CURVATURE-ADAPTED REAL HYPERSURFACES
IN QUATERNIONIC SPACE FORMS

TOSHIAKI ADACHI AND SADAHIRO MAEDA

Abstract

In this paper we study geodesics on curvature-adapted real hypersurfaces in non-flat quaternionic space forms. By observing the extrinsic shape of geodesics on these hypersurfaces we characterize them in the class of real hypersurfaces. We also investigate the length spectrum of geodesic spheres, which are the simplest curvature-adapted real hypersurfaces, in non-flat quaternionic space forms.

1. Introduction

The aim of this paper is to study curvature-adapted real hypersurfaces in quaternionic space forms from the viewpoint of the extrinsic shape of geodesics on these hypersurfaces. A hypersurface M of a Riemannian manifold $\tilde{M}$ is called *curvature-adapted* if the normal Jacobi operator K and the shape operator A of M with respect to a unit normal vector field $\mathcal{N}$ are simultaneously diagonalizable; $K \circ A = A \circ K$. Here the normal Jacobi operator $K : TM \to TM$ of M with respect to $\mathcal{N}$ is defined by $K(\cdot) = \tilde{R}(\cdot, \mathcal{N})\mathcal{N}$ with the curvature tensor $\tilde{R}$ of $\tilde{M}$. In his paper [B], J. B. Berndt completely classified curvature-adapted real hypersurfaces M^{4n-1} in a quaternionic projective space HP^n of quaternionic dimension n ($\geqq 2$). They are locally congruent to tubes around the canonically embedded quaternionic projective spaces of quaternionic dimension k ($0 \leqq k \leqq n-1$) or tubes around the canonically embedded complex projective space of complex dimension n. He also classified curvature-adapted real hypersurfaces M in a quaternionic hyperbolic space HH^n of quaternionic dimension n ($\geqq 2$) under the additional condition that all principal curvatures of M in HH^n are constant. They are locally congruent to either a horosphere, tubes around canonically embedded quaternionic hyperbolic spaces of quaternionic dimension k ($0 \leqq k \leqq$

$n-1$) or tubes around the canonically embedded complex hyperbolic space of complex dimension n.

In the first half of this paper we characterize these curvature-adapted real hypersurfaces in non-flat quaternionic space forms, which are quaternionic projective spaces and quaternionic hyperbolic spaces, by the extrinsic shape of geodesics. It is known that non-flat quaternionic space forms do not admit real hypersurfaces all of whose geodesics are circles in them. We consider a weaker condition. When the ambient manifold $\tilde{M}$ is a quaternionic Kähler manifold we have a natural orthogonal decomposition $\mathscr{D} \oplus \mathscr{D}^{\perp}$ of the tangent bundle TM of a real hypersurface M: The subbundle $\mathscr{D}$ is the maximal subbundle of TM which is invariant by the quaternionic structure of $\tilde{M}$. We characterize these curvature-adapted real hypersurfaces in non-flat quaternionic space forms by the existence of an orthonormal basis of $\mathscr{D}_x$ at each point x such that the extrinsic shape of geodesics of those directions are circles of positive curvature in the ambient manifolds.

When the ambient manifold is a non-flat complex space form, curvature-adapted real hypersurfaces are nothing but real hypersurfaces whose structure vector field $J\mathscr{N}$ is a principal curvature vector field. In preceding papers [AKM] and [CM] these real hypersurfaces with constant principal curvatures are characterized by a similar condition in our results Theorems 3.1 and 3.2. Our theorems in this paper can be regarded as quaternionic versions of preceding results.

In the second half of this paper we study the extrinsic shape of all geodesics on geodesic spheres in a non-flat quaternionic space form, which are the simplest examples in the class of real hypersurfaces. By direct computation we can see that each geodesic on these geodesic spheres is regarded as a helix of order 4 which lies on a totally geodesic complex space form of complex dimension 2. As a consequence we clarify properties on lengths of all closed geodesics on these geodesic spheres.

2. Modeled real hypersurfaces in quaternionic space forms

A quaternionic Kähler structure $\mathscr{J}$ on a Riemannian manifold $\tilde{M}$ of real dimension $4n$ is a rank 3 vector subbundle of the bundle of endmorphisms of the tangent bundle $T\tilde{M}$ with the following properties:

1) For each point $\tilde{x} \in \tilde{M}$ there is an open neighborhood $\tilde{G}$ of $\tilde{x}$ in $\tilde{M}$ and sections J_1, J_2, J_3 of the restriction $\mathscr{J}|_{\tilde{G}}$ over $\tilde{G}$ such that

    i) each J_i is an almost Hermitian structure on $\tilde{G}$, that is, $J_i^2 = -\mathrm{id}$ and $\langle J_i \tilde{X}, \tilde{Y} \rangle + \langle \tilde{X}, J_i \tilde{Y} \rangle = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ on $\tilde{G}$, where $\langle\ ,\ \rangle$ is the Riemannian metric of $\tilde{M}$.

    ii) $J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i$ ($i \bmod 3$) for $i = 1, 2, 3$.

2) $\tilde{\nabla}_{\tilde{X}} J$ is a section of $\mathscr{J}$ holds for each vector field $\tilde{X}$ on $\tilde{M}$ and section J of the bundle $\mathscr{J}$, where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

This triple $\{J_1, J_2, J_3\}$ is called a canonical local basis of $\mathscr{J}$. For each canonical local basis of quaternionic structure, there exist three 1-forms q_1, q_2 and q_3 on $\tilde{G}$ satisfying

$$(2.1) \qquad \tilde{\nabla}_{\tilde{X}} J_i = q_{i+2}(\tilde{X}) J_{i+1} - q_{i+1}(\tilde{X}) J_{i+2} \quad (i \bmod 3)$$

for each vector field $\tilde{X}$ on $\tilde{G}$ and $i = 1, 2, 3$.

Let M be a real hypersurface of a quaternionic Kähler manifold $\tilde{M}$, which is a Riemannian manifold equipped with a quaternionic Kähler structure $\mathcal{J}$, and $\mathcal{N}$ be a unit normal vector field of M. We denote by $\mathcal{D}$ the maximal subbundle of TM which is invariant by $\mathcal{J}$: At a point $x \in M$ the subspace $\mathcal{D}_x$ is the maximal subspace of $T_x M$ with the property that $Jv \in \mathcal{D}_x$ for each $v \in \mathcal{D}_x$ and $J \in \mathcal{J}_x$. Let $\mathcal{D}^\perp$ denote the orthogonal complement of $\mathcal{D}$ in TM. It is a rank 3 vector subbundle of TM. By using a canonical local basis $\{J_1, J_2, J_3\}$ of $\mathcal{J}$ over an open subset $\tilde{G}$ of $\tilde{M}$ containing x, we find that $\mathcal{D}_x^\perp$ is the real linear subspace of $T_x M$ spanned by $J_1 \mathcal{N}, J_2 \mathcal{N}, J_3 \mathcal{N}$. On an open set $G = \tilde{G} \cap M$ we set $\xi_i = -J_i \mathcal{N}$ and define $\phi_i : TM \to TM$ by $\phi_i = \pi \circ J_i|_{TM}$ for $i = 1, 2, 3$, where $\pi : T\tilde{M}|_M \to TM$ is the canonical projection. Then the following identities hold on G for $i = 1, 2, 3$:

$$(2.2) \qquad \begin{aligned} &\phi_i \xi_i = 0, \quad \phi_i \xi_{i+1} = \xi_{i+2}, \quad \phi_i \xi_{i+2} = -\xi_{i+1}, \\ &\phi_i \circ \phi_{i+1}|_{\mathcal{D}} = \phi_{i+2}|_{\mathcal{D}} = -\phi_{i+1} \circ \phi_i|_{\mathcal{D}} \quad (i \bmod 3). \end{aligned}$$

Let ∇ denote the Riemannian connection of M and A the shape operator of M with respect to $\mathcal{N}$. It follows from (2.1) and the Weingarten formula for a real hypersurface M that

$$(2.3) \qquad \nabla_X \xi_i = q_{i+2}(X) \xi_{i+1} - q_{i+1}(X) \xi_{i+2} + \phi_i A X$$

for each vector field X on G.

We call a connected quaternionic Kähler manifold $\tilde{M}$ a *quaternionic space form* of quaternionic sectional curvature $c \ (\in \mathbf{R})$ if the Riemannian sectional curvature of $\tilde{M}$ is equal to c for all tangent 2-planes spanned by $u \in T_{\tilde{x}} \tilde{M}$ and Ju with $J \in \mathcal{J}_{\tilde{x}}$ at each point $\tilde{x} \in \tilde{M}$. The standard models of quaternionic space forms are a quaternionic projective space $HP^n(c)$ of quaternionic sectional curvature $c \ (> 0)$, a quaternionic Euclidean space H^n and a quaternionic hyperbolic space $HH^n(c)$ of quaternionic sectional curvature $c \ (<0)$.

In his paper [B], Berndt characterized curvature-adapted real hypersurfaces in quaternionic space forms. Following his results, when M is a real hypersurface in a non-flat quaternionic space form, the following three conditions are equivalent each other:

(1) M is curvature-adapted.
(2) The subbundle $\mathcal{D}$ is invariant by the shape operator of M.
(3) The subbundle $\mathcal{D}^\perp$ is invariant by the shape operator of M.

The above condition (3) guarantees the existence of an open and dense subset $\mathcal{U}_0$ of a curvature-adapted real hypersurface M with the following property: For every $x \in \mathcal{U}_0$ there exists a canonical local basis $\{J_1, J_2, J_3\}$ of $\mathcal{J}$ on a neighborhood $\tilde{G}$ of x in a non-flat quaternionic space form $\tilde{M}$ satisfying that $G(= \tilde{G} \cap M) \subset \mathcal{U}_0$ and ξ_i is a principal curvature vector of M at each $x \in G$. Moreover, each of the principal curvatures α_i associated with ξ_i is locally constant on G.

