

ON THE SYMPLECTIC VOLUME OF THE MODULI SPACE OF SPHERICAL AND EUCLIDEAN POLYGONS

VU THE KHOI

Abstract

In this paper, we study the symplectic volume of the moduli space of polygons by using Witten's formula. We propose to use this volume as a measure for the flexibility of a polygon with fixed side-lengths. The main result of our is that among all the polygons with fixed perimeter in \mathbf{S}^3 or \mathbf{E}^3 the regular one is the most flexible and that among all the spherical polygons the regular one with side-length $\pi/2$ is the most flexible.

1. Introduction

An polygon in \mathbf{S}^3 or \mathbf{E}^3 is specified by its set of vertices $v = (v_1, \dots, v_n)$. This vertices are joined in cyclic order by edges e_1, \dots, e_n , where e_i is the directed geodesic segment from v_i to v_{i+1} . Two polygons $P = (v_1, \dots, v_n)$ and $Q = (w_1, \dots, w_n)$ are identified if there exists an orientation preserving isometry sending each v_i to w_i .

The side-length r_i of a polygon is defined to be the length of the geodesic segment e_i . We say that a polygon is *regular* if all of its side-lengths are equal.

Let $r = (r_1, \dots, r_n)$ be a tuple of real numbers such that $0 < r_i < \pi \forall i$, following [13], we will denote by $\mathcal{P}_r^{\mathbf{S}^3}$ the *configuration space of all polygons in \mathbf{S}^3 with side-lengths r* . The *moduli space of polygons in \mathbf{S}^3 with side-lengths r* is defined to be $\mathcal{P}_r^{\mathbf{S}^3}/\text{Iso} = \mathcal{P}_r^{\mathbf{S}^3}/\text{SO}(4)$.

In \mathbf{E}^3 , similarly, if $r = (r_1, \dots, r_n)$ is a tuple of positive real numbers, we denote by $\mathcal{P}_r^{\mathbf{E}^3}$ the *configuration space of all polygons in \mathbf{E}^3 with side-lengths r* . The *moduli space of polygons in \mathbf{E}^3 with side-lengths r* is defined to be $\mathcal{P}_r^{\mathbf{E}^3}/\text{Iso} = \mathcal{P}_r^{\mathbf{E}^3}/\mathbf{E}^+(3)$.

We will drop the superscript and simply write \mathcal{P}_r when we want to talk about polygons in both \mathbf{S}^3 and \mathbf{E}^3 .

We are interested in the question: *for which tuple r the polygon with side-*

2000 *Mathematics Subject Classification.* 53D30.

Key words and phrases. moduli space of polygons, symplectic volume.

The author was partially supported by the National Basic Research Program of Vietnam.

Received August 31, 2004; revised November 2, 2004.

lengths r is the most flexible when it is moved in the space provided that its side-lengths are fixed? First of all we need to specify what is the exact meaning of “flexible”. When a polygon is moved, it always lies in its configuration space. Therefore, it is natural to think that a polygon with side-lengths r is more flexible than another one with side-lengths r' if the corresponding configuration space \mathcal{P}_r is “bigger” than $\mathcal{P}_{r'}$ in some sense. Fortunately, there is a standard symplectic structure on the moduli spaces \mathcal{P}_r/Iso , see [2, 4], and we will measure the flexibility of a polygon with side-lengths r by the symplectic volume of its moduli space.

The main results are the followings:

THEOREM 1.1. *Among all the spherical or Euclidean polygons with fixed perimeter, the regular polygon is one of the most flexible. Moreover in the case n is even, the regular n -gon is the unique one with this property.*

THEOREM 1.2. *Among all the spherical polygons the regular one with side-length $\pi/2$ is the unique one which is the most flexible.*

The rest of this paper is organized as follows. In section 2, we will recall the symplectic structure on the moduli space $\mathcal{P}_r^{\mathbb{S}^3}/\text{SO}(4)$ and using Witten’s formula to find its symplectic volume in terms of Bernoulli polynomials. As a corollary we also derive the positivity of certain trigonometric sums which has been proved by analytic method. Section 3 is devoted to the study of the moduli space $\mathcal{P}_r^{\mathbb{E}^3}/\mathbb{E}^+(3)$, in particular we derive an explicit formula for its symplectic volume. The main results are proved in section 4 by a detail study of the symplectic volume of as a function of r .

