(f_2-x_j)=(T-\alpha_{1j})(T-\alpha_{2j})^2,\ j=1,\ 2.$$

Let us consider the case $m=4$. Then by Theorem B we have only three cases:

i) $F(z)-b=A_4(f_4(z)-d_1)(f_4(z)-d_2)(f_4(z)-d_3)^2$,

ii) $F(z)-b=A_4(f_4(z)-d_1)(f_4(z)-d_2)^3$,

iii) $F(z)-b=A_4(f_4(z)-d_1)^2(f_4(z)-d_2)^2$.

The case $F(z)-b=A_4(f_4(z)-d_1)^4$ does not occur, since $F-b=A_3(f_3-w_1)(f_3-w_2)^2$.

In this step we shall consider the cases i) and ii). Since $F(z)-b=A_2(f_2(z)-w_0)^2$, we may put

$$f_4-d_1=T^2,\qquad f_4-d_2=S^2$$

with entire functions T and S. Therefore

$$\frac{T+S}{\sqrt{d_2-d_1}}\frac{T-S}{\sqrt{d_2-d_1}}=1$$

and hence with a suitable entire function H

$$\frac{T+S}{\sqrt{d_2-d_1}}=e^H,\qquad \frac{T-S}{\sqrt{d_2-d_1}}=e^{-H}$$

which imply

$$T^2=\frac{d_2-d_1}{4}(e^H+e^{-H})^2,$$

$$S^2=\frac{d_2-d_1}{4}(e^H-e^{-H})^2$$

Case i).

$$F-b=A_2(f_2-w_0)^2=A_4\left(\frac{d_2-d_1}{4}\right)^4(e^{2H}-e^{-2H})^2(e^{2H}+y_3+e^{-2H})^2,$$

where

$$y_3=2-\frac{4(d_3-d_1)}{d_2-d_1}\neq\pm 2.$$

Thus

$$\left(\frac{A_2}{A_4}\right)^{1/2}\left(\frac{4}{d_2-d_1}\right)^2(f_2-w_0)=(e^{2H}-e^{-2H})(e^{2H}+y_3+e^{-2H}).$$

On the other hand

$$\left(\frac{A_2}{A_3}\right)^{1/2}(f_2-x_1)=(T-\alpha_{11})(T-\alpha_{21})^2,$$

$$x_1=w_0-\left(\frac{A_3}{A_2}\right)^{1/2}C_1.$$

Hence

$$\left(\frac{A_2}{A_4}\right)^{1/2}\left(\frac{4}{d_2-d_1}\right)^2\left(f_2-w_0+\left(\frac{A_3}{A_2}\right)^{1/2}C_1\right)$$

$$=(e^{2H}-e^{-2H})(e^{2H}+y_3+e^{-2H})+\left(\frac{A_3}{A_4}\right)^{1/2}C_1\left(\frac{4}{d_2-d_1}\right)^2$$

$$=\frac{1}{x^2}(x^4+y_3x^3+dx^2-y_3x-1)$$

with $x=e^{2H}$,

$$d=\left(\frac{A_3}{A_4}\right)^{1/2}C_1\left(\frac{4}{d_2-d_1}\right)^2.$$

We now put

$$x^4+y_3x^3+dx^2-y_3x-1=\prod_{j=1}^{4}(x-\beta_j).$$

Thus

$$G\equiv\frac{1}{x^2}\prod_{j=1}^{4}(x-\beta_j)=C'(T-\alpha_{11})(T-\alpha_{21})^2.$$

If β_j are all different, then $\beta_j\neq 0$ implies

$$\frac{2}{3}m(r, G)\sim 2m(r, T)\geqq \bar{N}(r, \alpha_{11}, T)+\bar{N}(r, a_{21}, T)$$

$$=\sum_{j=1}^{4}\bar{N}(r, \beta_j, x)\sim 4m(r, x)\sim m(r, G)$$

excepting a set of finite measure, which is impossible. If $\beta_1, \beta_2, \beta_3$ are different and $\beta_3=\beta_4$, then

$$\frac{2}{3}m(r, G)\sim 2m(r, T)\geqq \bar{N}(r, \alpha_{11}, T)+\bar{N}(r, \alpha_{21}, T)$$

$$=\sum_{j=1}^{3}\bar{N}(r, \beta_j, x)\sim 3m(r, x)\sim \frac{3}{4}m(r, G)$$

excepting a set of finite measure, which is impossible. If $\beta_1\neq\beta_2$ but $\beta_2=\beta_3=\beta_4$, then

$$G=C'(T-\alpha_{11})(T-\alpha_{21})^2$$
$$=\frac{1}{x^2}(x^4+y_3x^3+dx^2-y_3x-1)=\frac{1}{x^2}(x-\beta_1)(x-\beta_2)^3 .$$

In this case

$$\begin{cases} \beta_1+3\beta_2=-y_3 \\ 3\beta_2^2+3\beta_1\beta_2=d \\ \beta_2^3+3\beta_1\beta_2^2=y_3 \\ \beta_1\beta_2^3=-1 . \end{cases}$$

In referring $d\neq 0$, $\beta_2\neq 0$ we have

$$y_3^2=-32,\ 4\beta_2^2+y_3\beta_2+1=0 .$$

If $y_3=4\sqrt{2}\,i$,

$$\beta_2=\frac{-\sqrt{2}+\sqrt{6}}{2}i,\ \frac{-\sqrt{2}-\sqrt{6}}{2}i .$$

If $y_3=-4\sqrt{2}\,i$,

$$\beta_2=\frac{\sqrt{2}+\sqrt{6}}{2}i,\ \frac{\sqrt{2}-\sqrt{6}}{2}i .$$

In these cases we have

$$\beta_1=-\frac{i}{2}(5\sqrt{2}+3\sqrt{6}),\ -\frac{i}{2}(5\sqrt{2}-3\sqrt{6}),$$

$$\frac{i}{2}(5\sqrt{2}-3\sqrt{6}),\quad \frac{i}{2}(5\sqrt{2}+3\sqrt{6})$$

respectively. Let us consider the derivative G'. Then

$$G'=\frac{4H'}{x^2}(x-\beta_2)^2\Big(x^2+\frac{\beta_2-\beta_1}{2}x-\beta_1\beta_2\Big).$$

$$=\frac{4H'}{x^2}(x-\beta_2)^2(x-\gamma_2)^2,$$

where

$$\gamma_2=-\frac{\sqrt{2}+\sqrt{6}}{2}i,\ \frac{\sqrt{6}-\sqrt{2}}{2}i,\ -\frac{\sqrt{6}-\sqrt{2}}{2}i,\ \frac{\sqrt{6}+\sqrt{2}}{2}i$$

respectively. Hence $\beta_2\neq\gamma_2$, $\beta_2\gamma_2\neq 0$. On the other hand

$$G'=C'(3T^2-(w_2-w_1))T'.$$

Since $m(r, H')=o(m(r, e^H))$, almost all zeros of $T-\alpha_{21}$, $T+\alpha_{21}$ are of even order. Thus

$$2m(r, T)\leqq\bar{N}(r, \alpha_{11}, T)+\bar{N}(r, \alpha_{21}, T)+\bar{N}(r, -\alpha_{21}, T)$$

$$\leqq\bar{N}(r, \alpha_{11}, T)+(1+o(1))m(r, T),$$

$$(1-o(1))m(r, T)\leqq\bar{N}(r, \alpha_{11}, T)\leqq m(r, T).$$

This means that $\bar{N}(r, \alpha_{11}, T)\sim N(r, \alpha_{11}, T)\sim m(r, T)$ and so almost all zeros of $T-\alpha_{11}$ are simple. Let N_m denote the counting function of multiple roots being counted multiply. Then

$$\frac{3}{4}m(r, G)\sim 3m(r, x)\sim 3N(r, \beta_2, x)$$

$$\sim N_m(r, 0, G)\sim 2N(r, \alpha_{21}, T)$$

$$\leqq 2m(r, T)\sim\frac{2}{3}m(r, G).$$

This is impossible.