THEOREM A ([B], cf. [MP]).   (1) *A connected curvature-adapted real hyper-surface in a quaternionic projective space $HP^n(4)$ ($n \geqq 2$) is locally congruent to one of the following real hypersurfaces*:

(P₁) *A tube $P_1^k(r)$ of some radius $r \in (0, \pi/2)$ around the canonically (totally geodesic) embedded HP^k for some $k \in \{0, \ldots, n-1\}$.*

(P₂) *A tube $P_2(r)$ of some radius $r \in (0, \pi/4)$ around the canonically (totally geodesic) embedded complex projective space CP^n.*

(2) *A connected curvature-adapted real hypersurface with constant principal curvatures in a quaternionic hyperbolic space $HH^n(-4)$ ($n \geqq 2$) is locally congruent to one of the following real hypersurfaces*:

(H₁) *A tube $H_1^k(r)$ of some radius $r \in (0, \infty)$ around the canonically (totally geodesic) embedded HH^k for some $k \in \{0, \ldots, n-1\}$.*

(H₂) *A tube $H_2(r)$ of some radius $r \in (0, \infty)$ around the canonically (totally geodesic) embedded complex hyperbolic space CH^n.*

(H₃) *A horosphere H_3 in HH^n.*

(3) *Conversely, each of these modeled real hypersurfaces is a curvature-adapted real hypersurface with constant principal curvatures in the ambient manifold.*

We here list the principal curvatures and their multiplicities of these modeled real hypersurfaces.

	$P_1^k(r)$	$P_2(r)$	$H_1^k(r)$	$H_2(r)$	H_3
λ_1	$\cot r$	$\cot r$	$\coth r$	$\coth r$	1
λ_2	$-\tan r$	$-\tan r$	$\tanh r$	$\tanh r$	—
μ_1	$2 \cot 2r$	$2 \cot 2r$	$2 \coth 2r$	$2 \coth 2r$	2
μ_2	—	$-2 \tan 2r$	—	$2 \tanh 2r$	—
$m(\lambda_1)$	$4(n-k-1)$	$2(n-1)$	$4(n-k-1)$	$2(n-1)$	$4(n-1)$
$m(\lambda_2)$	$4k$	$2(n-1)$	$4k$	$2(n-1)$	—
$m(\mu_1)$	3	1	3	1	3
$m(\mu_2)$	—	2	—	2	—

Here λ_j is an eigenvalue of $A|_{\mathscr{D}}$ and μ_j is that of $A|_{\mathscr{D}^\perp}$, and $m(\cdot)$ denotes the multiplicity of corresponding eigenvalue.

For real hypersurfaces of type (P₁), (H₁) and (H₃) in Theorem A we find the following properties hold.

PROPOSITION 2.1 (cf. [P]).   *Every curvature-adapted real hypersurface M of type (P₁), (H₁) or (H₃) in the lists of Theorem A satisfies the following*:

(1) *The structure tensor ϕ_i and the shape operator A of M are commutative*: $\phi_i A = A\phi_i$ $(i = 1, 2, 3)$.

(2) *The covariant derivative of the shape operator A satisfies*

$$(2.4) \qquad (\nabla_X A)\, Y = \mp \sum_{i=1}^{3} \{\langle \phi_i X,\, Y\rangle \xi_i + \langle \xi_i,\, Y\rangle \phi_i X\},$$

where the double sign depends on the case that either the ambient space is $HP^n(4)$ or $HH^n(-4)$.

For real hypersurfaces of type (P_1) in HP^n this proposition is due to [P]. For real hypersurfaces of type (H_1) or (H_3) in HH^n we can check these properties by the same argument as for Lemma 4.2 in [P] and Theorem 4.3 in [NR].

In the last stage of this section we write down the Codazzi equation for a curvature adapted real hypersurface in a quaternionic projective space $HP^n(c)$ and a quaternionic hyperbolic space $HH^n(c)$, which will be usefull in the following sections:

$$(2.5) \quad (\nabla_X A)\, Y - (\nabla_Y A)X = \frac{c}{4}\sum_{i=1}^{3} \{\langle X, \xi_i\rangle \phi_i Y - \langle Y, \xi_i\rangle \phi_i X - 2\langle \phi_i X, Y\rangle \xi_i\}.$$

3.   Characterization of curvature-adapted real hypersurfaces

A smooth curve $\gamma = \gamma(s)$ on a Riemannian manifold N parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{X_1 = \dot\gamma, \ldots, X_d\}$ along γ and positive functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ which satisfy the following system of ordinary equations

$$(3.1) \qquad \nabla_{\dot\gamma} X_j(s) = -\kappa_{j-1}(s)X_{j-1}(s) + \kappa_j(s)X_{j+1}(s), \quad j = 1, \ldots, d,$$

where $X_0 \equiv X_{d+1} \equiv 0$ and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of N. Equation (3.1) is called the Frenet formula for a Frenet curve γ. The functions $\kappa_j(s)$ $(j = 1, \ldots, d-1)$ and the orthonormal frame $\{X_1, \ldots, X_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. We call a smooth curve a *helix of order d* when it is a Frenet curve of proper order r $(\leqq d)$ and all its curvatures are constant. A helix of order 1 is nothing but a geodesic. A helix of order 2 with the first curvature $\kappa_1 = \kappa$ is called a *circle of curvature κ*. A smooth curve γ parametrized by its arclength is a circle of curvature κ if and only if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma$.

Let ι denote the isometric immersion of a real hypersurface M into a non-flat quaternionic space form $\tilde M$. For a smooth curve γ on M we call the curve $\iota \circ \gamma$ on $\tilde M$ the *extrinsic shape* of γ. For simplicity we usually denote the curve $\iota \circ \gamma$ by γ. The aim of this section is to characterize curvature-adapted real hypersurfaces in non-flat quaternionic space forms in Theorem A by the extrinsic shape of some geodesics on these hypersurfaces.

THEOREM 3-1. *Let M be a connected real hypersurface of a quaternionic projective space HP^n. Then M is locally congruent to a real hypersurface of type (P_1) or (P_2) if and only if at each point $x \in M$ there exists an orthonormal basis $\{v_1, \ldots, v_{4n-4}\}$ of $\mathscr{D}_x$ such that all geodesics $\gamma_j = \gamma_j(s)$ on M with $\gamma_j(0) = x$ and $\dot{\gamma}_j(0) = v_j$ $(1 \leqq j \leqq 4n-4)$ are circles of positive curvature in HP^n.*

THEOREM 3-2. *Let M be a connected real hypersurface of a quaternionic hyperbolic space HH^n. Then M is locally congruent to a real hypersurface of type (H_1), (H_2) or (H_3) if and only if at each point $x \in M$ there exists an orthonormal basis $\{v_1, \ldots, v_{4n-4}\}$ of $\mathscr{D}_x$ such that all geodesics $\gamma_j = \gamma_j(s)$ on M with $\gamma_j(0) = x$ and $\dot{\gamma}_j(0) = v_j$ $(1 \leqq j \leqq 4n-4)$ are circles of positive curvature in HH^n.*

We shall prove these theorems simultaneously. In this section, let $\tilde{M}$ denote a non-flat quaternionic space form. Namely, $\tilde{M}$ denotes either a quaternionic projective space or a quaternionic hyperbolic space. We denote by $\tilde{\nabla}$ the Riemannian connection of $\tilde{M}$. We use the following technical results due to Berndt [B] on curvature-adapted real hypersurfaces in a non-flat quaternionic space form.

PROPOSITION B ([B]). *Let M be a curvature-adapted real hypersurface in a quaternionic space form of quaternionic sectional curvature c $(\neq 0)$. We choose a canonical local basis of $\mathscr{J}$ on $\tilde{G}$ such that each ξ_i is a principal curvature vector; $A\xi_i = \alpha_i \xi_i$.*
   (1) At each point $x \in G = \tilde{G} \cap M$, for each principal curvature vector $v \in \mathscr{D}_x$ with $Av = \lambda v$,

$$(2\lambda - \alpha_i)A\phi_i v = \left(\lambda\alpha_i + \frac{c}{2}\right)\phi_i v,$$

holds for every $i = 1, 2, 3$.
   (2) $\alpha_i = \alpha_j$ or $q_k|\mathscr{D}_x = 0$ for all distinct $i, j, k \in \{1, 2, 3\}$.
   (3) If $\alpha_i \neq \alpha_{i+1} = \alpha_{i+2}$, then $q_{i+1}(\xi_i) = q_{i+2}(\xi_i) = 0$ $(i \bmod 3)$.

["*Only if*" *part of the proof of Theorems* 3.1 *and* 3.2.]
   In the first place we prove the only if part. Let M be a real hypersurface in a non-flat quaternionic space form $\tilde{M}$ satisfying the following condition $(*)$:

   At each point $x \in M$ there exists an orthonormal basis $\{v_1, \ldots, v_{4n-4}\}$
$(*)$ of $\mathscr{D}_x$ such that all geodesics $\gamma_j = \gamma_j(s)$ on M with $\gamma_j(0) = x$ and $\dot{\gamma}_j(0) = v_j$ $(1 \leqq j \leqq 4n-4)$ are circles of positive curvature in $\tilde{M}$.