2. Volume of the moduli space of spherical polygons

We first briefly recall the identification between $\mathcal{P}_r^{\mathbb{S}^3}/\text{SO}(4)$ and the moduli space of flat $\text{SU}(2)$ connections on a punctured sphere with fixed holonomies around the punctures. Let $S_n := S^2 \setminus \{p_1, \dots, p_n\}$ be an n -punctured sphere. For a tuple of numbers $r = (r_1, \dots, r_n)$ such that $0 < r_i < \pi \ \forall i$, we will denote by $\mathcal{M}(S_n, r)$ the moduli space of flat $\text{SU}(2)$ connections on S_n modulo gauge

equivalence such that the holonomy around p_i is conjugate to $\begin{pmatrix} e^{ir_i} & 0 \\ 0 & e^{-ir_i} \end{pmatrix}$.

It is well-known that $\mathcal{M}(S_n, r) \cong \mathcal{R}(S_n, r)/\text{SU}(2)$, where

$$\mathcal{R}(S_n, r) := \{(g_1, \dots, g_n) \in \text{SU}(2)^n \mid g_1 \cdots g_n = \text{Id}, \text{Tr}(g_j) = 2 \cos(r_j) \ \forall j\}$$

is the representation space consists of all representations from $\pi_1(S_n)$ to $\text{SU}(2)$ such that the image of the loop around the puncture p_i is conjugate to

$\begin{pmatrix} e^{ir_i} & 0 \\ 0 & e^{-ir_i} \end{pmatrix}$ and the group $\text{SU}(2)$ act diagonally by conjugation.

On the other hand, following [13], we may identify $\mathcal{R}(S_n, r)$ with the configuration space $\mathcal{P}_{r,0}^{\mathbb{S}^3}$ of *based polygons*, i.e. polygons having the first vertex $v_1 = \text{Id} \in \text{SU}(2) \cong \mathbb{S}^3$. For $(g_1, \dots, g_n) \in \mathcal{R}(S_n, r)$ we construct a based polygon by defining $g_0 := \text{Id}$ and let $v_j := g_0 \cdots g_{j-1}$ for $1 \leq j \leq n$. Under this identification, we have

$$\mathcal{R}(S_n, r)/\text{SU}(2) \cong \mathcal{P}_{r,0}^{\mathbb{S}^3}/\text{SO}(3) \cong \mathcal{P}_r^{\mathbb{S}^3}/\text{SO}(4).$$

Therefore, we have a natural identification $\mathcal{P}_r^{\mathbb{S}^3}/\text{SO}(4) \cong \mathcal{M}(S_n, r)$. According to Atiyah-Bott [2], the moduli space of flat connections on a surface of genus g with specified holonomies around the boundary is a finite dimensional symplectic manifold, possibly with singularities. Witten [15] compute the symplectic volume of the moduli space of flat connections. Since then, other people have given simplified and rigorous proofs (see [3, 7, 11, 12]). As a special case of Witten’s theorem, we have the followings (see [15, 12]):

THEOREM 2.1 (Witten’s formula). *Denote by $V^{\mathbb{S}^3}(r)$ the symplectic volume of $\mathcal{P}_r^{\mathbb{S}^3}/\text{SO}(4)$ —the moduli space of polygons of side-lengths $r = (r_1, \dots, r_n)$. We have:*

$$V^{\mathbb{S}^3}(r) = \frac{2^{n-1}}{\pi} \sum_{k=1}^{\infty} \frac{\sin kr_1 \cdots \sin kr_n}{k^{n-2}}.$$

For $I \subset \{1, 2, \dots, n\}$, let \bar{I} denote the complement of I , $|I|$ be the cardinality of I and $r_I := \sum_{i \in I} r_i$. It has been shown that the moduli space of polygons of side-lengths $r = (r_1, \dots, r_n)$ is non-empty if and only if:

$$r_I \leq r_{\bar{I}} + (|I| - 1)\pi \quad \forall I \subset \{1, 2, \dots, n\}, \quad |I| \text{ odd}.$$

See [1] for a proof using quantum cohomology. This result can also be proved in a similar line as in the case of spherical polygons in \mathbb{S}^2 in [9].