If $\beta_1=\beta_2=\beta_3=\beta_4$, then

$$C'(T-\alpha_{11})(T-\alpha_{21})^2=\frac{1}{x^2}(x-\beta_1)^4.$$

This is impossible by Theorem B.

If $\beta_1=\beta_3$ and $\beta_2=\beta_4$, $\beta_1\neq\beta_2$, then

$$C'(T-\alpha_{11})(T-\alpha_{21})^2=\frac{1}{x^2}(x-\beta_1)^2(x-\beta_2)^2.$$

Therefore

$$(x^2-1)(x^2+y_3x+1)+dx^2=(x-\beta_1)^2(x-\beta_2)^2.$$

This is possible if and only if $y_3=0$, $d=\pm 2i$ and

$$\begin{cases} \beta_1=e^{\pi i/4} \\ \beta_2=-e^{\pi i/4} \end{cases} \quad \text{or} \quad \begin{cases} \beta_1=e^{-\pi i/4} \\ \beta_2=-e^{-\pi i/4} \end{cases}$$

Hence

$$\left(\frac{A_2}{A_4}\right)^{1/2}(f_2-w_0)=\left(\frac{d_2-d_1}{4}\right)^2(e^{4H}-e^{-4H}),$$

which leads us to

$$F-b=A_2(f_2-w_0)^2=A_4\left(\frac{d_2-d_1}{4}\right)^4(e^{8H}+e^{-8H}-2).$$

Let us put $8H=iK$. Then

$$F-b=A\cos K-A,$$

which gives a part of the final result.

Case ii). In this case with $x=e^{2H}$

$$G\equiv T(T^2-w_2+w_1)=a\frac{1}{x^2}(x+1)(x-1)^3.$$

Thus

$$\begin{aligned}\frac{2}{3}m(r, G)&\sim 2m(r, T)\\ &\leqq \bar{N}(r, 0, T)+\bar{N}(r, \sqrt{w_2-w_1}, T)+\bar{N}(r, -\sqrt{w_2-w_1}, T)\\ &=\bar{N}(r, 1, x)+\bar{N}(r, -1, x)\leqq 2m(r, x)\\ &\sim\frac{1}{2}m(r, G),\end{aligned}$$

which is absurd.

The fourth step. In this step we shall discuss the case iii). Hence we start from

$$F-b=A_2(f_2-w_0)^2=A_4(f_4-d_1)^2(f_4-d_2)^2.$$

This gives

$$\left(\frac{A_2}{A_4}\right)^{1/2}(f_2-w_0)=\left(f_4-\frac{d_1+d_2}{2}\right)^2-\frac{(d_1-d_2)^2}{4}$$

and

$$f_2-w_0+\frac{(d_1-d_2)^2}{4}\left(\frac{A_4}{A_2}\right)^{1/2}=\left(f_4-\frac{d_1+d_2}{2}\right)^2\left(\frac{A_4}{A_2}\right)^{1/2}.$$

Assume that

$$y_0=w_0-\frac{(d_1-d_2)^2}{4}\left(\frac{A_4}{A_2}\right)^{1/2}\neq x_j,\ j=1,\ 2.$$

Then by Theorem A and by (1)

$$(1+o(1))2m(r, f_2)\leqq\bar{N}(r, x_1, f_2)+\bar{N}(r, x_2, f_2)+\bar{N}(r, y_0, f_2)$$
$$=\bar{N}(r, \alpha_{11}, T)+\bar{N}(r, \alpha_{21}, T)+\bar{N}(r, \alpha_{12}, T)+\bar{N}(r, \alpha_{22}, T)$$
$$+\bar{N}(r, y_0, f_2).$$

Evidently

$$\bar{N}(r, y_0, f_2)\leqq\frac{1}{2}N(r, y_0, f_2)\leqq\frac{1}{2}m(r, f_2).$$

Hence

$$\left(\frac{3}{2}+o(1)\right)m(r, f_2)$$
$$\leqq\bar{N}(r, \alpha_{11}, T)+\bar{N}(r, \alpha_{21}, T)+\bar{N}(r, \alpha_{12}, T)+\bar{N}(r, \alpha_{22}, T)$$
$$\leqq 4m(r, T)\sim\frac{4}{3}m(r, f_2).$$

This is a contradiction. Hence $y_0=x_1$, unless $y_0=x_2$. It is sufficient to consider the case $y_0=x_1$. By the way we have

$$\frac{(d_1-d_2)^2}{4}\left(\frac{A_4}{A_2}\right)^{1/2}=\left(\frac{A_3}{A_2}\right)^{1/2}\sqrt{\frac{4}{27}(w_2-w_1)^3} \tag{2}$$
$$=\frac{x_2-x_1}{2}.$$

Hence we have

$$f_2-x_1=M_1^2,$$
$$f_2-x_2=LM_2^2.$$

Here L is an entire function whose zeros are all simple and satisfy $T=\alpha_{12}$. M_1, M_2 are also entire functions. Let f be the entire function satisfying $f+1=a_1(f_2-x_1)$, $f-1=a_1(f_2-x_2)$. Then

$$a_1(x_2-x_1)=2,$$
$$f+1=a_1M_1^2,$$
$$f-1=a_1LM_2^2,$$
$$f_2-w_0=\frac{1}{a_1}f.$$

Hence

$$f^2-a_1^2LM_1^2M_2^2=1.$$

Let us consider the logarithmic derivative of $f+a_1\sqrt{L}\,M_1M_2$:

$$\frac{d}{dz}\log(f+a_1\sqrt{L}M_1M_2)$$

$$=\frac{1}{f+a_1\sqrt{L}M_1M_2}\Big(f'+\frac{a_1L'}{2\sqrt{L}}M_1M_2+a_1\sqrt{L}(M_1M_2)'\Big)$$

$$=\Big(f'+\frac{a_1L'}{2\sqrt{L}}M_1M_2+a_1\sqrt{L}(M_1M_2)'\Big)(f-a_1\sqrt{L}M_1M_2)$$

$$=ff'-\frac{a_1{}^2}{2}L'(M_1M_2)^2-a^2L\,M_1M_2(M_1M_2)'$$

$$+a_1\sqrt{L}(f(M_1M_2)'-f'M_1M_2)+\frac{a_1L'}{2\sqrt{L}}M_1M_2f$$

$$=a_1\sqrt{L}(f(M_1M_2)'-f'M_1M_2)+\frac{a_1L'}{2\sqrt{L}}M_1M_2f\,.$$

Let $\{\alpha_j\}$ be the set of zeros of L. Then $f(\alpha_1)-a_1\sqrt{L(\alpha_1)}\times M_1(\alpha_1)M_2(\alpha_1)=1$. Hence

$$\log(f(z)+a_1\sqrt{L(z)}M_1(z)M_2(z))$$

$$=a_1\int_{\alpha_1}^{z}\frac{L'}{2\sqrt{L}}M_1M_2f\,dz+a_1\int_{\alpha_1}^{z}\sqrt{L}(f(M_1M_2)'-f'M_1M_2)dz$$

$$=a_1M_1M_2f\sqrt{L}\,\Big|_{\alpha_1}^{z}-2a_1\int_{\alpha_1}^{z}\sqrt{L}M_1M_2f'dz$$

$$=a_1\sqrt{L(z)}M_1(z)M_2(z)f(z)-2a_1\int_{\alpha_1}^{z}\sqrt{L}M_1M_2f'dz\,.$$

Of course this depends on paths of integration connecting with α_1 to z. Evidently with an integer p_j depending on paths connecting with α_1 to α_j

$$2p_j\pi i=\log(f(\alpha_j)+a_1\sqrt{L(\alpha_j)}M_1(\alpha_j)M_2(\alpha_j))$$

$$=-2a_1\int_{\alpha_1}^{\alpha_j}\sqrt{L}M_1M_2f'dz\,.$$

Let $\Theta(z)$ be

$$\frac{1}{i}a_1\sqrt{L(z)}M_1(z)M_2(z)f(z)-2a_1\frac{1}{i}\int_{\alpha_1}^{z}\sqrt{L}M_1M_2f'dz\,.$$