By this condition the geodesic γ_j whose initial vector is v_j satisfies

(3.2) $$\tilde{\nabla}_{\dot{\gamma}_j}(\tilde{\nabla}_{\dot{\gamma}_j}\dot{\gamma}_j) = -\kappa_j^2 \dot{\gamma}_j$$

for some positive constant κ_j.
   On the other hand, from the formulae of Gauss and Weingarten;

$$\tilde{\nabla}_X Z = \nabla_X Z + \langle AX, Z \rangle \mathcal{N} \quad \text{and} \quad \tilde{\nabla}_X \mathcal{N} = -AX$$

we have

$$(3.3) \qquad \tilde{\nabla}_{\dot\gamma_j}(\tilde{\nabla}_{\dot\gamma_j}\dot\gamma_j) = \langle (\nabla_{\dot\gamma_j} A)\dot\gamma_j, \dot\gamma_j \rangle \mathcal{N} - \langle A\dot\gamma_j, \dot\gamma_j \rangle A\dot\gamma_j.$$

By comparing the tangential components of the equalities (3.2) and (3.3) we find

$$\langle A\dot\gamma_j(s), \dot\gamma_j(s) \rangle A\dot\gamma_j(s) = \kappa_j^2 \dot\gamma_j(s),$$

in particular, at $s = 0$

$$\langle Av_j, v_j \rangle Av_j = \kappa_j^2 v_j.$$

By noticing that $\kappa_j \neq 0$ we get

$$(3.4) \qquad Av_j = \kappa_j v_j \quad \text{or} \quad Av_j = -\kappa_j v_j \quad \text{for } j = 1, \ldots, 4n - 4.$$

This implies that $\mathcal{D}_x$ is invariant by the shape operator A. Since $x \in M$ is arbitrary, we find that our real hypersurface M is curvature-adapted to $\tilde{M}$. When $\tilde{M}$ is a quaternionic projective space, we obtain M is locally congruent to a real hypersurface of type (P_1) or (P_2) by Berndt's classification.

What we have to do is to prove that when $\tilde{M}$ is a quarternionic hyperbolic space all principal curvatures of our curvature-adapted real hypersurface M are constant on M. Consider an open and dense subset $\mathcal{U}$ of M given by

$$\mathcal{U} = \left\{ x \in \mathcal{U}_0 \,\middle|\, \begin{array}{l} \text{the multiplicity of each principal curvature of } M \\ \text{in } \boldsymbol{HH}^n \text{ is constant on some neighborhood } \mathcal{V}_x \text{ of } x \end{array} \right\}.$$

Then all principal curvatures are differentiable on $\mathcal{U}$. For each point $x \in \mathcal{U}$, if we choose sufficiently small neighborhood $\mathcal{V}_x$, we can choose principal curvature vectors to be smooth and the principal curvatures α_i $(i = 1, 2, 3)$ associated with ξ_i are constant on this neighborhood.

At a fixed point $x \in \mathcal{U}$, we denote by $\lambda_1, \lambda_2, \ldots, \lambda_g$ the distinct principal curvatures at x, which can be extended to differentiable functions on $\mathcal{V}_x$. Since the principal curvatures associated with ξ_i $(i = 1, 2, 3)$ are constant on $\mathcal{V}_x$, we shall prove that each λ_l associated with $A|_{\mathcal{D}}$ is constant on some neighborhood of x. By (3.4) there is j_l such that either $\lambda_l(x) = \kappa_{j_l}$ or $\lambda_l(x) = -\kappa_{j_l}$ holds. We only treat the former case; $Av_{j_l} = \kappa_{j_l} v_{j_l}$. Since the geodesic γ_{j_l} with the initial vector $v_{j_l} \in T_x M$ on M is a circle of curvature κ_{j_l} on $\tilde{M}$, we have $\lambda_l(\gamma_{j_l}(s)) = \kappa_{j_l}$ for each s, hence find $v_{j_l}\lambda_l = 0$. As A is symmetric, we also have

$$(3.5) \qquad \langle (\nabla_{v_{j_l}} A)v_k, v_{j_l} \rangle = \langle v_k, (\nabla_{v_{j_l}} A)v_{j_l} \rangle, \quad 1 \leqq k \neq j_l \leqq 4n - 4.$$

In order to compute (3.5) easily, we extend the tangent vectors $v_k, v_{j_l} \in T_x M$ to vector fields V_k, V_{j_l} on some sufficiently small neighborhood $\mathcal{W}_x \, (\subset \mathcal{V}_x)$ in the following manner: We denote by W_{j_l} the smooth vector field on $\mathcal{V}_x$ defined by parallel displacement of the vector v_{j_l} along each geodesic through x. Although

W_{j_l} is not principal on $\mathscr{V}_x$ in general, we have $AW_{j_l}(\gamma_{j_l}(s)) = \kappa_{j_l}W_{j_l}(\gamma_{j_l}(s))$. We define a vector field on $\mathscr{V}_x$ by

$$U_{j_l} = \left(\prod_{\substack{1 \leqq m \leqq g \\ m \neq l}} (A - \lambda_m I) \right) W_{j_l}.$$

Clearly, at each point $y \in \mathscr{V}_x$ the vector $U_{j_l}(y)$ is $\prod_{m \neq l}(\lambda_l - \lambda_m)$ times of the projection of $W_{j_l}(y)$ onto the eigenspace corresponding to $\lambda_l(y)$. Hence $AU_{j_l} = \lambda_l U_{j_l}$ on $\mathscr{V}_x$. We choose a neighborhood $\mathscr{W}_x$ ($\subset \mathscr{V}_x$) of x such that every U_{j_l} does not vanish on it, and set $V_{j_l} = U_{j_l}/\|U_{j_l}\|$. Our construction guarantees that $AV_{j_l} = \lambda_l V_{j_l}$ on $\mathscr{W}_x$ and $V_{j_l}(x) = v_{j_l}$, and that the integral curve of V_{j_l} through the point x is a geodesic on M. In particular, we obtain $(\nabla_{V_{j_l}} V_{j_l})(x) = 0$. For v_k we choose a section V_k of $\mathscr{D}|_{\mathscr{W}_x}$ with $V_k(x) = v_k$.

Since the Codazzi equation (2.5) implies $\langle (\nabla_X A)Y, Z \rangle = \langle (\nabla_Y A)X, Z \rangle$ for arbitrary sections X, Y, Z of $\mathscr{D}$, we find at the point x

$$\begin{aligned}
\text{(The left hand side of (3.5))} &= \langle (\nabla_{v_k} A)v_{j_l}, v_{j_l} \rangle \\
&= \langle (\nabla_{V_k} A)V_{j_l}, V_{j_l} \rangle_x \\
&= \langle (V_k \lambda_l)V_{j_l} + (\lambda_l I - A)\nabla_{V_k} V_{j_l}, V_{j_l} \rangle_x \\
&= v_k \lambda_l.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
\text{(The right hand side of (3.5))} &= \langle V_k, (\nabla_{V_{j_l}} A)V_{j_l} \rangle_x \\
&= \langle V_k, \nabla_{V_{j_l}}(\lambda_l V_{j_l}) - A\nabla_{V_{j_l}} V_{j_l} \rangle_x \\
&= \langle v_k, (v_{j_l} \lambda_l)v_{j_l} \rangle = 0.
\end{aligned}$$

These imply that $X\lambda_l = 0$ for every $X \in \mathscr{D}_x$.

Next, we shall prove $\xi_i(x)\lambda_l = 0$ ($i = 1, 2, 3$). We may suppose either one of the following conditions holds:
(I) $2\lambda_l - \alpha_i \neq 0$ for every $i = 1, 2, 3$ on some neighborhood of x,
(II) $2\lambda_l - \alpha_i \equiv 0$ holds on a neighborhood of x for some i,
because the set of points $y \in \mathscr{U}$ with such a property is a dense open subset of M. Since α_i is constant on $\mathscr{V}_x$, we have only to study the case (I). In this case, we obtain from Proposition B

$$A\phi_i V_{j_l} = \left(\frac{\alpha_i \lambda_{j_l} - 2}{2\lambda_{j_l} - \alpha_i} \right) \phi_i V_{j_l}$$

on some neighborhood $\mathscr{W}_x'$ ($\subset \mathscr{W}_x$) of x. This, together with Proposition B and (2.3), yields

$$(\nabla_{\xi_i} A) V_{j_l} - (\nabla_{V_{j_l}} A) \xi_i$$

$$= \nabla_{\xi_i} (A V_{j_l}) - A \nabla_{\xi_i} V_{j_l} - \nabla_{V_{j_l}} (\alpha_i \xi_i) + A \nabla_{V_{j_l}} \xi_i$$

$$= \nabla_{\xi_i} (\lambda_l V_{j_l}) - A \nabla_{\xi_i} V_{j_l} - \alpha_i \phi_i A V_{j_l} + A \phi_i A V_{j_l}$$

$$+ q_{i+2} (V_{j_l})(\alpha_{i+1} - \alpha_i) \xi_{i+1} - q_{i+1} (V_{j_l})(\alpha_{i+2} - \alpha_i) \xi_{i+2}$$

$$= (\xi_i \lambda_l) V_{j_l} + (\lambda_l I - A) \nabla_{\xi_i} V_{j_l} - \lambda_l \left(\alpha_i - \frac{\alpha_i \lambda_l - 2}{2\lambda_l - \alpha_i} \right) \phi_i V_{j_l}.$$

On the other hand, (2.5) yields

$$\langle (\nabla_{\xi_i} A) V_{j_l} - (\nabla_{V_{j_l}} A) \xi_i, V_{j_l} \rangle = 0.$$

By combining these two equalities we find $\xi_i \lambda_l = 0$ for every i. Consequently, λ_l is constant on $\mathscr{W}'_x$.

Thus we know that every principal curvature of M is locally constant on an open and dense subset of M. Since M is assumed to be connected, each principal curvature is constant on M. We therefore find by Berndt's classification that our real hypersurface M in a quaternionic hyperbolic space is locally congruent to either a real hypersurface of type of (H$_1$), (H$_2$) or (H$_3$).
["*If*" *part of the proof of Theorems* 3.1 *and* 3.2.]

In the second place we prove the if part. We shall prove each member of real hypersurfaces in the lists of Theorem A satisfies the condition (∗) case by case. At an arbitrary point $x \in M$ we choose orthonomal principal curvature vectors $v_1, \ldots, v_{4n-4} \in \mathscr{D}_x$. We denote by λ_j the principal curvature associated with v_j (i.e. $A v_j = \lambda_j v_j$) and by γ_j the geodesic with initial vector v_j.