Combining this result with Witten’s formula we obtained the positivity of a trigonometric series.

COROLLARY 2.2. *If $r = (r_1, \dots, r_n)$ be a tuple of real numbers such that $0 < r_i < \pi$ for all i , then*

$$\sum_{k=1}^{\infty} \frac{\sin kr_1 \cdots \sin kr_n}{k^{n-2}} \geq 0.$$

Moreover, the inequality is strict if and only if $r_I \leq r_{\bar{I}} + (|I| - 1)\pi$ for all $I \subset \{1, 2, \dots, n\}$, $|I|$ odd.

We can find a closed form expression of the volume function in terms of the Bernoulli polynomials.

PROPOSITION 2.3. *The symplectic volumes of the moduli space of $2n$ -gons and $(2n+1)$ -gons are given respectively by:*

$$V^{\mathbb{S}^3}(r) = \frac{(2\pi)^{2n-3}}{2(2n-2)!} \sum_{I \subset \{1,2,\dots,2n\}} (-1)^{|I|} B_{2n-2} \left(\left\{ \left(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \right) / 2\pi \right\} \right)$$

$$V^{\mathbb{S}^3}(r) = \frac{(2\pi)^{2n-2}}{(2n-1)!} \sum_{I \subset \{1,2,\dots,2n+1\}, |I| \text{ odd}} B_{2n-1} \left(\left\{ \left(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \right) / 2\pi \right\} \right).$$

Where, B_n are the Bernoulli polynomials defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

and $\{ \}$ denote the fractional part.

Proof. In the proof of this proposition we will use the following formulae (see [5], 1.443):

$$(1) \quad \sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \left(\frac{x}{2} \right)$$

$$(2) \quad \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} \frac{(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1} \left(\frac{x}{2} \right)$$

Where, B_n is the Bernoulli polynomials defined above and the formulae hold for $0 \leq x \leq 2$.

We first prove the $2n$ -gon case. Notice that we can write,

$$\begin{aligned} \sin kr_1 \cdots \sin kr_{2n} &= \frac{(e^{ikr_1} - e^{-ikr_1}) \cdots (e^{ikr_{2n}} - e^{-ikr_{2n}})}{(2i)^{2n}} \\ &= \frac{\sum_{I \subset \{1,2,\dots,2n\}} (-1)^{|I|} e^{ik(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i)}}{(2i)^{2n}} \\ &= \frac{\sum_{I \subset \{1,2,\dots,2n\}} (-1)^{|I|} \cos k(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i)}{(2i)^{2n}}. \end{aligned}$$

In the last line, we have replaced $|I'|$ by $|I|$ since $|I| + |I'| = 2n$. Substitute this in to Witten's formula we get:

$$V^{\mathbb{S}^3}(r) = \frac{(-1)^n}{2\pi} \sum_{I \subset \{1,2,\dots,2n\}} (-1)^{|I|} \sum_{k=1}^{\infty} \frac{\cos k(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i)}{k^{2n-2}}$$

Now using (1) above, we can deduce the result.

Now consider the $2n + 1$ -gon case. Similarly, we can write

$$\begin{aligned} \sin kr_1 \cdots \sin kr_{2n+1} &= \frac{(e^{ikr_1} - e^{-ikr_1}) \cdots (e^{ikr_{2n+1}} - e^{-ikr_{2n+1}})}{(2i)^{2n+1}} \\ &= \frac{\sum_{I \subset \{1, 2, \dots, 2n+1\}} (-1)^{|I|} e^{ik(\sum_{i \in I} r_i - \sum_{i \in I'} r_i)}}{(2i)^{2n+1}} \\ &= \frac{\sum_{I \subset \{1, 2, \dots, 2n+1\}, |I| \text{ odd}} \sin k(\sum_{i \in I} r_i - \sum_{i \in I'} r_i)}{(2i)^{2n}}. \end{aligned}$$

We can obtain the formula by plugging this into Witten’s formula and then use formula (2). □

3. Volume of the moduli space of Euclidean polygons

In Euclidean space, the moduli space of polygons $\mathcal{P}_r^{\mathbf{E}^3}/\mathbf{E}^+(3)$ also has a symplectic structure [9, 13] which can be briefly described as follows.