Then

$$f+a_1\sqrt{L}M_1M_2=e^{i\Theta}\,.$$

Further with the same path of integration

$$f-a_1\sqrt{L}M_1M_2=e^{-i\Theta}\,.$$

Hence

$$f(z)=\cos\Theta .$$

Returning back to f_2, we have

$$f_2-w_0=\frac{x_2-x_1}{2}\cos\Theta , \tag{3}$$

$$F(z)-b=A_2\frac{(x_2-x_1)^2}{8}(\cos 2\Theta+1) .$$

If it is possible to prove

$$\int_{\alpha_1}^{\alpha_j}\sqrt{L}\,M_1M_2f'dz=0$$

for all j, then

$$\int_{\alpha_1}^{z}\sqrt{L}\,M_1M_2f'dz=\sqrt{L}\,S$$

with an entire function S. This can be proved by the standard method. Then

$$\frac{\Theta}{\sqrt{L}}=K$$

reduces to an entire function. Therefore

$$F-b=A_2\frac{(x_2-x_1)^2}{8}(\cos 2\sqrt{L}\,K+1) .$$

which gives the desired result. Thus it is sufficient to prove that

$$\int_{\alpha_1}^{\alpha_j}\sqrt{L}\,M_1M_2f'dz$$

with a fixed path of integration connecting with α_1 to α_j vanishes for every j. This will be done by making a detour.

The fifth step. In order to go further we need the following:

LEMMA. *Let f, g and h be entire functions satisfying*

$$f(z)^2=a(g(z)-\gamma_1)^2(g(z)-\gamma_2)^2=P_m(h(z))$$

with $\gamma_1\neq\gamma_2$ and a polynomial P_m of degree m. Assume that f has infinitely many zeros and that m is an even integer $2n$ ($n\geqq 3$). Then either

$$f(z)^2=A\prod_{j=1}^{s}(h(z)-\alpha_j)^{2\nu_j}$$

with integers ν_j satisfying $\sum_{j=1}^{s}\nu_j=n$ or

$$f(z)^2=A\cos K-A$$

with a constant A and an entire function K.

Proof. Suppose that the former does not hold. Then by Theorem B

$$f(z)^2=A(h(z)-\alpha_1)^{\nu_1}(h(z)-\alpha_2)^{\nu_2}\prod_{j=3}^{s}(h(z)-\alpha_j)^{2\nu_j}$$

with odd integers ν_1, ν_2 and integers $\nu_j(3\leqq j\leqq s)$ such that

$$\nu_1+\nu_2+\sum_{j=3}^{s}2\nu_j=2n$$

and with different α_j. In the first place we shall prove $s\geqq 3$. If this is not the case, then

$$a(g-\gamma_1)^2(g-\gamma_2)^2=A(h-\alpha_1)^{\nu_1}(h-\alpha_2)^{\nu_2}$$

with odd integers ν_1 and ν_2 satisfying $\nu_1+\nu_2=2n$. In this case

$$\bar{N}(r, \gamma_1, g)+\bar{N}(r, \gamma_2, g)=\bar{N}(r, \alpha_1, h)+\bar{N}(r, \alpha_2, h)$$

and

$$\bar{N}(r, \alpha_j, h)\leqq\frac{1}{2}N(r, \alpha_j, h)\leqq\frac{1}{2}m(r, h)$$

and further $4m(r, g)\sim 2n\, m(r, h)$. Hence

$$m(r, g)\leqq m(r, h)\sim\frac{2}{n}m(r, g)\,,$$

which implies $n\leqq 2$. This contradicts $n\geqq 3$. Thus $s\geqq 3$. We may assume that $\nu_1\leqq\nu_2$. Let us put

$$h-\alpha_1=T^2\,,\qquad h-\alpha_2=S^2$$

and

$$\frac{T+S}{\sqrt{\alpha_2-\alpha_1}}=e^H\,,\qquad\frac{T-S}{\sqrt{\alpha_2-\alpha_1}}=e^{-H}\,.$$

Then with $y_j\neq\pm 2$

$$f^2=A\left(\frac{\alpha_2-\alpha_1}{4}\right)^{2n}(e^H+e^{-H})^{2\nu_1}(e^H-e^{-H})^{2\nu_2}\prod_{j=3}^{s}(e^{2H}+y_j+e^{-2H})^{2\nu_j}$$

Hence

$$f=A^{1/2}\left(\frac{\alpha_2-\alpha_1}{4}\right)^{n}(e^{2H}-e^{-2H})^{\nu_1}(e^{2H}-2+e^{-2H})^{\frac{\nu_2-\nu_1}{2}}$$

$$\prod_{j=3}^{s}(e^{2H}+y_j+e^{-2H})^{\nu_j}\,.$$

We put $x=e^{2H}$. Then

$$b(g-\gamma_1)(g-\gamma_2)=\left(x-\frac{1}{x}\right)^{\nu_1}\left(x-2+\frac{1}{x}\right)^{\frac{\nu_2-\nu_1}{2}}\prod_{j=3}^{s}\left(x+y_j+\frac{1}{x}\right)^{\nu_j}.$$

Hence with $C=b(\gamma_1-\gamma_2)^2/4\neq 0$

$$b\left(g-\frac{\gamma_1+\gamma_2}{2}\right)^2=\left(x-\frac{1}{x}\right)^{\nu_1}\left(x-2+\frac{1}{x}\right)^{\frac{\nu_2-\nu_1}{2}}\prod_{j=3}^{s}\left(x+y_j+\frac{1}{x}\right)^{\nu_j}+c$$

$$\equiv\frac{1}{x^n}(L(x)+cx^n).$$

Let $L(x)+cx^n$ be $\prod_{l=1}^{t}(x-\beta_l)^{\mu_l}$, $\sum_{l=1}^{t}\mu_l=2n$. Here $L(x)+cx^n$ has only zeros of even order. If μ_l is odd, then $(x-\beta_l)^{\mu_l}$ has infinitely many zeros of order μ_l, since $x-\beta_l$ has infinitely many simple zeros for $\beta_l\neq 0$. Hence we may put $\mu_l=2m_l$ for all l. We put

$$\prod_{l=1}^{t}(x-\beta_l)^{2m_l}=(x^n-u_{n-1}x^{n-1}+\cdots\pm u_0)^2.$$

Evidently $x^m L(1/x)=-L(x)$. Hence we may put

$$L(x)+cx^n=x^{2n}+a_{2n-1}x^{2n-1}+\cdots+a_{n+1}x^{n+1}+cx^n$$

$$-a_{n+1}x^{n-1}-\cdots-a_{2n-1}x-1.$$

Comparing with the corresponding coefficients we have firstly $u_0{}^2=-1$. If $u_0=i$, then $u_{n-j}=-iu_j$ $1\leqq j\leqq n-1$ and hence $u_j=0$ for $1\leqq j\leqq n-1$. If $u_0=-i$, then $u_{n-j}=iu_j$ $1\leqq j\leqq n-1$ and hence $u_j=0$ for $1\leqq j\leqq n-1$. Hence in each cases $u_j=0$ for $1\leqq j\leqq n-1$ and

$$C=\pm 2i.$$

Hence

$$\frac{L(x)+cx^n}{x^n}=\frac{(x^n\pm i)^2}{x^n}=x^n\pm 2i-\frac{1}{x^n}.$$

Therefore

$$b(g-\gamma_1)(g-\gamma_2)=x^n-\frac{1}{x^n}$$

and

$$f^2=A\left(\frac{\alpha_2-\alpha_1}{4}\right)^{2n}\left(x^{2n}-2+\frac{1}{x^{2n}}\right).$$

Hence putting $4nH=iK$ we have

$$f^2=2A\left(\frac{\alpha_2-\alpha_1}{4}\right)^{2n}(\cos K-1)$$

This is the desired result.