First, we study the case that M is a real hypersurface of type (P$_1$), (H$_1$) or (H$_3$) in a non-flat quaternionic space form. We shall show that $\dot{\gamma}_j$ is a principal curvature vector field along γ_j for every j. It follows from Proposition 2.1(2) that

$$\nabla_{\dot{\gamma}_j} \|A \dot{\gamma}_j - \lambda_j \dot{\gamma}_j\|^2 = \nabla_{\dot{\gamma}_j} \langle A \dot{\gamma}_j, A \dot{\gamma}_j \rangle - 2\lambda_j \nabla_{\dot{\gamma}_j} \langle A \dot{\gamma}_j, \dot{\gamma}_j \rangle$$

$$= 2 \langle (\nabla_{\dot{\gamma}_j} A) \dot{\gamma}_j, A \dot{\gamma}_j \rangle - 2\lambda_j \langle (\nabla_{\dot{\gamma}_j} A) \dot{\gamma}_j, \dot{\gamma}_j \rangle$$

$$= \mp 2 \left\langle \sum_{i=1}^{3} \{ \langle \phi_i \dot{\gamma}_j, \dot{\gamma}_j \rangle \xi_i + \langle \xi_i, \dot{\gamma}_j \rangle \phi_i \dot{\gamma}_j \}, A \dot{\gamma}_j - \lambda_j \dot{\gamma}_j \right\rangle$$

$$= \mp 2 \sum_{i=1}^{3} \langle \xi_i, \dot{\gamma}_j \rangle \langle \phi_i \dot{\gamma}_j, A \dot{\gamma}_j \rangle.$$

Since $A\phi_i = \phi_i A$, we find $\langle \phi_i \dot{\gamma}_j, A \dot{\gamma}_j \rangle = \langle A \phi_i \dot{\gamma}_j, \dot{\gamma}_j \rangle = -\langle A \dot{\gamma}_j, \phi_i \dot{\gamma}_j \rangle$, which leads us to $\langle \phi_i \dot{\gamma}_j, A \dot{\gamma}_j \rangle = 0$, and hence to $\nabla_{\dot{\gamma}_j} \|A \dot{\gamma}_j - \lambda_j \dot{\gamma}_j\|^2 = 0$. As $A \dot{\gamma}_j(0) = \lambda_j \dot{\gamma}_j(0)$, this shows that $A \dot{\gamma}_j(s) = \lambda_j \dot{\gamma}_j(s)$ for every s. By use of the formulae of Gauss and Weingarten, we can see that $\tilde{\nabla}_{\dot{\gamma}_j} \tilde{\nabla}_{\dot{\gamma}_j} \dot{\gamma}_j = \lambda_j \tilde{\nabla}_{\dot{\gamma}_j} \mathscr{N} = -\lambda_j^2 \dot{\gamma}_j$, which means that the extrinsic shape of the geodesic γ_j is a circle of curvature $|\lambda_j|$ in a non-flat quaternionic space form.

Next we study the case that M is a real hypersurface of type (P_2) in HP^n. We decompose the tangent bundle TM into the bundles of principal curvature vectors:

$$TM = \mathscr{D} \oplus \mathscr{D}^\perp = (\mathscr{F}_{\lambda_1} \oplus \mathscr{F}_{\lambda_2}) \oplus (\mathscr{F}_{\mu_1} \oplus \mathscr{F}_{\mu_2}).$$

Here, when the ambient space is $HP^n(4)$, the principal curvatures are $\lambda_1 = \cot r$, $\lambda_2 = -\tan r$, $\mu_1 = 2\cot 2r$, $\mu_2 = -2\tan 2r$, and $\mathscr{F}_\nu$ denotes the subbundle of principal curvature vectors associated with ν. For simplicity we choose canonical local basis of $\mathscr{J}$ on $\tilde{G}$ in Proposition B so that $\alpha_1 = \mu_1$ and $\alpha_2 = \alpha_3 = \mu_2$. It follows from Proposition B and (2.2) that

$$(3.6) \quad \begin{cases} \phi_1(\mathscr{F}_{\lambda_1}) = \mathscr{F}_{\lambda_1}, & \phi_1(\mathscr{F}_{\lambda_2}) = \mathscr{F}_{\lambda_2}, & \phi_1(\mathscr{F}_{\mu_1}) = \{0\}, & \phi_1(\mathscr{F}_{\mu_2}) = \mathscr{F}_{\mu_2}; \\ \phi_2(\mathscr{F}_{\lambda_1}) = \mathscr{F}_{\lambda_2}, & \phi_2(\mathscr{F}_{\lambda_2}) = \mathscr{F}_{\lambda_1}, & \phi_2(\mathscr{F}_{\mu_1}) \subset \mathscr{F}_{\mu_2}, & \phi_2(\mathscr{F}_{\mu_2}) = \mathscr{F}_{\mu_1}; \\ \phi_3(\mathscr{F}_{\lambda_1}) = \mathscr{F}_{\lambda_2}, & \phi_3(\mathscr{F}_{\lambda_2}) = \mathscr{F}_{\lambda_1}, & \phi_3(\mathscr{F}_{\mu_1}) \subset \mathscr{F}_{\mu_2}, & \phi_3(\mathscr{F}_{\mu_2}) = \mathscr{F}_{\mu_1}. \end{cases}$$

Since $\alpha_1 \neq \alpha_2 = \alpha_3$, again by using Proposition B we have $q_2|_{\mathscr{D}_x} = q_3|_{\mathscr{D}_x} = 0$ and $q_2(\xi_1(x)) = q_3(\xi_1(x)) = 0$ at each point $x \in G = \tilde{G} \cap M$.

We shall verify that both $\mathscr{F}_{\lambda_1} \oplus \boldsymbol{R}\xi_1$ and $\mathscr{F}_{\lambda_2} \oplus \boldsymbol{R}\xi_1$ over G are integrable and moreover that each leaf of these local distributions is a totally geodesic submanifold of M. Since by (2.3) we have

$$\nabla_{\xi_1}\xi_1 = q_3(\xi_1)\xi_2 - q_2(\xi_1)\xi_3 + \phi_1 A\xi_1 = \alpha_1 \phi_1 \xi_1 = 0,$$

we shall check that $\nabla_X \xi_1$, $\nabla_{\xi_1} X$, $\nabla_X Y$ are sections of $\mathscr{F}_{\lambda_j} \oplus \boldsymbol{R}\xi_1$ for all sections X, Y of $\mathscr{F}_{\lambda_j}$. First, we see that

$$\nabla_X \xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX = \lambda_j \phi_1 X$$

hence $\nabla_X \xi_1$ is a section of $\mathscr{F}_{\lambda_j}$. Next we have

$$\begin{aligned} (\nabla_{\xi_1} A)X - (\nabla_X A)\xi_1 &= \nabla_{\xi_1}(AX) - A\nabla_{\xi_1} X - \nabla_X(A\xi_1) + A\nabla_X \xi_1 \\ &= (\lambda_j I - A)\nabla_{\xi_1} X - \lambda_j(\alpha_1 - \lambda_j)\phi_1 X \\ &= (\lambda_j I - A)\nabla_{\xi_1} X + \phi_1 X. \end{aligned}$$

On the other hand, the Codazzi equation (2.5) leads us to

$$(\nabla_{\xi_1} A)X - (\nabla_X A)\xi_1 = \sum_{i=1}^{3} \{\langle \xi_1, \xi_i \rangle \phi_i X - \langle X, \xi_i \rangle \phi_i \xi_1 - 2\langle \phi_i \xi_1, X \rangle \xi_i\} = \phi_1 X.$$

Thus we obtain $(\lambda_j I - A)\nabla_{\xi_1} X = 0$, hence $\nabla_{\xi_1} X$ is a section of $\mathscr{F}_{\lambda_j}$. Finally, we shall check $\nabla_X Y - \langle \nabla_X Y, \xi_1 \rangle \xi_1$. We have

$$\begin{aligned} A(\nabla_X Y - \langle \nabla_X Y, \xi_1 \rangle \xi_1) &= \nabla_X(AY) - (\nabla_X A)Y - \alpha_1 \langle \nabla_X Y, \xi_1 \rangle \xi_1 \\ &= \lambda_j(\nabla_X Y - \langle \nabla_X Y, \xi_1 \rangle \xi_1) \\ &\quad - (\nabla_X A)Y + (\lambda_j - \alpha_1)\langle \nabla_X Y, \xi_1 \rangle \xi_1. \end{aligned}$$

Here we see that

$$\langle(\nabla_X A)Y, \xi_1\rangle = \langle Y, (\nabla_X A)\xi_1\rangle = \langle\nabla_X(A\xi_1) - A(\nabla_X\xi_1), Y\rangle$$
$$= \langle(\alpha_1 I - A)\nabla_X\xi_1, Y\rangle = \langle\nabla_X\xi_1, (\alpha_1 I - A)Y\rangle$$
$$= (\alpha_1 - \lambda_j)\langle\nabla_X\xi_1, Y\rangle = (\lambda_j - \alpha_1)\langle\nabla_X Y, \xi_1\rangle,$$

hence $\langle-(\nabla_X A)Y + (\lambda_j - \alpha_1)\langle\nabla_X Y, \xi_1\rangle\xi_1, \xi_1\rangle = 0$. For each section Z which is orthogonal to ξ_1 we get by making use of (3.6)

$$\langle(\nabla_X A)Y, Z\rangle = \langle Y, (\nabla_X A)Z\rangle$$
$$= \left\langle (\nabla_Z A)X + \sum_{i=1}^{3}\{\langle X, \xi_i\rangle\phi_i Z - \langle Z, \xi_i\rangle\phi_i X - 2\langle\phi_i X, Z\rangle\xi_i\}, Y \right\rangle$$
$$= \langle(\nabla_Z A)X - \langle Z, \xi_2\rangle\phi_2 X - \langle Z, \xi_3\rangle\phi_3 X, Y\rangle$$
$$= \langle\nabla_Z(AX) - A\nabla_Z X, Y\rangle$$
$$= \langle(\lambda_j I - A)\nabla_Z X, Y\rangle$$
$$= \langle\nabla_Z X, (\lambda_j I - A)Y\rangle = 0.$$