Let $r = (r_1, \dots, r_n)$ be a tuple of positive numbers, denote by $\mathcal{P}_{r,0}^{\mathbf{E}^3}$ the configuration space of *based polygons*, i.e. polygons having the first vertex $v_1 = 0 \in \mathbf{E}^3$. Consider the map:

$$\begin{aligned} \Phi_r : \mathcal{P}_{r,0}^{\mathbf{E}^3} &\rightarrow (S^2)^n \\ (v_1, \dots, v_n) &\mapsto \left(\frac{v_2}{r_1}, \frac{v_3 - v_2}{r_2}, \dots, \frac{v_n - v_{n-1}}{r_{n-1}}, \frac{-v_n}{r_n} \right). \end{aligned}$$

If we defined $\mu_r : (S^2)^n \rightarrow \mathbf{R} : (u_1, \dots, u_n) \mapsto r_1 u_1 + \dots + r_n u_n$, then Φ_r define an $\text{SO}(3)$ -equivariant map between $\mathcal{P}_{r,0}^{\mathbf{E}^3}$ and $\mu_r^{-1}(0)$ which results in a diffeomorphism between $\mathcal{P}_{r,0}^{\mathbf{E}^3}/\text{SO}(3)$ and $\mu_r^{-1}(0)/\text{SO}(3)$.

It is well-known that the unit sphere S^2 is a symplectic manifold where the symplectic 2-form is the volume form. Consider the symplectic manifold $(S^2)^n$ with the weighted symplectic structure r_1, \dots, r_n , i.e., the symplectic 2-form is $r_1 \omega_1 + \dots + r_n \omega_n$, where ω_i is the volume form on the i^{th} component. It is well-known that μ_r is the moment map for $(S^2)^n$ with the diagonal action of $\text{SO}(3)$. Therefore, we can identify $\mathcal{P}_r^{\mathbf{E}^3}/\mathbf{E}^+(3) \cong \mathcal{P}_{r,0}^{\mathbf{E}^3}/\text{SO}(3)$ with the weighted symplectic quotient of $(S^2)^n$ by the diagonal action of $\text{SO}(3)$.

Moreover, we can deduce from a result of L. Jeffrey [6], Theorem 6.6 (see also the discussion at the end of [13]) that the moduli space of polygons in \mathbf{S}^3 and \mathbf{E}^3 with the same side-lengths r are symplectomorphic provided that r_i are sufficiently small. So the volume of the moduli space of polygons in \mathbf{S}^3 and \mathbf{E}^3 are the same if the side-lengths are sufficiently small. Using this fact, we can express the volume of $\mathcal{P}_r^{\mathbf{E}^3}/\mathbf{E}^+(3)$ in a quite explicit form.

PROPOSITION 3.1. *The symplectic volumes of the moduli space of normalized Euclidean $2n$ -gon and $(2n + 1)$ -gon are given respectively by:*

$$V^{\mathbf{E}^3}(r) = -\frac{1}{4(2n-3)!} \sum_{I \subset \{1,2,\dots,2n\}} (-1)^{|I|} \left| \sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \right|^{2n-3}$$

$$V^{\mathbf{E}^3}(r) = -\frac{1}{2(2n-2)!} \sum_{I \subset \{1,2,\dots,2n+1\}, |I| \text{ odd}} \text{sign} \left(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \right) \left(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \right)^{2n-2}.$$

Proof. Notice that if we scale r by a scalar $\lambda > 0$ then $\mathcal{P}_{\lambda r}^{\mathbf{E}^3} / \mathbf{E}^+(3)$ is the weighted symplectic quotient of $(S^2)^n$ for the weight λr . Moreover, we know that $\mathcal{P}_r^{\mathbf{E}^3} / \mathbf{E}^+(3)$ is of dimension $2(n-3)$, it follows that $V^{\mathbf{E}^3}(\lambda r) = \lambda^{n-3} V^{\mathbf{E}^3}(r)$.