The sixth step. We shall denote our original problem by P for F. In the second and third steps we have proved from P for F that

$$F-b=A_2(f_2-w_0)^2=A_3(f_3-w_1)(f_3-w_2)^2 \qquad (w_1\neq w_2)$$
$$=A_4(f_4-d_1)^2(f_4-d_2)^2 \qquad (d_1\neq d_2)$$

excepting $F-b=A\cos K-A$ with a constant A and an entire function K. By Lemma

$$F-b=A_{2n}\prod_{j=1}^{s_{2n}}(f_{2n}-\beta_j)^{2\nu_j}, \qquad \sum_{j=1}^{s_{2n}}\nu_j=n$$

excepting the case $F-b=A\cos K-A$. This gives

$$f_2-w_0=\left(\frac{A_4}{A_2}\right)^{1/2}(f_4-d_1)(f_4-d_2)$$
$$=\left(\frac{A_{2n}}{A_2}\right)^{1/2}\prod_{j=1}^{s_{2n}}(f_{2n}-\beta_j)^{\nu_j}.$$

This is nothing but P for f_2. Hence we correspondingly have

$$f_2-x_1=\left(\frac{A_4}{A_2}\right)^{1/2}\left(f_4-\frac{d_1+d_2}{2}\right)^2=\left(\frac{A_6}{A_2}\right)^{1/2}(f_6-w_1')(f_6-w_2')^2$$
$$=\left(\frac{A_8}{A_2}\right)^{1/2}(f_8-d_1')^2(f_8-d_2')^2$$

with $w_1'\neq w_2'$, $d_1'\neq d_2'$ except for $f_2-x_1=A(e^H+e^{-H}-2)$, which comes from

$$f_2-x_1=\left(\frac{A_8}{A_2}\right)^{1/2}(f_8-d_1')(f_8-d_2')(f_8-d_3')^2,$$
$$A=\left(\frac{A_8}{A_2}\right)^{1/2}\left(\frac{d_2'-d_1'}{4}\right)^4$$

and

$$f_8-d_1'=\frac{d_2'-d_1'}{4}(e^{H/8}+e^{-H/8})^2.$$

We shall now discuss this exceptional case. In this case

$$\left(\frac{A_4}{A_2}\right)^{1/2}(f_4-d_1)(f_4-d_2)$$
$$=\left(\frac{A_4}{A_2}\right)^{1/2}\left(f_4-\frac{d_1+d_2}{2}\right)^2-\left(\frac{A_4}{A_2}\right)^{1/2}\frac{(d_1-d_2)^2}{4}$$
$$=f_2-w_0=\left(\frac{A_8}{A_2}\right)^{1/2}\left(\frac{d_2'-d_1'}{4}\right)^4(e^H+e^{-H}-2)-\left(\frac{A_4}{A_2}\right)^{1/2}\frac{(d_1-d_3)^2}{4}.$$

Let us put

$$D=\left(\frac{A_4}{A_8}\right)^{1/2}\frac{(d_1-d_3)^2}{4}\frac{4^4}{(d_2'-d_1')^4}\neq 0\,.$$

Then

$$A_2(f_2-w_0)^2=A_8\left(\frac{d_2'-d_1'}{4}\right)^8(e^H+e^{-H}-2-D)^2$$

$$=A_3(f_3-w_1)(f_3-w_2)^2\,.$$

Our present problem is when the above relation is possible. Let us put $f_3-w_1=T^2$. Then

$$T^3-b^2T-M(2+D-e^H-e^{-H})=0\,,$$

$$b^2=w_2-w_1,\ M=\left(\frac{A_8}{A_3}\right)^{1/2}\left(\frac{d_2'-d_1'}{4}\right)^4.$$

Let us consider the cubic equation

$$y^3-b^2y-M(2+D-e^H-e^{-H})=0\,.$$

This should have an entire solution T, which means that the above cubic equation is reducible in the ring of entire functions. Hence the discriminant Δ of the above cubic form satisfies

$$\Delta=-4b^6+27M^2(2+D-e^H-e^{-H})^2$$

$$=(3T^2-3b^2)(3T^2-b^2)^2\,.$$

Evidently

$$\Delta=27M^2(e^H+e^{-H}-\alpha_1)(e^H+e^{-H}-\alpha_2)\,,$$

$$\alpha_1=2+D+\sqrt{C}\,,\qquad \alpha_2=2+D-\sqrt{C}\,,$$

$$C=\frac{4}{27}\frac{b^6}{M^2}\neq 0\,.$$

It is not difficult to prove that

$$\bar{N}(r,\ \alpha_1,\ e^H+e^{-H})\sim N(r,\ \alpha_1,\ e^H+e^{-H})\sim m(r,\ e^H+e^{-H})$$

if $\alpha_1\neq\pm 2$ and

$$\bar{N}(r,\ \alpha_1,\ e^H+e^{-H})\sim\frac{1}{2}N(r,\ \alpha_1,\ e^H+e^{-H})\sim\frac{1}{2}m(r,\ e^H+e^{-H})$$

if $\alpha_1=\pm 2$. Now assume that $\alpha_1\neq\pm 2$, $\alpha_2\neq\pm 2$. Then

$$2m(r,\ e^H+e^{-H})\sim\bar{N}(r,\ \alpha_1,\ e^H+e^{-H})+\bar{N}(r,\ \alpha_2,\ e^H+e^{-H})$$

$$=\bar{N}(r,\ 4b^2,\ 3T^2)+\bar{N}(r,\ b^2,\ 3T^2)$$

$$\leqq 2m(r,\ 3T^2) \sim \frac{4}{3} m(r,\ e^H + e^{-H}) .$$

This is impossible. If $\alpha_1=2$, $\alpha_2 \neq -2$, then

$$\frac{3}{2} m(r,\ e^H+e^{-H}) \sim \bar{N}(r,\ \alpha_1,\ e^H+e^{-H}) + \bar{N}(r,\ \alpha_2,\ e^H+e^{-H})$$

$$= \bar{N}(r,\ 4b^2,\ 3T^2) + \bar{N}(r,\ b^2,\ 3T^2)$$

$$\leqq 2m(r,\ 3T^2) \sim \frac{4}{3} m(r,\ e^H+e^{-H}) .$$

Again this is impossible. Hence we have either

$$\begin{cases} \alpha_1 = 2 \\ \alpha_2 = -2 \end{cases} \quad \text{or} \quad \begin{cases} \alpha_1 = -2 \\ \alpha_2 = 2 . \end{cases}$$

In these cases we have $2+D=0$, $C=4$. Hence

$$F-b = A_8 \left(\frac{d_2'-d_1'}{4} \right)^8 (e^{2H}+e^{2H}+2)$$

$$= A \cos K + A\ ,\ 2H = iK .$$

This is again a part of the desired result.

Further by Lemma

$$f_2 - x_1 = \left(\frac{A_{4n}}{A_2} \right)^{1/2} \prod_{j=1}^{s_{4n}} (f_{4n} - \beta_j')^{2\nu_j'}$$

$$\sum_{j=1}^{s_{4n}} \nu_j' = n$$

except for the case $f_2 - x_1 = A(e^H+e^{-H}-2)$. This exceptional case gives again $F-b=A'(\cos K+1)$ similarly. Thus we have

$$f_4 - \frac{d_1+d_2}{2} = \left(\frac{A_8}{A_4} \right)^{1/4} (f_8 - d_1')(f_8 - d_2')$$

$$= \left(\frac{A_{4n}}{A_4} \right)^{1/4} \prod_{j=1}^{s_{4n}} (f_{4n} - \beta_j')^{\nu_j'},\ \sum_{j=1}^{s_{4n}} \nu_j' = n .$$

Still there are exceptional cases of the same type. This indicates just P for f_4. Therefore

$$f_4 - x_1' = \left(\frac{A_8}{A_4} \right)^{1/4} \left(f_8 - \frac{d_1'+d_2'}{2} \right)^2$$

$$= \left(\frac{A_{12}}{A_4} \right)^{1/4} (f_{12} - w_1'')(f_{12} - w_2'')^2$$

$$=\left(\frac{A_{16}}{A_4}\right)^{1/4}(f_{16}-d_1'')^2(f_{16}-d_2'')^2$$

with $w_1''\neq w_2''$, $d_1''\neq d_2''$ with the exception of the case

$$f_2-x_1=A(e^{2H}+e^{2H}+2)\,.$$

In this exceptional case comparing with $A_3(f_3-w_1)(f_3-w_2)^2$ we have

$$F-b=A'(\cos K+1)$$

similarly. Further with the same exceptional case

$$f_4-x_1'=\left(\frac{A_{8n}}{A_4}\right)^{1/4}\prod_{j=1}^{s_{8n}}(f_{8n}-\beta_j'')^{2\nu_j''}\,,$$

$$\sum_{j=1}^{s_{8n}}\nu_j''=n\,.$$

Hence

$$f_8-\frac{d_1'+d_2'}{2}=\left(\frac{A_{16}}{A_8}\right)^{1/8}(f_{16}-d_1'')(f_{16}-d_2'')$$

$$=\left(\frac{A_{8n}}{A_8}\right)^{1/8}\prod_{j=1}^{s_{8n}}(f_{8n}-\beta_j'')^{\nu_j''},\ \sum_{j=1}^{s_{8n}}\nu_j''=n\,.$$

This is just P for f_8. We evidently can repeat this process ad infinitum.