Therefore we find that $\nabla_X Y - \langle\nabla_X Y, \xi_1\rangle\xi_1$ is a section of $\mathscr{F}_{\lambda_j}$, so that $\nabla_X Y$ is a section of $\mathscr{F}_{\lambda_j} \oplus \mathbf{R}\xi_1$. Consequently, the local distribution $\mathscr{F}_{\lambda_j} \oplus \mathbf{R}\xi_1$ is integrable and moreover each leaf L of $\mathscr{F}_{\lambda_j} \oplus \mathbf{R}\xi_1$ is a totally geodesic submanifold of our real hypersurface M for $j = 1, 2$. When $j = 1$, the manifold L is locally congruent to a geodesic sphere of radius r in a complex projective space $\mathbf{C}P^n(4)$ which is a totally complex totally geodesic submanifold of $\mathbf{H}P^n$, and when $j = 2$, the manifold L is locally congruent to a geodesic sphere of radius $\pi/2 - r$ in $\mathbf{C}P^n(4)$.

We here recall the extrinsic shape of geodesics on a geodesic sphere in a complex projective space (see p. 540 of [MO]). Every geodesic γ on a leaf L of $\mathscr{F}_{\lambda_j} \oplus \mathbf{R}\xi_1$ with the initial vector $\dot{\gamma}(0) \in \mathscr{F}_{\lambda_j}$ is a circle of curvature $|\lambda_j|$ in $\mathbf{C}P^n$. Therefore, at each point $x \in M$ taking an orthonormal basis $\{v_1, \ldots, v_{2n-2}\}$ of $(\mathscr{F}_{\lambda_1})_x$ and an orthonormal basis $\{v_{2n-1}, \ldots, v_{4n-4}\}$ of $(\mathscr{F}_{\lambda_2})_x$, we find that a real hypersurface of type (P_2) in a quaternionic projective space satisfies the condition $(*)$.

By just the same way as in the above discussion we can check that a real hypersurface of type (H_2) in a quaternionic hyperbolic space satisfies the condition $(*)$, and complete our proof for Theorems 3-1 and 3-2.

4. Geodesics on geodesic spheres in a quaternionic space form

In the preceding section, we characterized all curvature-adapted real hypersurfaces with constant principal curvatures in non-flat quaternionic space forms in terms of the extrinsic shape of geodesics of directions of principal curvature vectors. So, next it is natural to consider the extrinsic shape of all geodesics. In this context we devote the rest of this paper to study all geodesics on geodesic spheres in a quaternionic projective space, and on geodesic spheres, tubes around

quaternionic hyperplane and horospheres in a quaternionic hyperbolic space. We shall start by observing an invariant for geodesics on these real hypersurfaces.

Let M be a real hypersurface of type (P_1), (H_1) or (H_3) in a non-flat quaternionic space form $\tilde{M}$. We denote by $\mathcal{N}$ a unit normal vector field on M and put $\xi_i = -J_i\mathcal{N}$ $(i = 1, 2, 3)$ for an arbitrary local basis $\{J_1, J_2, J_3\}$ of the quaternionic structure on $\tilde{M}$. For a tangent vector $v \in T_x M$ we define non-negative $\rho(v)$ by

$$\rho(v) = \sqrt{\langle v, \xi_1 \rangle^2 + \langle v, \xi_2 \rangle^2 + \langle v, \xi_3 \rangle^2},$$

which is the norm of the projection of v onto the space $\mathcal{D}_x^\perp$. We put $\rho_\gamma = \rho(\dot{\gamma})$ for a geodesic γ on M. By using Weingarten formula and the equalities $A\phi_i = \phi_i A$ $(i = 1, 2, 3)$ we find it is constant along γ: Indeed,

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\rho_\gamma^2 &= \langle \dot{\gamma}, q_3(\dot{\gamma})\xi_2 - q_2(\dot{\gamma})\xi_3 + \phi_1 A\dot{\gamma}\rangle\langle \dot{\gamma}, \xi_1 \rangle \\
&\quad + \langle \dot{\gamma}, q_1(\dot{\gamma})\xi_3 - q_3(\dot{\gamma})\xi_1 + \phi_2 A\dot{\gamma}\rangle\langle \dot{\gamma}, \xi_2 \rangle \\
&\quad + \langle \dot{\gamma}, q_2(\dot{\gamma})\xi_1 - q_1(\dot{\gamma})\xi_2 + \phi_3 A\dot{\gamma}\rangle\langle \dot{\gamma}, \xi_3 \rangle \\
&= \langle \dot{\gamma}, \phi_1 A\dot{\gamma}\rangle\langle \dot{\gamma}, \xi_1 \rangle + \langle \dot{\gamma}, \phi_2 A\dot{\gamma}\rangle\langle \dot{\gamma}, \xi_2 \rangle + \langle \dot{\gamma}, \phi_3 A\dot{\gamma}\rangle\langle \dot{\gamma}, \xi_3 \rangle = 0.
\end{aligned}$$

We shall call this constant ρ_γ (fibre) *structure torsion* of γ. Clearly it satisfies $0 \leqq \rho_\gamma \leqq 1$. When M is one of a geodesic sphere in a quaternionic projective space, a geodesic sphere, a tube around quaternionic hyperplane and a horosphere in a quaternionic hyperbolic space, both of each vector $v \in \mathcal{D}$ and each vector $u \in \mathcal{D}^\perp$ are principal curvature vectors. So the structure torsion plays an important role in classifying geodesics on these real hypersurfaces.

First we study geodesics on a geodesic sphere $M = P_1^0(r)$ of radius r in a quaternionic projective space $HP^n(4)$ of quaternionic sectional curvature 4. Let $\tilde{\nabla}$ and ∇ denote the Riemannian connections of $HP^n(4)$ and M, respectively. For a geodesic γ we also denote the curve $\iota \circ \gamma$ by γ for simplicity. As a first step, we have

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma} + \langle A\dot{\gamma}, \dot{\gamma}\rangle\mathcal{N} = \langle A\dot{\gamma}, \dot{\gamma}\rangle\mathcal{N}.$$

This shows that the curve $\iota \circ \gamma$ is a geodesic on HP^n when $\langle A\dot{\gamma}, \dot{\gamma}\rangle = \cot r - \rho_\gamma^2 \tan r = 0$. If it is not this case, we set as

$$\kappa_1 = |\langle A\dot{\gamma}, \dot{\gamma}\rangle| = |\cot r - \rho_\gamma^2 \tan r|,$$

$$X_2 = \begin{cases} \mathcal{N} & \text{if } \langle A\dot{\gamma}, \dot{\gamma}\rangle > 0, \\ -\mathcal{N} & \text{if } \langle A\dot{\gamma}, \dot{\gamma}\rangle < 0. \end{cases}$$

Differentiating X_2, we have $\tilde{\nabla}_{\dot{\gamma}}X_2 = -\kappa_1\dot{\gamma} + \kappa_2 X_3$ with

$$\kappa_2 = \sqrt{\|A\dot{\gamma}\|^2 - \langle A\dot{\gamma}, \dot{\gamma}\rangle^2} = \rho_\gamma\sqrt{1 - \rho_\gamma^2}\cot r,$$

$$X_3 = \begin{cases} \dfrac{1}{\kappa_2}(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma}) & \text{if } \langle A\dot{\gamma}, \dot{\gamma}\rangle > 0, \\[2mm] -\dfrac{1}{\kappa_2}(\langle A\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} - A\dot{\gamma}) & \text{if } \langle A\dot{\gamma}, \dot{\gamma}\rangle < 0. \end{cases}$$

Thus, when $\rho_\gamma = 0$ and $\rho_\gamma = 1$, we see that the curve $\iota \circ \gamma$ is a circle of curvature $\cot r$ and $2|\cot 2r|$, respectively.

In order to continue our calculation we set $\phi(v) = \sum_{i=1}^{3} \langle v, \xi_i\rangle\phi_i(v)$ for a tangent vector $v \in TM$, which does not depend on the choice of canonical local basis. One can easily compute that $\|\phi(v)\|^2 = \rho^2(v) - \rho^4(v)$. By making use of (2.3) we obtain $\tilde{\nabla}_{\dot{\gamma}}X_3 = -\kappa_2 X_2 + \kappa_3 X_4$ and $\tilde{\nabla}_{\dot{\gamma}}X_4 = -\kappa_3 X_3$ with $\kappa_3 = \cot r$ and

$$X_4 = \begin{cases} \dfrac{1}{\rho_\gamma\sqrt{1 - \rho_\gamma^2}}\phi(\dot{\gamma}) & \text{if } \langle A\dot{\gamma}, \dot{\gamma}\rangle > 0, \\[3mm] \dfrac{-1}{\rho_\gamma\sqrt{1 - \rho_\gamma^2}}\phi(\dot{\gamma}) & \text{if } \langle A\dot{\gamma}, \dot{\gamma}\rangle < 0. \end{cases}$$

Thus we get

PROPOSITION 4.1. *For each geodesic γ on a geodesic sphere $P_1^0(r)$ of radius r $(0 < r < \pi/2)$ in $\boldsymbol{HP}^n(4)$, the curve $\iota \circ \gamma$ lies on a totally geodesic $\boldsymbol{CP}^2(4)$. Moreover, the extrinsic shape of γ is as follows*:

(1) *Suppose the radius r satisfies $\pi/4 \leqq r < \pi/2$. When $\rho_r = \cot r$, the curve $\iota \circ \gamma$ is a geodesic.*

(2) *When $r \neq \pi/4$, the curve $\iota \circ \gamma$ is a circle of curvature $2|\cot 2r|$ if $\rho_\gamma = 1$. This circle lies on a totally geodesic $\boldsymbol{CP}^1(4)$.*

(3) *If γ has null structure torsion, the curve $\iota \circ \gamma$ is a circle of curvature $\cot r$. This circle lies on a totally geodesic $\boldsymbol{RP}^2(1)$.*

(4) *Generally, if $\rho_\gamma \neq 0, 1$, then the curve $\iota \circ \gamma$ is a helix of proper order 4 whose curvatures are described as*

$$\kappa_1 = |\cot r - \rho_\gamma^2\tan r|, \quad \kappa_2 = \rho_r\sqrt{1 - \rho_\gamma^2}\tan r, \quad \kappa_3 = \cot r.$$

Every geodesic γ on $P_1^0(r)$ is a simple curve.