We first, assume that all r_i are sufficiently small, from the discussion above, we know that the volume of $\mathcal{P}_r^{\mathbf{E}^3} / \mathbf{E}^+(3)$ can also be expressed in terms of Bernoulli polynomials as in Proposition 2.3. When all r_i are sufficiently small, we have:

$$\left\{ \left(\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \right) / 2\pi \right\} = \begin{cases} (\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i) / 2\pi, & \text{if } \sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i \geq 0 \\ 1 + (\sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i) / 2\pi, & \text{if } \sum_{i \in I} r_i - \sum_{i \in \bar{I}} r_i < 0. \end{cases}$$

We recall the two following property of Bernoulli polynomials [14]:

- a) $B_n(1-x) = (-1)^n B_n(x)$
- b) $B_n(x) = (B+x)^n$, where the symbol B^n is the Bernoulli number $B_n(0)$.

By the homogeneity of $V^{\mathbf{E}^3}(r)$, we know that to find $V^{\mathbf{E}^3}(r)$ it is enough to collect the terms of order $(n-3)$ from the formulae of Proposition 2.3. Now using this two properties of Bernoulli polynomials and notice that $B_1 = -1/2$, one can easily obtain the required results. So our formulae hold in the case where all r_i are sufficiently small. But as notice before, $V^{\mathbf{E}^3}(r)$ are homogenous of degree $n-3$ with respect to multiplication by a positive scalar, therefore the formulae hold for all r . □

As a special case of our result, when all $r_i = 1$, we obtain the following formula which has been shown by direct computation in [8].

COROLLARY 3.2. *The volume of the moduli space of regular polygon with side-length 1 in \mathbf{E}^3 is given by:*

$$V^{\mathbf{E}^3}(1) = -\frac{1}{2(n-3)!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} (n-2k)^{n-3}.$$

Proof. The $2n$ -gon case is straightforward, we will give the proof for the case of $2n + 1$ -gon. From Proposition 3.1, we get

$$V^{E^3}(1) = -\frac{1}{2(2n-2)!} \left(\sum_{k \text{ odd}, k > n} \binom{2n+1}{k} (2n+1-2k)^{2n-2} - \sum_{k \text{ odd}, k \leq n} \binom{2n+1}{k} (2n+1-2k)^{2n-2} \right).$$

By changing the index in the first sum from k to $(2n+1-k)$ we get,

$$\begin{aligned} V^{E^3}(1) &= -\frac{1}{2(2n-2)!} \left(\sum_{k \text{ even}, k \leq n} \binom{2n+1}{k} (2n+1-2k)^{2n-2} - \sum_{k \text{ odd}, k \leq n} \binom{2n+1}{k} (2n+1-2k)^{2n-2} \right) \\ &= -\frac{1}{2(2n-2)!} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n+1-2k)^{2n-2}. \end{aligned}$$

This finishes the proof. □

4. Proof of the main results

In this section we will give the proofs of Theorem 1.1 and 1.2 by study the maximum of the volume function which is the trigonometric series given by Theorem 2.1. From Proposition 2.3, we can easily find out that for $n = 3$, the moduli space of polygons has volume 1, no matter what the side-lengths are. So we only need to deal with the non-trivial cases and may assume that $n > 3$.

Proof of Theorem 1.1. It is enough to prove the spherical case.

For a fixed n , consider the set of all polygon whose perimeter is P . To prove the first part of our theorem, we must show the following:

Claim: Let

$$f(x_1, \dots, x_n) = \sum_{k=1}^{\infty} \frac{\sin kx_1 \cdots \sin kx_n}{k^{n-2}}$$

be a function on the domain $D = \{(x_1, \dots, x_n) \mid 0 < x_i < \pi, \sum_1^n x_i = P\}$. Then $f(x) \leq f(x^*)$ for all $x \in D$ where $x^* = (\frac{P}{n}, \dots, \frac{P}{n})$.

To prove this claim we will show that starting from any point $x \in D$, $x \neq x^*$ we can build a sequence $\{x^k\}_{k=1 \dots \infty}$, $x^k \in D$, such that:

- i) $x^1 = x$.
- ii) $f(x)$ is non-decreasing on the segment connecting x^i and x^{i+1} .
- iii) $\lim_{k \rightarrow \infty} x^k = x^*$.