The seventh step. Especially we have P for f_{2^p}, that is,

$$f_{2^p}-\frac{d_1^{(p-2)}+d_2^{(p-2)}}{2}=\left(\frac{A_{2 2^p}}{A_{2^p}}\right)^{1/2^p}(f_{2^{p+1}}-d_1^{(p-1)})$$

$$\times(f_{2^{p+1}}-d_2^{(p-1)})=\left(\frac{A_{n2^p}}{A_{2^p}}\right)^{1/2^p}\prod_{j=1}^{s_{n2^p}}(f_{n2^p}-\beta_j^{(p-1)})^{\nu_j^{(p-1)}}$$

with $d_1^{(p-1)}\neq d_2^{(p-1)}$ and

$$\sum_{j-1}^{s_{n2^p}}\nu_j^{(p-1)}=n\,.$$

Then

$$f_{2^p}-x_1^{(p-1)}=\left(\frac{A_{2 2^p}}{A_{2^p}}\right)^{1/2^p}\left(f_{2^{p+1}}-\frac{d_1^{(p-1)}+d_2^{(p-1)}}{2}\right)^2$$

$$=\left(\frac{A_{3 2^p}}{A_{2^p}}\right)^{1/2^p}(f_{3 2^p}-w_1^{(p)})(f_{3 2^p}-w_2^{(p)})^2$$

$$=\left(\frac{A_{4 2^p}}{A_{2^p}}\right)^{1/2^p}(f_{4 2^p}-d_1^{(p)})^2(f_{4 2^p}-d_2^{(p)})^2\,.$$

Exceptional cases give $F-b=A\cos K+A$ finally. So we may omit them.

In the final step we need a relation between $d_1^{(p-1)}-d_2^{(p-1)}$ and $d_1^{(p)}-d_2^{(p)}$. For f_{2p-1} we have the similar relation. Especially

$$\left(\frac{A_{22p-1}}{A_{2p-1}}\right)^{1/2p-1}\left(f_{2p}-\frac{d_1^{(p-2)}+d_2^{(p-2)}}{2}\right)^2$$

$$=\left(\frac{A_{32p-1}}{A_{2p-1}}\right)^{1/2p-1}(f_{32p-1}-w_1^{(p-1)})(f_{32p-1}-w_2^{(p-1)})^2 .$$

Hence by putting $T=f_{32p-1}-w_1^{(p-1)}$ we have

$$f_{2p}-\frac{d_1^{(p-2)}+d_2^{(p-1)}}{2}=\left(\frac{A_{32p-1}}{A_{2p}}\right)^{1/2p}T(T^2-w_2^{(p-1)}+w_1^{(p-1)}) .$$

Hence

$$f_{2p}-x_1^{(p-1)}=\left(\frac{A_{32p-1}}{A_{2p}}\right)^{1/2p}T(T^2-w_2^{(p-1)}+w_1^{(p-1)})$$

$$+\left(\frac{A_{22p}}{A_{2p}}\right)^{1/2p}\frac{(d_1^{(p-2)}-d_2^{(p-2)})^2}{4} .$$

This gives

$$f_{2p}-x_1^{(p-1)}=\left(\frac{A_{32p-1}}{A_{2p}}\right)^{1/2p}(T-\alpha_1)(T-\alpha_2)^2 .$$

Then we have

$$(4)\qquad \begin{cases} \alpha_1+2\alpha_2=0 \\ \alpha_2^2+2\alpha_1\alpha_2=-w_2^{(p-1)}+w_1^{(p-1)} \\ \alpha_1\alpha_2^2=-\dfrac{(d_1^{(p-2)}-d_2^{(p-2)})^2}{4}\left(\dfrac{A_{22p}}{A_{32p-1}}\right)^{1/2p} \end{cases}$$

Further we have

$$\left(\frac{A_{32p-1}}{A_{2p}}\right)^{1/2p}(T-\alpha_1)(T-\alpha_2)^2$$

$$=\left(\frac{A_{32p}}{A_{2p}}\right)^{1/2p}(f_{32p}-w_1^{(p)})(f_{32p}-w_2^{(p)})^2$$

$$=\left(\frac{A_{42p}}{A_{2p}}\right)^{1/2p}(f_{42p}-d_1^{(p)})^2(f_{42p}-d_2^{(p)})^2$$

Hence we can put $T-\alpha_1=U^2$, $f_{32p}-w_1^{(p)}=V^2$. Then

$$U(U^2-\alpha_2+\alpha_1)=\left(\frac{A_{32p}}{A_{32p-1}}\right)^{1/2p+1}V(V^2-w_2^{(p)}+w_1^{(p)})$$

$$=\left(\frac{A_{42p}}{A_{32p}}\right)^{1/2p+1}(f_{42p}-d_1^{(p)})(f_{42p}-d_2^{(p)})\,.$$

Let c be

$$\left(\frac{A_{42p}}{A_{32p}}\right)^{1/2p+1}\frac{(d_1^{(p)}-d_2^{(p)})^2}{4}\,.$$

Then

$$\left(\frac{A_{42p}}{A_{32p}}\right)^{1/2p+1}\left(f_{42p}-\frac{d_1^{(p)}+d_2^{(p)}}{2}\right)^2$$

$$=U(U^2-\alpha_2+\alpha_1)+c$$

$$=\left(\frac{A_{32p}}{A_{32p-1}}\right)^{1/2p+1}V(V^2-w_2^{(p)}+w_1^{(p)})+c\,.$$

This implies that

$$U(U^2-\alpha_2+\alpha_1)+c=(U-\delta_1)(U-\delta_2)^2$$

and

$$\left(\frac{A_{32p}}{A_{32p-1}}\right)^{1/2p+1}V(V^2-w_2^{(p)}+w_1^{(p)})+c$$

$$=\left(\frac{A_{32p}}{A_{32p-1}}\right)^{1/2p+1}(V-\varepsilon_1)(V-\varepsilon_2)^2\,.$$

Hence

$$\text{(5)}\qquad\left\{\begin{array}{l}\delta_1+2\delta_2=0\\ \delta_2^2+2\delta_1\delta_2=-\alpha_2+\alpha_1\\ \delta_1\delta_2^2=-c\end{array}\right.$$

and

$$\text{(6)}\qquad\left\{\begin{array}{l}\varepsilon_1+2\varepsilon_2=0\\ \varepsilon_2^2+2\varepsilon_1\varepsilon_2=-w_2^{(p)}+w_1^{(p)}\\ \varepsilon_1\varepsilon_2^2=-c\left(\dfrac{A_{32p-1}}{A_{32p}}\right)^{1/2p+1}.\end{array}\right.$$

By (4), (5) and (6)

$$c=\mp2\sqrt{\frac{(\alpha_2-\alpha_1)^3}{27}}\,,\quad c=\mp2\sqrt{\frac{1}{27}(w_2^{(p)}-w_1^{(p)})^3\left(\frac{A_{32p}}{A_{32p-1}}\right)^{1/2p}}$$

and

$$w_2^{(p-1)}-w_1^{(p-1)}=\frac{1}{3}(\alpha_2-\alpha_1)^2\,.$$

Hence

$$3A_{32p-1}^{1/32^{p-1}}(w_2^{(p-1)}-w_1^{(p-1)})=A_{32p}^{1/32^{p-1}}(w_2^{(p)}-w_1^{(p)})^2. \tag{7}$$