We call a smooth curve γ closed if there exists s_0 $(\neq 0)$ with $\gamma(s + s_0) = \gamma(s)$ for all s. The minimum positive s_0 with such a property is called the length of γ and is denoted by $\text{length}(\gamma)$. When γ is not closed we put $\text{length}(\gamma) = \infty$. As a direct consequence of this proposition we obtain the following with the aid of Theorem 2.5 in [AMY].

PROPOSITION 4.2. *Let γ be a geodesic on a geodesic sphere $P_1^0(r)$ of radius r $(0 < r < \pi/2)$ in $\boldsymbol{HP}^n(4)$.*

(1) *If the structure torsion of γ is 1, then γ is closed and its length is $\pi\sin 2r$.*

(2) *If γ has null structure torsion, then γ is also closed and its length is* $2\pi \sin r$.

(3) *When the structure torsion ρ_γ of γ satisfies $0 < \rho_\gamma < 1$, it is closed if and only if*

$$\rho_\gamma = \frac{q}{\sin r \sqrt{p^2 \tan^2 r + q^2}}$$

with some relatively prime positive integers p and q with $q < p \tan^2 r$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}, & \text{if } pq \text{ is even} \\ \pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}, & \text{if } pq \text{ is odd}. \end{cases}$$

We now make mention of the length spectrum of geodesic spheres in $\boldsymbol{HP}^n$. We denote by $\text{Geod}(N)$ the moduli space of geodesics on a Riemannian manifold N, which is the set of all congruency classes of geodesics on N under the action of the isometry group of N. The length spectrum $\mathscr{L}_N : \text{Geod}(N) \to \boldsymbol{R} \cup \{\infty\}$ is defined by $\mathscr{L}_N([\gamma]) = \text{length}(\gamma)$. We also call the image $\text{Lspec}(N) = \mathscr{L}_N(\text{Geod}(N)) \cap \boldsymbol{R}$ the length spectrum of N. For a length spectrum $\lambda \in \text{Lspec}(N)$ we call the cardinality $m_N(\lambda)$ of the set $\mathscr{L}_N^{-1}(\lambda)$ the *multiplicity* of λ. When the multiplicity of a length spectrum is 1 we say it is *simple*. For congruency of geodesics on a geodesic sphere in $\boldsymbol{HP}^n$ we obtain the following.

PROPOSITION 4.3. *On a geodesic sphere M in a quaternionic projective space, two geodesics are congruent with respect to the isometry group of M if and only if their structure torsions coincide.*

Proof. Let γ_1 and γ_2 be geodesics on M. Since the isometric embedding ι is equivariant, for each isometry φ on M there is an isometry $\tilde{\varphi}$ on $\boldsymbol{HP}^n$ with $\tilde{\varphi} \circ \iota = \iota \circ \varphi$. This implies that $\iota \circ \gamma_1$ and $\iota \circ \gamma_2$ are congruent if γ_1 and γ_2 are congruent. Comparing their curvatures we find $\rho_{\gamma_1} = \rho_{\gamma_2}$ if γ_1 and γ_2 are congruent.

Conversely, when γ_1 and γ_2 have the same structure torsion ρ, their initial vectors are of the form $\dot{\gamma}_1(0) = \sqrt{1 - \rho^2}u_1 + \rho\eta_1$ and $\dot{\gamma}_2(0) = \sqrt{1 - \rho^2}u_2 + \rho\eta_2$ with unit tangent vectors $u_1, u_2 \in \mathscr{D}$ and $\eta_1, \eta_2 \in \mathscr{D}^\perp$. By the following Lemma 4.4 we can choose an isometry φ of M with $d\varphi_{\gamma_1(0)}(u_1) = u_2$ and $d\varphi_{\gamma_1(0)}(\eta_1) = \eta_2$. Hence, $d\varphi_{\gamma_1(0)}(\dot{\gamma}_1(0)) = \dot{\gamma}_2(0)$, so that $\varphi \circ \gamma_1(s) = \gamma_2(s)$ for every s.

LEMMA 4.4. *Let $M = P_1^0(r)$ be a geodesic sphere in $\boldsymbol{HP}^n$. For any tangent vectors $u_1 + \eta_1 \in T_xM = \mathscr{D}_x \oplus \mathscr{D}_x^\perp$ and $u_2 + \eta_2 \in T_yM = \mathscr{D}_y \oplus \mathscr{D}_y^\perp$ of M with $|u_1| = |u_2| = |\eta_1| = |\eta_2| = 1$ at arbitrary points x, y, there exists an isometry $\tilde{\varphi}$ of $\boldsymbol{HP}^n$ with*

i) $\tilde{\varphi}(M) = M$ and $\tilde{\varphi}(x) = y$,
ii) $d\tilde{\varphi}_x(u_1) = u_2$, $d\tilde{\varphi}_x(\eta_1) = \eta_2$, $d\tilde{\varphi}_x(\mathscr{N}_x) = \mathscr{N}_y$,
iii) $d\tilde{\varphi}_x(\mathscr{D}_x) = \mathscr{D}_y$ and $d\tilde{\varphi}_x(\mathscr{D}_x^\perp) = \mathscr{D}_y^\perp$.

Proof. Let $\varpi : S^{4n+3} \to HP^n$ denote the S^3-fibration of a unit sphere S^{4n+3} in a right vector space H^{n+1}. We decompose the tangent space $T_w S^{4n+3}$ of S^{4n+3} at w into the horizontal and vertical spaces: $T_w S^{4n+3} = \mathscr{H}_w S^{4n+3} \oplus \mathscr{V}_w S^{4n+3}$. They are given by

$$\mathscr{H}_w S^{4n+3} = \{(w, v) \in \{w\} \times H^{n+1} \mid \bar{w}_0 v_0 + \cdots + \bar{w}_n v_n = 0\},$$

$$\mathscr{V}_w S^{4n+3} = \{(w, w\lambda) \in \{w\} \times H^{n+1} \mid \lambda \in H, \mathrm{Re}(\lambda) = 0\}.$$

Here for $\lambda \in H = \{a_0 + a_1 j_1 + a_2 j_2 + a_3 j_3 \mid a_i \in R\}$ we denote by $\bar{\lambda}$ its quaternionic conjugate and by $\mathrm{Re}(\lambda)$ its real part, which is given as $(\lambda + \bar{\lambda})/2$. Let $\hat{M}$ denote a hypersurface in S^{4n+3} given by

$$\{w = (w_0, w_1, \ldots, w_n) \in H^{n+1} \mid |w_0| = \cos r, |w_1|^2 + \cdots + |w_n|^2 = \sin^2 r\},$$

where $|\lambda| = (\bar{\lambda}\lambda)^{1/2}$. We see that $\varpi^{-1}(M)$ is isometric to $\hat{M}$.

For simplicity we only treat the case $n = 2$ and $M = \varpi(\hat{M})$. At a point $w = (w_0, w_1, w_2) \in \hat{M}$ the tangent space of $\hat{M}$ is represented as

$$T_w \hat{M} = \{(w, v) \in \{w\} \times H^3 \mid \mathrm{Re}(\bar{w}_0 v_0) = \mathrm{Re}(\bar{w}_1 v_1 + \bar{w}_2 v_2) = 0\}.$$

We denote by $\hat{\mathscr{N}}_w \in (T_w \hat{M})^\perp \cap \mathscr{H}_w S^{11}$ the horizontal lift of the unit normal $\mathscr{N}$ at $\varpi(w)$, where $(T_w \hat{M})^\perp$ denotes the orthogonal complement of $T_w \hat{M}$ in $T_w S^{11}$. When $w = (\cos r, w_1, w_2)$, it is of the form

$$\hat{\mathscr{N}}_w = (w, (-\sin r, w_1 \cot r, w_2 \cot r)),$$

hence the horizontal lifs of $\mathscr{D}_{\varpi(w)}$ and $\mathscr{D}_{\varpi(w)}^\perp$ are of the following forms:

$$\hat{\mathscr{D}}_w = \{(w, (0, v_1, v_2)) \mid \bar{w}_1 v_1 + \bar{w}_2 v_2 = 0\},$$

$$\hat{\mathscr{D}}_w^\perp = \{(w, (-\sin r, w_1 \cot r, w_2 \cot r) \cdot \lambda) \mid \mathrm{Re}\,\lambda = 0\}.$$

Put $z = (\cos r, \sin r, 0)(\in \hat{M})$. For $(w, (0, v_1, v_2)) \in \hat{\mathscr{D}}_w$ the othogonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w_1(\sin r)^{-1} & v_1 \\ 0 & w_2(\sin r)^{-1} & v_2 \end{pmatrix} \in GL(3, H) \cong GL(12, R)$$

induces an isometry $\hat{\psi}$ of S^{4n+3} such that

 i) $\hat{\psi}(z) = w$ and $\hat{\psi}(\hat{M}) = \hat{M}$,
 ii) $d\hat{\psi}_z(\hat{\mathscr{D}}_z) = \hat{\mathscr{D}}_w$ and $d\hat{\psi}_z(\hat{\mathscr{N}}_z \cdot \lambda) = \hat{\mathscr{N}}_w \cdot \lambda$ for every $\lambda \in H$,
 iii) $d\hat{\psi}_z((z, (0, 0, 1))) = (w, (0, v_1, v_2))$.