Once we have done this, it follows immediately that $f(x) \leq \lim_{k \rightarrow \infty} f(x^k) = f(x^*)$.

We construct the sequence $\{x^k\}$ as follows. Put $x^1 = x$. Suppose that we already have x^k , we will find x^{k+1} as follows:

- If $x^k = x^*$, put $x^{k+1} = x^*$.
- If $x^k = (x_1^k, \dots, x_n^k) \neq x^*$, suppose that $x_m^k = \min_{i=1 \dots n} x_i^k$ and $x_M^k = \max_{i=1 \dots n} x_i^k$, we then specified x^{k+1} by:

$$x_i^{k+1} := \begin{cases} x_i^k & \text{for } i \neq m, M \\ \frac{x_m^k + x_M^k}{2} & \text{for } i = m \text{ or } i = M. \end{cases}$$

It is not hard to see that the sequence constructed this way converges to x^* . It remains to prove that it satisfies (ii) above. Without loss of generality, it is enough to show that if $x = (x_1, x_2, \dots, x_n)$ and $x_1 = \max_{i=1 \dots n} x_i$, $x_2 = \min_{i=1 \dots n} x_i$ then $f(x)$ is non-decreasing on the segment $l(t) := \left(x_1 - \frac{t(x_1 - x_2)}{2}, x_2 + \frac{t(x_1 - x_2)}{2}, x_3, \dots, x_n \right)$, $0 \leq t \leq 1$.

When restricted to the segment $l(t)$, our function has the form:

$$\begin{aligned} f(t) &= \sum_{k=1}^{\infty} \frac{\sin k \left(x_1 - \frac{t(x_1 - x_2)}{2} \right) \sin k \left(x_2 + \frac{t(x_1 - x_2)}{2} \right) \cdots \sin kx_n}{k^{n-2}} \\ &= \sum_{k=1}^{\infty} \frac{(\cos k(x_1 - x_2 - t(x_1 - x_2))) - \cos k(x_1 + x_2)) \sin kx_3 \cdots \sin kx_n}{2k^{n-2}}. \end{aligned}$$

We compute:

$$f'(t) = (x_1 - x_2) \sum_{k=1}^{\infty} \frac{\sin k(x_1 - x_2 - t(x_1 - x_2)) \sin kx_3 \cdots \sin kx_n}{2k^{n-3}}.$$

Since $x_1 > x_2$, by Corollary 2.2, we deduce that $f'(t) \geq 0$, $\forall 0 \leq t \leq 1$. So we obtain the first part of the theorem.

For n even, by Corollary 2.2, it is easy to see that $f'(t)$ above is strictly positive for x sufficiently close to x^* . We conclude that x^* is the unique point where f attains its maximum value.

For n odd, in order for the function $f(x)$ not to be identically zero on the domain D , by Corollary 2.2, we must have $P \leq (n-1)\pi$. Even with this assumption, in certain cases, x^* may not be the unique point where f attains its maximum value. For example, when $n = 5$ and $P = 4\pi$, then it is not hard to check that

$$f\left(\frac{4\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}\right) = f\left(\frac{4\pi}{5} + \varepsilon, \frac{4\pi}{5} - \varepsilon, \frac{4\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}\right)$$

for all $\varepsilon > 0$ sufficiently small.

Proof of Theorem 1.2. It follows from the proof of Theorem 1.1 that a regular polygon is the unique one which is the most flexible in the set of all polygons with the same perimeter. So, by Theorem 1.1, it is enough to restrict ourself to the regular polygons.

Let $0 < x < \pi$, and denote by $V_n(x)$ the symplectic volume of the moduli space of regular spherical n -gons with side x . From Theorem 2.1, we know that

$$V_n(x) = \frac{2^{n-1}}{\pi} \sum_{k=1}^{\infty} \frac{(\sin kx)^n}{k^{n-2}}.$$

As

$$V'_n(x) = \frac{2^{n-1}}{\pi} \sum_{k=1}^{\infty} \frac{n(\sin kx)^{n-1} \cos kx}{k^{n-3}} = \frac{2^{n-2}n}{\pi} \sum_{k=1}^{\infty} \frac{(\sin kx)^{n-2} \sin 2kx}{k^{n-3}}.$$

If $0 < x < \pi/2$, Corollary 2.2 tells us that $V'_n(x) \geq 0$.