By the same way as in the proof of (2) we have

$$\frac{(d_1^{(p-1)}-d_2^{(p-1)})^2}{4}=\sqrt{\frac{4}{27}(w_2^{(p-1)}-w_1^{(p-1)})^3}\left(\frac{A_{32p-1}}{A_{42p-1}}\right)^{1/2p}$$

and

$$\left(\frac{d_1^{(p)}-d_2^{(p)}}{2}\right)^4\left(\frac{A_{42p}}{A_{32p}}\right)^{1/2p}=\frac{4}{27}(w_2^{(p)}-w_1^{(p)})^3.$$

Hence by (7) we have

$$\frac{(d_1^{(p-1)}-d_2^{(p-1)})^2}{2}=\left(\frac{A_{42p}}{A_{42p-1}}\right)^{1/2p}\left(\frac{d_1^{(p)}-d_2^{(p)}}{2}\right)^4. \tag{8}$$

The eighth step. Let C be a cycle in $\{|z|<\infty\}-\{\alpha_j\}$, which bounds only two points α_1, α_k of $\{\alpha_j\}$. Then by a suitable choice of path of integration

$$a_2\int_C \sqrt{L}\, M_1M_2f'dz=-2a_2\int_{\alpha_1}^{\alpha_k}\sqrt{L}\, M_1M_2f'dz$$
$$=2p_k\pi\imath.$$

Let us consider

$$f_2-x_1=\left(\frac{A_4}{A_2}\right)^{1/2}\left(f_4-\frac{d_1+d_2}{2}\right)^2.$$

This is the same as

$$f_2-w_0=\left(\frac{A_4}{A_2}\right)^{1/2}\frac{(d_1-d_2)^2}{4}[2f_4^{*2}-1],$$

$$f_4^*=\frac{\sqrt{2}}{d_1-d_2}\left(f_4-\frac{d_1+d_2}{2}\right).$$

By remembering

$$\frac{x_2-x_1}{2}=\frac{(d_1-d_2)^2}{4}\left(\frac{A_4}{A_2}\right)^{1/2}$$

and

$$f_2-w_0=\frac{x_2-x_1}{2}\cos\Theta,$$

we have

$$\cos\Theta=2f_4^{*2}-1.$$

Hence

$$f_4^*=\cos\frac{\Theta+2\pi j}{2},\ j=0,\ 1.$$

Inductively we assume that we have proved

$$f_{2^p}^*=\cos\frac{\Theta+2\pi j}{2^{p-1}},\ j=0,\ 1,\cdots,\ 2^{p-1}-1,$$

$$f_{2^p}^*=\frac{\sqrt{2}}{d_1^{(p-2)}-d_2^{(p-2)}}\Big(f_{2^p}-\frac{d_1^{(p-2)}+d_2^{(p-2)}}{2}\Big).$$

Then by

$$f_{2^p}-x_1^{(p-1)}=\Big(\frac{A_{22^p}}{A_{2^p}}\Big)^{1/2^p}\Big(f_{2^{p+1}}-\frac{d_1^{(p-1)}+d_2^{(p-1)}}{2}\Big)^2$$

we have

$$f_{2^p}-\frac{d_1^{(p-2)}+d_2^{(p-2)}}{2}$$

$$=\Big(\frac{A_{22^p}}{A_{2^p}}\Big)^{1/2^p}\frac{(d_1^{(p-1)}-d_2^{(p-1)})^2}{4}(2f_{22^p}^{*2}-1),$$

$$f_{22^p}^*=f_{2^{p+1}}^*=\frac{\sqrt{2}}{d_1^{(p-1)}-d_2^{(p-1)}}$$

$$\times\Big(f_{2^{p+1}}-\frac{d_1^{(p-1)}+d_2^{(p-1)}}{2}\Big).$$

Hence

$$\frac{d_1^{(p-2)}-d_2^{(p-2)}}{\sqrt{2}}\cos\frac{\Theta+2\pi j}{2^{p-1}}$$

$$=\Big(\frac{A_{2^{p+1}}}{A_{2^p}}\Big)^{1/2^p}\frac{(d_1^{(p-1)}-d_2^{(p-1)})^2}{4}(2f_{2^{p+1}}^{*2}-1)$$

and

$$\frac{d_1^{(p-2)}-d_2^{(p-2)}}{\sqrt{2}}\Big(2\cos^2\frac{\Theta+2\pi j}{2^p}-1\Big)$$

$$=\Big(\frac{A_{2^{p+1}}}{A_{2^p}}\Big)^{1/2^p}\frac{(d_1^{(p-1)}-d_2^{(p-1)})^2}{4}(2f_{2^{p+1}}^{*2}-1).$$

Hence by (8) with $p-1$ instead of p we have

$$f_{2^{p+1}}^*=\cos\frac{\Theta+2\pi j}{2^p},\ j=0,\ 1,\cdots,\ 2^p-1.$$

Let us consider the period of $(\Theta+2\pi j)/2^p$ along the cycle C. Then this is just $4p_k\pi/2^p$. Assume that $p_k\neq 0$. Then $f_{2^{p+1}}^*$ is not one-valued along this cycle

C, since $(\Theta+2\pi j)/2^p$ has an arbitrary small period along C. This contradicts the one-valuedness of f_{2p+1}. Therefore

$$\int_{\alpha_1}^{\alpha_k} \sqrt{L}\, M_1 M_2 f' dz=0$$

for every k and for every path of integration. This completes the proof of Theorem.

4. There exists the following conjecture: Let $F(z)$ be an entire function, for which there is an infinite sequence of different integers $\{m_j\}$ such that $F(z)=P_{m_j}(f_{m_j}(z))$ with a polynomial P_{m_j} of degree m_j and an entire function f_{m_j}. Then either

$$F(z)=P(Ae^{H(z)}+B)$$

or

$$F(z)=P(A\cos\sqrt{H(z)}+B)$$

with constants A, B and an entire function $H(z)$ and a suitable polynomial P.

Amemiya has constructed counter-examples showing that this conjecture is not true in general. His example shows that, if $\{m_j\}$ is equal to $\{2^j\}$, this conjecture is not true. We have made use of only $\{2^j, 3\cdot 2^j\}_{j=0}^{\infty}$ and obtained the result.

5. We shall now discuss the case that $\{2^j, 3\cdot 2^j\}_{j=1}^{\infty}$ is given as $\{m_j\}$. In this case firstly we have

$$F-b=A_2(f_2-w_0)^2.$$

Let us consider

$$F-b=A_4\prod_{j=1}^{4}(f_4-d_j).$$

In this case we only have the following possibilities:

1) $A_4(f_4-d_1)(f_4-d_2)(f_4-d_3)^2$

2) $A_4(f_4-d_1)(f_4-d_2)^3$

3) $A_4(f_4-d_1)^2(f_4-d_2)^2$

4) $A_4(f_4-d_1)^4$.

Case 1). In this case we may put

$$f_4-d_1=T^2, \qquad f_4-d_2=s^2,$$

$$\frac{T+S}{\sqrt{d_2-d_1}}=e^H$$

and $x=e^{2H}$. Then

$$A_4(f_4-d_1)(f_4-d_2)(f_4-d_3)^2$$
$$=A\left(\frac{d_2-d_1}{4}\right)^4\left(x-\frac{1}{x}\right)^2\left(x+u+\frac{1}{x}\right)^2,$$

$$u=2-\frac{4(d_3-d_1)}{d_2-d_1}\neq\pm 2\,.$$

Let us consider

$$F-b=A_6\prod_{j=1}^{6}(f_6-e_j)\,.$$

By Theorem B we have the following possibilities:

i) $A_6(f_6-e_1)(f_6-e_2)(f_6-e_3)^2(f_6-e_4)^2$

ii) $A_6(f_6-e_1)(f_6-e_2)(f_6-e_3)^4$

iii) $A_6(f_6-e_1)(f_6-e_2)^3(f_6-e_3)^2$

iv) $A_6(f_6-e_1)(f_6-e_2)^5$

v) $A_6(f_6-e_1)^2(f_6-e_2)^2(f_6-e_3)^2$

iv) $A_6(f_6-e_1)^2(f_6-e_2)^4$.