Next for $v = a_1 j_1 + a_2 j_2 + a_3 j_3 \in H$ with $|v| = 1$ we choose an othogonal matrix

$$A = \begin{pmatrix} -a_1 & * & * \\ -a_2 & * & * \\ -a_3 & * & * \end{pmatrix} \in O(3)$$

and consider an othogonal matrix

$$
\begin{pmatrix}
1 & & & & 0 \\
& A & & & \\
& & 1 & & \\
& & & A & \\
0 & & & & I_4
\end{pmatrix} \in O(12),
$$

where $I_4 \in GL(4, \boldsymbol{R})$ denotes the identity matrix. This induces an isometry $\hat{\Psi}$ of S^{4n+3} such that

i) $\hat{\Psi}(z) = z$ and $\hat{\Psi}(\hat{M}) = \hat{M}$,

ii) $d\hat{\Psi}_z(\hat{\mathscr{D}}_z) = \hat{\mathscr{D}}_z$ and $d\hat{\Psi}_z(\hat{J}_1\hat{\mathscr{N}}_z) = \hat{\mathscr{N}}_z \cdot v$,

iii) $d\hat{\Psi}_z((z, (0, 0, 1))) = (z, (0, 0, 1))$,

where the real linear transformation $\hat{J}_i$ on $\boldsymbol{H}^3$ is defined by $\hat{J}_i(u) = u(-j_i)$. These guarantee the existence of an isometry $\tilde{\varphi}$ of $\boldsymbol{HP}^n$ with desirable conditions.

As a consequence of Proposition 4.2, for a geodesic sphere M of radius $2r/\sqrt{c}$ $(0 < r < \pi/2)$ in $\boldsymbol{HP}^n(c)$, we see that its length spectrum coinsides with the length spectrum of a geodesic sphere of radius $2r/\sqrt{c}$ in a complex projective space $\boldsymbol{CP}^n(c)$ of holomorphic sectional curvature c:

$$
\mathrm{Lspec}(M) = \left\{\frac{2\pi}{\sqrt{c}} \sin 2r\right\} \cup \left\{\frac{4\pi}{\sqrt{c}} \sin r\right\}
$$

$$
\cup \left\{4\pi\sqrt{\frac{1}{c}(p^2 \sin^2 r + q^2 \cos^2 r)} \;\middle|\;
\begin{array}{l}
p \text{ and } q \text{ are relatively prime} \\
\text{positive integers which satisfy} \\
pq \text{ is even and } q < p \tan^2 r
\end{array}
\right\}
$$

$$
\cup \left\{2\pi\sqrt{\frac{1}{c}(p^2 \sin^2 r + q^2 \cos^2 r)} \;\middle|\;
\begin{array}{l}
p \text{ and } q \text{ are relatively prime} \\
\text{positive integers which satisfy} \\
pq \text{ is odd and } q < p \tan^2 r
\end{array}
\right\}.
$$

Therefore we obtain the following.

THEOREM 4.5. *Let M be a geodesic sphere of radius $2r/\sqrt{c}$ $(0 < r < \pi/2)$ in $\boldsymbol{HP}^n(c)$ of quaternionic sectional curvature c.*

(1) *The length spectrum $\mathrm{Lspec}(M)$ of M is a discrete unbounded subset in the real line $\boldsymbol{R}$. In particular, there exist infinitely many congruency classes of closed geodesics.*

(2) *If $\tan^2 r$ is irrational, every length spectrum of M is simple.*

(3) *If $\tan^2 r$ is rational, the multiplicity of each length spectrum of M is finite. But it is not uniformly bounded; $\limsup_{\lambda \to \infty} m_M(\lambda) = \infty$. In this case, the growth order of m_M is not so rapid. It satisfies $\lim_{\lambda \to \infty} \lambda^{-\delta} m_M(\lambda) = 0$ for arbitrary positive δ.*

(4) *The cardinality $n_M(\lambda)$ of the set $\{[\gamma] \in \text{Geod}(M) \mid \mathscr{L}_M([\gamma]) \leqq \lambda\}$ satisfies*

$$\lim_{\lambda \to \infty} \frac{n_M(\lambda)}{\lambda^2} = \frac{3cr}{4\pi^4 \sin 2r}.$$

Next we study geodesics on a horosphere H_3, on a geodesic sphere $H_1^0(r)$ of radius r and on a tube $H_1^{n-1}(r)$ of radius r around quaternionic hyperplane in a quaternionic hyperbolic space $HH^n(-4)$. Let M be one of $H_1^0(r)$, $H_1^{n-1}(r)$ and H_3, and ι be an isometric embedding of M into $HH^n(-4)$. We denote by $\tilde{\nabla}$ the Riemannian connection of $HH^n(-4)$. For each geodesic γ on M we find that $\iota \circ \gamma$ is a helix of order 4 in $HH^n(-4)$:

$$(4.1) \quad \begin{cases} \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa_1 X_2, \\ \tilde{\nabla}_{\dot{\gamma}}X_2 = -\kappa_1 \dot{\gamma} + \kappa_2 X_3, \\ \tilde{\nabla}_{\dot{\gamma}}X_3 = -\kappa_2 X_2 + \kappa_3 X_4, \\ \tilde{\nabla}_{\dot{\gamma}}X_4 = -\kappa_3 X_3, \end{cases}$$

where

$$\kappa_1 = \langle A\dot{\gamma}(s), \dot{\gamma}(s) \rangle, \quad \kappa_2 = \sqrt{\|A\dot{\gamma}\|^2 - \langle A\dot{\gamma}, \dot{\gamma} \rangle^2}, \quad \kappa_3 = \frac{1}{\kappa_2} \rho_\gamma \sqrt{1 - \rho_\gamma^2},$$

$$X_2 = \mathscr{N}, \quad X_3 = \frac{1}{\kappa_2} (\langle A\dot{\gamma}, \dot{\gamma} \rangle \dot{\gamma} - A\dot{\gamma}), \quad X_4 = \frac{1}{\rho_\gamma \sqrt{1 - \rho_\gamma^2}} \sum_{i=1}^{3} \langle \dot{\gamma}, \xi_i \rangle \phi_i(\dot{\gamma})$$

with a unit normal vector field $\mathscr{N}$ and the shape operator A of M in HH^n. Thus we get the following result on the extrinsic shape of geodesics on those real hypersurfaces in HH^n.

PROPOSITION 4.6. *Let M be a real hypersurface in $HH^n(-4)$ which is one of a horosphere, a geodesic sphere of radius r $(0 < r < \infty)$, and a tube of radius r $(0 < r < \infty)$ around quaternionic hyperplane HH^{n-1}. For a geodesic γ on M the curve $\iota \circ \gamma$ lies on a totally geodesic $CH^2(-4)$. Moreover, the extrinsic shape $\iota \circ \gamma$ is as follows*:

(1) *If $\rho_\gamma = 1$, the curve $\iota \circ \gamma$ is a circle in HH^n which lies on a totally geodesic $CH^1(-4)$. Its curvature is 2 if M is a horosphere, $2 \coth 2r$ if M is a geodesic sphere of radius r or a tube of radius r around quaternionic hyperplane.*

(2) *If $\rho_\gamma = 0$, the curve $\iota \circ \gamma$ is also a circle which lies on a totally geodesic $RH^2(-1)$. Its curvature is 1 if M is a horosphere, $\coth r$ if M is a geodesic sphere of radius r, and $\tanh r$ if M is a tube of radius r around quaternionic hyperplane.*

(3) *Generally, if $0 < \rho_\gamma < 1$, the curve $\iota \circ \gamma$ is a helix of proper order* 4. *Its curvatures are as the following table*:

	H_3	$H_1^0(r)$	$H_1^{n-1}(r)$
κ_1	$1 + \rho_\gamma^2$	$\coth r + \rho_\gamma^2 \tanh r$	$\tanh r + \rho_\gamma^2 \coth r$
κ_2	$\rho_\gamma \sqrt{1 - \rho_\gamma^2}$	$\rho_\gamma \sqrt{1 - \rho_\gamma^2} \tanh r$	$\rho_\gamma \sqrt{1 - \rho_\gamma^2} \coth r$
κ_3	1	$\coth r$	$\tanh r$

Every geodesic on these manifolds is a simple curve.

As a consequence of this proposition we obtain the following (see [AMY]).

THEOREM 4.7. *Let γ be a geodesic on a geodesic sphere M of radius r $(0 < r < \infty)$ in $HH^n(-4)$.*
 (1) *When $\rho_\gamma = 1$, it is closed with length $\pi \sinh 2r$.*
 (2) *When $\rho_\gamma = 0$, it is also closed and its length is $2\pi \sinh r$.*
 (3) *When $0 < \rho_\gamma < 1$, it is closed if and only if*

$$\rho_\gamma = \frac{q}{\sinh r \sqrt{p^2 \tanh^2 r - q^2}}$$

with some relatively prime positive integers p and q with $q < p \tanh^2 r$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2 \sinh^2 r - q^2 \cosh^2 r}, & \text{if } pq \text{ is even} \\ \pi\sqrt{p^2 \sinh^2 r - q^2 \cosh^2 r}, & \text{if } pq \text{ is odd}. \end{cases}$$

We call a smooth curve σ on a quaternionic hyperbolic space CH^n *unbounded in both directions* if both $\sigma([0, \infty))$ and $\sigma((-\infty, 0])$ are unbounded sets. Considering the ideal boundary ∂HH^n of HH^n as a Hadamard manifold, we can define its limit points at infinity

$$\sigma(\infty) = \lim_{t \to \infty} \sigma(t), \quad \sigma(-\infty) = \lim_{t \to -\infty} \sigma(t) \in \partial HH^n,$$

if they exist. We shall call a smooth curve σ on HH^n *horocyclic* if the following conditions hold.
 i) It has single point at infinity; $\sigma(\infty) = \sigma(-\infty)$.
 ii) If a geodesic ρ on HH^n with $\rho(\infty) = \sigma(\infty)$ crosses σ at some point $\rho(s_0)$, then they cross orthogonally at $\rho(s_0)$.