In the case $\pi/2 < x < \pi$, we write:

$$V'_n(x) = \frac{2^{n-2}n}{\pi} \sum_{k=1}^{\infty} - \frac{(\sin kx)^{n-2} \sin 2k(\pi - x)}{k^{n-3}}$$

and again by Corollary 2.2, we get $V'_n(x) \leq 0$.

Notice that $V'_n(\pi/2) = 0$ and moreover, by using the second part of Corollary 2.2, it is not hard to check that $V'_n(x)$ is strictly positive (resp. negative) when x is sufficiently close to $\pi/2$ on the left (resp. right). From all this we conclude that $\pi/2$ is the unique global maximum of $V_n(x)$ on the interval $(0, \pi)$. So, we get the conclusion of Theorem 1.2.

Remark. Theorem 1.2 seems to agree with the intuitive reasoning that as the side-lengths are $\pi/2$, each vertex moves on the great sphere, i.e., on the biggest possible space, therefore the polygon is the most flexible.

REFERENCES

- [1] AGNIHOTRI, S., WOODWARD, C., Eigenvalues of products of unitary matrices and quantum Schubert calculus. *Math. Res. Lett.*, Vol. 5 (1998), 817–836.
- [2] ATIYAH, M. F., BOTT, R., The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A* 308 (1983), no. 1505, 523–615.
- [3] DONALDSON, S. K., *Gluing techniques in the cohomology of moduli spaces. Topological methods in modern mathematics* (Stony Brook, NY, 1991), 137–170, Publish or Perish, Houston, TX, 1993.
- [4] GOLDMAN, WILLIAM, The symplectic nature of fundamental groups of surfaces. *Adv. in Math.* 54 (1984), no. 2, 200–225.
- [5] GRADSHTEYN, I. S., RYZHIK, I. M., *Table of integrals, series, and products. Corrected and enlarged edition* edited by Alan Jeffrey. Translated from the Russian. Academic Press, New York-London-Toronto, Ont., 1980.

- [6] JEFFREY, LISA, Extended moduli spaces of flat connections on Riemann surfaces. *Math. Ann.* **298** (1994), no. 4, 667–692.
- [7] JEFFREY, LISA, WEITSMAN, JONATHAN, Toric structures on the moduli space of flat connections on a Riemann surface. II. Inductive decomposition of the moduli space. *Math. Ann.* **307** (1997), no. 1, 93–108.
- [8] KAMIYAMA, YASUHIKO, TEZUKA, MICHISHIGE, Symplectic volume of the moduli space of spatial polygons. *J. Math. Kyoto Univ.* **39** (1999), no. 3, 557–575.
- [9] KAPOVICH, MICHAEL, MILLSON, JOHN, The symplectic geometry of polygons in Euclidean space. *J. Differential Geom.* **44** (1996), no. 3, 479–513.
- [10] KAPOVICH, MICHAEL, MILLSON, JOHN, On the moduli space of a spherical polygonal linkage. *Canad. Math. Bull.* **42** (1999), no. 3, 307–320.
- [11] LIU, KEFENG, Heat kernel and moduli space. *Math. Res. Lett.* **3** (1996), no. 6, 743–762.
- [12] MEINRENKEN, E., WOODWARD, C., Moduli spaces of flat connections on 2-manifolds, cobordism, and Witten’s volume formulas. *Advances in geometry*, 271–295, *Progr. Math.*, 172, Birkhäuser Boston, Boston, MA, 1999.
- [13] MILLSON, JOHN, PORITZ, JONATHAN, Around polygons in \mathbf{R}^3 and S^3 . *Comm. Math. Phys.* **218** (2001), no. 2, 315–331.
- [14] WEISSTEIN, Eric “Bernoulli Polynomial.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/BernoulliPolynomial.html>.
- [15] WITTEN, EDWARD, On quantum gauge theories in two dimensions. *Comm. Math. Phys.* **141** (1991), no. 1, 153–209.

INSTITUTE OF MATHEMATICS
18 HOANG QUOC VIET ROAD
10307, HANOI, VIETNAM
E-mail address: vtKhoi@math.ac.vn