In the cases i), ii), iii) and iv) we can put

$$f_6-e_1=U^2,\ f_6-e_2=V^2,\ \frac{U+V}{\sqrt{e_2-e_1}}=e^L,\ y=e^{2L}\,.$$

Then ii) gives

$$A_6\left(\frac{e_2-e_1}{4}\right)^6\left(y-\frac{1}{y}\right)^2\left(y+v+\frac{1}{y}\right)^4,\ v\neq\pm 2\,.$$

Hence

$$\frac{1}{x^2}(x^2-1)(x^2+ux+1)=c\frac{1}{y^3}(y^2-1)(y^2+vy+1)^2\,.$$

Since $x=\alpha$, $y=\beta$ have almost simple zeros respectively, this is impossible. Case iii) gives

$$A_6\left(\frac{e_2-e_1}{4}\right)^6\left(y-\frac{1}{y}\right)^2\left(y-2+\frac{1}{y}\right)^2\left(y+v+\frac{1}{y}\right)^2,\ v\neq\pm 2\,.$$

Hence

$$\frac{1}{x^2}(x^2-1)(x^2+ux+1)=c\frac{1}{y^3}(y+1)(y-1)^3(y^2+vy+1)\,.$$

This is again impossible. Similarly we have the impossibility of iv). Case i) gives

$$\frac{1}{x^2}(x^2-1)(x^2+ux+1)$$

$$=\left(\frac{A_6}{A_4}\right)^{1/2}\left(\frac{e_2-e_1}{4}\right)^3\left(\frac{4}{d_2-d_1}\right)^2\frac{1}{y^3}(y^2-1)(y^2+v_1y+1)(y^2+v_2y+1),$$

$$v_1,\ v_2\neq\pm 2.$$

In this case firstly $4m(r, x)\sim 6m(r, y)$. Then we can make use of the impossibility of Borel's identity. Then we have

$$4H=6L+k \quad \text{or} \quad 4H=-6L+k$$

and $u=0$, $v_1+v_2=0$, $1+v_1v_2=0$, $e^{2k}=1$. Hence

$$F-b=A_4\left(\frac{d_2-d_1}{4}\right)^4(e^{4H}-e^{-4H})^2$$

$$=2A_4\left(\frac{d_2-d_1}{4}\right)^4(\cos 4K-1),\ 8H=4Ki.$$

This gives a part of the desired result.

Case vi). Then

$$A_4^{1/2}\left(\frac{d_2-d_1}{4}\right)^2\frac{1}{x^2}(x^2-1)(x^2+ux+1)$$

$$=A_6^{1/2}(f_6-e_1)(f_6-e_2)^2.$$

This is impossible, since

$$4m(r, x)\sim\bar{N}(r, 1, x)+\bar{N}(r, -1, x)+\bar{N}(r, \alpha_1, x)+\bar{N}(r, \alpha_2, x)$$

$$=\bar{N}(r, e_1, f_6)+\bar{N}(r, e_2, f_6)\leqq 2m(r, f_6)\sim\frac{8}{3}m(r, x).$$

Case v). In this case we have

$$A_6^{1/2}(f_6-e_1)(f_6-e_2)(f_6-e_3)=A_4^{1/2}\left(\frac{d_2-d_1}{4}\right)^2\frac{1}{x^2}(x^2-1)$$

$$\times(x^2+ux+1).$$

There is a constant c such that

$$A_6^{1/2}\{(f_6-e_1)(f_6-e_2)(f_6-e_3)+c\}=A_6^{1/2}(f_6-\alpha_1)(f_6-\alpha_2)^2.$$

$$\begin{cases} \alpha_2=\dfrac{s_1+\sqrt{s_1^2-3s_2}}{3},\ \alpha_1=\dfrac{s_1-\sqrt{s_1^2-3s_2}}{3}, \\ s_1=e_1+e_2+e_3,\ s_2=e_1e_2+e_1e_3+e_2e_3,\ s_3=e_1e_2e_3. \\ c=s_3-\left(\dfrac{2}{3}s_2-\dfrac{2}{9}s_1^2\right)\alpha_2-\dfrac{s_1s_2}{9}. \end{cases}$$

Then

$$G\equiv A_6^{1/2}(f_6-\alpha_1)(f_6-\alpha_2)^2=A_4^{1/2}\left(\frac{d_2-d_1}{4}\right)^2\frac{1}{x^2}(x^4+ux^3+dx^2-ux-1)\,,$$

$$d=c\left(\frac{A_6}{A_4}\right)^{1/2}\left(\frac{4}{d_2-d_1}\right)^2\neq 0\,.$$

Hence the right hand side term is equal to

$$A_4^{1/2}\left(\frac{d_2-d_1}{4}\right)^2\frac{1}{x^2}\prod_{j=1}^{4}(x-\beta_j)\,.$$

as in the third step there are only two possibilities:

a) $\beta_1\neq\beta_2=\beta_3=\beta_4$,

b) $\beta_1=\beta_3\neq\beta_2=\beta_4$.

Case a). In this case

$$\begin{cases}\beta_1+3\beta_2=-u\\ 3\beta_2^2+3\beta_1\beta_2=d\\ \beta_2^3+3\beta_1\beta_2^2=u\\ \beta_1\beta_2^3=-1\,.\end{cases}$$

This was already solved in the third step. Hence

$$\begin{aligned}G'&=A_6^{1/2}\{(f_6-\alpha_2)(3f_6-\alpha_2-2\alpha_1)\}f_6'\\ &=A_4^{1/2}\left(\frac{d_2-d_1}{4}\right)^2\frac{4H'}{x^2}(x-\beta_2)^2(x-\gamma_2)^2\,.\end{aligned}$$

Therefore almost all the zeros of

$$f_6-\alpha_2,\ f_6-\frac{\alpha_2+2\alpha_1}{3}$$

are of even order. Hence

$$\begin{aligned}&2m(r,\ f_6)(1+o(1))\\ &\leqq\bar{N}(r,\ \alpha_1,\ f_6)+\bar{N}(r,\ \alpha_2,\ f_6)+\bar{N}\left(r,\ \frac{\alpha_2+2\alpha_1}{2},\ f_6\right)\\ &\leqq 2m(r,\ f_6)(1+o(1))\end{aligned}$$

and hence

$$\bar{N}(r,\ \alpha_1,\ f_6)\sim N(r,\ \alpha_1,\ f_6)\sim m(r,\ f_6)\,.$$

This gives

$$\frac{3}{4}m(r,\ G)\sim 3m(r,\ x)\sim 3N(r,\ \beta_2,\ x)$$

$$\sim N_m(r,\ 0,\ G)\leqq 2m(r,\ f_6)\sim\frac{2}{3}m(r,\ G).$$

This is impossible.

Case b). In this case we have $u=0$, $d=\pm 2i$ and

$$\begin{cases}\beta_1=e^{\pi i/4}\\ \beta_2=-e^{\pi i/4}\end{cases}\quad\text{or}\quad\begin{cases}\beta_1=e^{-\pi i/4}\\ \beta_2=-e^{-\pi i/4}.\end{cases}$$

Hence

$$F-b=A_4\left(\frac{d_2-d_1}{4}\right)^2\frac{1}{x^4}(x^4-1)^2$$

$$=A_4\left(\frac{d_2-d_1}{4}\right)^2\left(x^2-\frac{1}{x^2}\right)^2$$

$$=A_4\left(\frac{d_2-d_1}{4}\right)^2(e^{8H}+e^{-8H}-2)$$

$$=A(\cos 4K-1),\ 8H=4Ki.$$

This gives a part of the final result.