These conditions are equivalent to the condition that σ is unbounded in both directions and lies on a horosphere. We obtain the following along the same lines as in [AMY].

THEOREM 4.8. *Let γ be a geodesic on a tube M of radius r $(0 < r < \infty)$ around quaternionic hyperplane $\boldsymbol{H}H^{n-1}$ in $\boldsymbol{H}H^n(-4)$.*
(1) *When $\rho_\gamma = 1$, it is closed and its length is $\pi \sinh 2r$.*
(2) *When $\rho_\gamma < 1/\cosh r$, it is unbounded in both directions, and has two distinct points at infinity as a curve on $\boldsymbol{H}H^n$.*
(3) *When $\rho_\gamma = 1/\cosh r$, it is horocyclic as a curve on $\boldsymbol{H}H^n$.*
(4) *When $1/\cosh r < \rho_\gamma < 1$, it is bounded. Under this situation, it is closed if and only if*

$$\rho_\gamma = \frac{p}{\cosh r \sqrt{p^2 - q^2 \coth^2 r}}$$

with some relatively prime positive integers p and q with $p \tanh^2 r > q$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2 \sinh^2 r - q^2 \cosh^2 r}, & \text{if } pq \text{ is even} \\ \pi\sqrt{p^2 \sinh^2 r - q^2 \cosh^2 r}, & \text{if } pq \text{ is odd}. \end{cases}$$

PROPOSITION 4.9. *Every geodesic on a horosphere in a quaternionic hyperbolic space is unbounded in both directions, hence it is horocyclic as a curve on a quaternionic hyperbolic space.*

In order to make mention of length spectrum we need a congruence theorem. We set

$$H_1^{4n+3} = \{w = (w_0, \ldots, w_n) \in \boldsymbol{H}^{n+1} | -|w_0|^2 + |w_1|^2 + \cdots |w_n|^2 = -1\}$$

and denote by $\varpi : H_1^{4n+3} \to \boldsymbol{H}H^n$ the S^3-fibration. We consider the following hypersurfaces in H_1^{4n+3}:

$$\hat{M}_1 = \left\{ w \in \boldsymbol{H}^{n+1} \middle| w_0| = \cosh r, |w_1|^2 + \cdots + |w_n|^2 = \sinh^2 r \right\},$$

$$\hat{M}_2 = \left\{ w \in \boldsymbol{H}^{n+1} \middle| \begin{array}{c} -|w_0|^2 + |w_1|^2 + \cdots + |w_{n-1}|^2 = -\cosh^2 r, \\ |w_n| = \sinh r \end{array} \right\},$$

$$\hat{M}_3 = \left\{ w \in \boldsymbol{H}^{n+1} \middle| \begin{array}{c} -|w_0|^2 + |w_1|^2 + \cdots + |w_n|^2 = -1, \\ |w_0 - w_1| = 1 \end{array} \right\}.$$

According as M is a geodesic sphere of radius r, a tube of radius r around quaternionic hyperplane or a horosphere in $\boldsymbol{H}H^n(-4)$, the manifold $\varpi^{-1}(M)$ is isometric to $\hat{M}_1, \hat{M}_2$ or $\hat{M}_3$. By a similar argument to that in the proof of Lemma 4.4 and Proposition 4.3 we obtain the following.

PROPOSITION 4.10. *Let M be one of a horosphere, a geodesic sphere and a tube around quaternionic hyperplane in a quaternionic hyperbolic space. Two geodesics on M are congruent with respect to the isometry group of M if and only if their structure torsions coincide.*

With the aid of this propsition, Theorem 4.7 assures that the length spectrum of a geodesic sphere M of radius $2r/\sqrt{c}$ in $\boldsymbol{HH}^n(-c)$ is of the following form;

$$
\mathrm{Lspec}(M) = \left\{ \frac{4\pi}{\sqrt{c}} \sinh r \right\} \cup \left\{ \frac{2\pi}{\sqrt{c}} \sinh 2r \right\}
$$

$$
\cup \left\{ 4\pi\sqrt{\frac{1}{c}(p^2 \sinh^2 r - q^2 \cosh^2 r)} \;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p \tanh^2 r \end{array} \right\}
$$

$$
\cup \left\{ 2\pi\sqrt{\frac{1}{c}(p^2 \sinh^2 r - q^2 \cosh^2 r)} \;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p \tanh^2 r \end{array} \right\},
$$

and Theorem 4.8 assures that the length spectrum of a tube M of radius $2r/\sqrt{c}$ around quaternionic hyperplane $\boldsymbol{HH}^{n-1}$ in $\boldsymbol{HH}^n(-c)$ is of the following form;

$$
\mathrm{Lspec}(M) = \left\{ \frac{2\pi}{\sqrt{c}} \sinh 2r \right\}
$$

$$
\cup \left\{ 4\pi\sqrt{\frac{1}{c}(p^2 \sinh^2 r - q^2 \cosh^2 r)} \;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p \tanh^2 r \end{array} \right\}
$$

$$
\cup \left\{ 2\pi\sqrt{\frac{1}{c}(p^2 \sinh^2 r - q^2 \cosh^2 r)} \;\middle|\; \begin{array}{l} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p \tanh^2 r \end{array} \right\}.
$$

We obtain the following on the multiplicity of the length spectrum.

THEOREM 4.11. *Let M be either a geodesic sphere of radius $2r/\sqrt{c}$ or a tube of radius $2r/\sqrt{c}$ around quaternionic hyperplane $\boldsymbol{HH}^{n-1}$ in $\boldsymbol{HH}^n(-c)$ of quaternionic sectional curvature $-c$.*

(1) The length spectrum $\mathrm{Lspec}(M)$ is a discrete unbounded subset of the real line $\boldsymbol{R}$. In particular, there exist infinitely many congruency classes of closed geodesics.

(2) If $\coth^2 r$ is irrational of the form either

$$
\frac{1}{2q^2} \left\{ (p^2 + q^2 - 1) - \sqrt{\{p^2 - (q-1)^2\}\{p^2 - (q+1)^2\}} \right\}
$$

for some relatively prime positive integers p, q with even pq and $p \geq q + 3$, or

$$\frac{1}{2q^2}\left\{(p^2 + q^2 - 4) - \sqrt{\{p^2 - (q-2)^2\}\{p^2 - (q+2)^2\}}\right\}$$

for some relatively prime positive integers p, q with odd pq and $p \geq q + 4$, then the multiplicity of the spectrum $(2\pi/\sqrt{c})\sinh 2r$ is two and other length spectrum of M is simple.

(3) *If $\coth^2 r$ is irrational and is not of the form in* (2), *then every length spectrum of M is simple.*

(4) *If $\coth^2 r$ is rational, the multiplicity of each length spectrum is finite. But it is not uniformly bounded; $\limsup_{\lambda \to \infty} m_M(\lambda) = \infty$. The growth order of $m_M(\lambda)$ is not so rapid. It satisfies $\lim_{\lambda \to \infty} \lambda^{-\delta} m_M(\lambda) = 0$ for arbitrary positive δ.*

(5) *The number $n_M(\lambda)$ of congruency classes of geodesics on M with length not greater than λ satisfies*

$$\lim_{\lambda \to \infty} \frac{n_M(\lambda)}{\lambda^2} = \frac{3cr}{4\pi^4 \sinh 2r}.$$

We here go back to the viewpoint of characterizations of curvature-adapted real hypersurfaces with constant principal curvatures by the extrinsic shape of geodesics. By Propositions 4.1 and 4.6 and the same argument as in the proof of Propositions 4.1 and 4.2 in [AMY], we see the following.

PROPOSITION 4.12. *A real hypersurface M in a non-flat quaternionic space form $\tilde{M}$ of quaternionic dimension n (≥ 3) is locally congruent to one of a geodesic sphere, a tube around quaternionic hyperplane and a horosphere if and only if at each point $x \in M$ there exists an orthonormal basis $\{v_1, v_2, \ldots, v_{4n-4}\}$ of $\mathcal{D}_x$ $(\subset T_x M)$ such that all geodesics on M emanating x in the direction $v_j + v_k$ $(1 \leq j \leq k \leq 4n - 4)$ are circles of positive curvature in $\tilde{M}$.*

PROPOSITION 4.13. *A real hypersurface M in $HH^n(-c)$ $(n \geq 2)$ is locally congruent to a horosphere if and only if at each point $x \in M$ there exists an orthonormal basis $\{v_1, v_2, \ldots, v_{4n-4}\}$ of $\mathcal{D}_x$ such that all geodesics on M with the initial vector v_j $(1 \leq j \leq 4n - 4)$ are circles of curvature $\sqrt{c}/2$ on $HH^n(-c)$.*

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DEPARTMENT OF MATHEMATICS
NAGOYA INSTITUTE OF TECHNOLOGY
GOKISO, NAGOYA, 466-8555, JAPAN
e-mail: adachi@math.kyy.nitech.ac.jp

DEPARTMENT OF MATHEMATICS
SHIMANE UNIVERSITY
MATSUE, SHIMANE, 690-8504, JAPAN
e-mail: smaeda@math.shimane-u.ac.jp