Case 2). In this case

$$F-b=A_2(f_2-w_0)^2=A_4(f_4-d_1)(f_4-d_2)^3.$$

Let us consider

$$F-b=A_6\prod_{j=1}^{6}(f_6-e_j).$$

By Theorem B we have the following possibilities:

i) $A_6(f_6-e_1)(f_6-e_2)(f_6-e_3)^2(f_6-e_4)^2$

ii) $A_6(f_6-e_1)(f_6-e_2)(f_6-e_3)^4$

iii) $A_6(f_6-e_1)(f_6-e_2)^3(f_6-e_3)^2$

iv) $A_6(f_6-e_1)(f_6-e_2)^5$

v) $A_6(f_6-e_1)^3(f_6-e_2)^3$

vi) $A_6(f_6-e_1)^2(f_6-e_2)^2(f_6-e_3)^2$

vii) $A_6(f_6-e_1)^2(f_6-e_2)^4$.

In the first five cases we introduce x and y as in Case 1) and arrive at contradiction easily.

Case vi). In this case we have

$$\frac{1}{x^2}(x+1)(x-1)^3=c(f_6-e_1)(f_6-e_2)(f_6-e_3)\equiv G\,.$$

Then

$$\frac{2}{3}m(r,\ G)\sim 2m(r,\ f_6)\leqq \sum_{j=1}^{3}\bar{N}(r,\ e_j,\ f_6)$$

$$=\bar{N}(r,\ -1,\ x)+\bar{N}(r,\ 1,\ x)\sim 2m(r,\ x)\sim \frac{1}{2}m(r,\ G)\,.$$

This is absurd.

Case vii). In this case we have

$$\frac{1}{x^2}(x+1)(x-1)^3=d(f_6-e_1)(f_6-e_2)^2\equiv G\,.$$

Then there is a constant c such that

$$d(f_6-e_1)(f_6-e_2)^2+dc=d(f_6-e_2-\alpha_1)(f_6-e_2-\alpha_2)^2\,.$$

Here

$$c=-\frac{2}{9}(e_2-e_1)^2\neq 0\,,\ \alpha_1=\frac{e_2-e_1}{3},\ \alpha_2=-\frac{2}{3}(e_2-e_1)\,.$$

Then

$$\frac{1}{x^2}(x+1)(x-1)^3+dc=\frac{1}{x^2}(x^4-2x^3+dcx^2+2x-1)$$

$$=\frac{1}{x^2}\prod_{j=1}^{4}(x-\beta_j)\,.$$

Here we have four possibilities:

$$\frac{1}{x^2}(x-\beta_1)(x-\beta_2)(x-\beta_3)(x-\beta_4)\,,$$

$$\frac{1}{x^2}(x-\beta_1)(x-\beta_2)(x-\beta_3)^2\,,$$

$$\frac{1}{x^2}(x-\beta_1)(x-\beta_2)^3\,,$$

$$\frac{1}{x^2}(x-\beta_1)^2(x-\beta_2)^2\,.$$

In each cases we have the impossibility easily.

Case 4). In this case

$$F-b=A_2(f_2-w_0)^2=A_4(f_4-d_1)^4\,.$$

Then by Theorem B

$$F-b=A_{2n}\prod_{j=1}^{s_n}(f_{2n}-\beta_{nj})^{2\nu_j}\,,\ \nu_1+\cdots+\nu_{s_n}=n\,.$$

Therefore either

$$A_2^{1/2}(f_2-w_0)=A_4^{1/2}(f_4-d_1)^2=A_6^{1/2}(f_6-e_1)(f_6-e_2)^2$$
$$=A_{2n}^{1/2}(f_{2n}-\beta_{n1})^{\nu_1}\cdots(f_{2n}-\beta_{ns_n})^{\nu_{s_n}}$$

or

$$F-b=A_2(f_2-w_0)^2=A_4(f_4-d_1)^4=A_6(f_6-e_1)^6\,.$$

In the latter case we have either $F-b=Ae^H$ if $F-b$ has only finitely many zeros or there is no such F if $F-b$ has infinitely many zeros. This has been done already in the first and the second steps. In the former case the problem reduces to the problem of Theorem. Hence we can make use of our result. Therefore

$$A_2^{1/2}(f_2-w_0)=\begin{cases} Ae^H+B\,, \\ A\cos\sqrt{H}+B\,. \end{cases}$$

Hence

$$F-b=\begin{cases} (Ae^H+B)^2\,, \\ (A\cos\sqrt{H}+B)^2\,. \end{cases}$$

Case 3). In this case

$$F-b=A_2(f_2-w_0)^2=A_4(f_4-d_1)^2(f_4-d_2)^2\,.$$

Then

$$F-b=A_6(f_6-e_1)^2(f_6-e_2)^2(f_6-e_3)^2$$

or

$$F-b=A_6(f_6-e_1)^2(f_6-e_2)^4\,.$$

Hence with a suitable constant c

$$A_2^{1/2}(f_2-w_0+c)=A_4^{1/2}\left(f_4-\frac{d_1+d_2}{2}\right)^2$$

$$=\begin{cases} A_6^{1/2}(f_6-\alpha_1)(f_6-\alpha_2)^2 \\ A_6^{1/2}(f_6-\alpha_1)(f_6-\alpha_2)^2 \end{cases} \quad\text{or}\quad A_6^{1/2}(f_6-\alpha_1)^3\,.$$

respectively. In the latter case we have

$$A_2^{1/2}(f_2-w_0+c)=Ae^H+B$$

and hence

$$F-b=(Ae^H+B)^2\,.$$

In the former case we can make use of our theorem and then we have

$$F-b=(A\cos\sqrt{H}+B)^2.$$

Hence we have the following corollary.

COROLLARY. *Let $F(z)$ be an entire function, for which*

$$F(z)=P_m(f_m(z))$$

holds for $m=2^j$ and for $m=3\cdot 2^j$, $1\leqq j<\infty$. Here P_m is a polynomial of degree m and f_m is an entire function. Then either

$$F(z)=(Ae^H+B)^2+C$$

or

$$F(z)=(A\cos\sqrt{H}+B)^2+C,$$

where A, B, C are constants and H is an entire function.

6. In this section we shall discuss the case that $\{2^j, 3\}$ is given as $\{m_l\}$ and we shall prove our theorem. In the fourth step and by (1) we have proved

$$A_2^{1/2}(f_2-x_1)=A_4^{1/2}\left(f_4-\frac{d_1+d_2}{2}\right)^2=A_3^{1/2}(T-\alpha_{11})(T-\alpha_{21})^2$$
$$=A_8^{1/2}(f_8-d_1')^2(f_8-d_2')^2.$$

Hence $T-\alpha_{11}=T_2{}^2$ and so

$$A_4^{1/4}\left(f_4-\frac{d_1+d_2}{2}\right)=A_3^{1/4}T_2(T_2{}^2-\alpha_{21}+\alpha_{11})$$
$$=A_8^{1/4}(f_8-d_1')(f_8-d_2').$$

Further by Lemma

$$A_2^{1/2}(f-x_1)=A_{2^p}^{1/2}\prod_{j=1}^{s_{2^p}}(f_{2^p}-\beta_{2^p j})^{2\nu_j},\quad \sum_1^{s_{2^p}}\nu_j=2^{p-2}.$$

Hence

$$A_4^{1/4}\left(f_4-\frac{d_1+d_2}{2}\right)=A_{16}^{1/4}(f_{16}-\beta_{16,1})(f_{16}-\beta_{16,2})(f_{16}-\beta_{16,3})$$
$$\times(f_{16}-\beta_{16,4}).$$

Here $\beta_{16,j}$ may coincide with each other. Hence by the third and fourth steps we have

$$A_4^{1/4}(f_4-x_1')=A_8^{1/4}\left(f_8-\frac{d_1'+d_2'}{2}\right)^2$$

$$=A_3^{1/4}(T_2-\alpha_1')(T_2-\alpha_2')^2$$

$$=A_{16}^{1/4}(f_{16}-d_1'')^2(f_{16}-d_2'')^2$$

$$=A_{2p}^{1/4}\prod_{j=1}^{s'_{2p}}(f_{2p}-\beta'_{2p,j})^{2\nu_j'},\ \sum\nu_j'=2^{p-3}.$$

Again $T_2-\alpha_1'=T_3^2$. Therefore we go a step further. This process can be repeated ad infinitum. Hence we can get the desired result.

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