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A CLASS FUNCTION ON THE TORELLI GROUP

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Abstract

The Magnus representation of the Torelli group has been defined in virtue of Fox derivation. The Torelli group is a significant subgroup of the mapping class group of a surface. In this paper, we show some properties of the characteristic polynomials of matrices obtained from the Magnus representation of the Torelli group, which is a class function on the Torelli group.

1. Introduction

Let $\Sigma_{g,1}$ be an oriented surface obtained from a closed surface Σ_g of genus g by removing an open disk. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is the group of path components of the group of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_{g,1}$ be the Torelli group of $\Sigma_{g,1}$, namely the normal subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act on the first homology group of $\Sigma_{g,1}$ trivially.

We call the following mapping r_1 the Magnus representation of the Torelli group:

$$r_1 : \mathcal{I}_{g,1} \to \mathrm{GL}(2g; \boldsymbol{Z}[H])$$

where $H = H_1(\Sigma_{g,1}; \boldsymbol{Z})$. We will consider the characteristic polynomials of matrices obtained from the Magnus representation of the Torelli group. That is, we will investigate

$$R(\varphi) = \det(\lambda I_{2g} - r_1(\varphi))$$

for $\varphi \in \mathcal{I}_{g,1}$, where I_{2g} is the unit matrix and λ is an indeterminate. Then R is a class function on $\mathcal{I}_{g,1}$.

We will prove some properties of this class function R. For example, we will show that the restriction of R to $\mathcal{K}_{g,1}$ is non-trivial, where $\mathcal{K}_{g,1}$ is the normal subgroup of $\mathcal{I}_{g,1}$ generated by all the Dehn twists along bounding simple closed curves.

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2. Definition of the Magnus representation of the Torelli group

In this section, we recall the definition of the Magnus representation of the Torelli group.

Let F_n be a free group of rank n with free basis $z_1, \dots, z_n$. The following simple derivation on the integral group ring $\boldsymbol{Z}[F_n]$ is the main ingredient of Fox derivation.

DEFINITION 2.1 (Fox derivation). The Fox derivation is defined by the following equations:

$$\frac{\partial}{\partial z_j}(z_{\mu_1}^{\varepsilon_1} \cdots z_{\mu_r}^{\varepsilon_r}) = \sum_{i=1}^{r} \varepsilon_i \delta_{\mu_i, j} z_{\mu_1}^{\varepsilon_1} \cdots z_{\mu_{i-1}}^{\varepsilon_{i-1}} z_{\mu_i}^{(1/2)(\varepsilon_i - 1)}, \quad \varepsilon_i = \pm 1$$

$$\frac{\partial}{\partial z_j}\left(\sum a_w w\right) = \sum a_w \frac{\partial w}{\partial z_j}, \quad w \in F_n,\ a_w \in \boldsymbol{Z}.$$

We fix a system of generators $\alpha_1, \dots, \alpha_g$, $\beta_1, \dots, \beta_g$ of the free group $\Gamma_0 = \pi_1(\Sigma_{g,1})$ as shown in Figure 1. Let us simply write $\gamma_1, \dots, \gamma_{2g}$ for them. Moreover, we obtain a system of symplectic basis x_i, y_i of H by abelianizing α_i, β_i respectively.

DEFINITION 2.2. We call the mapping

$$r : \mathcal{M}_{g,1} \to \mathrm{GL}(2g; \boldsymbol{Z}[\Gamma_0])$$
$$\varphi \mapsto \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j}$$

the Magnus representation for the mapping class group. Here $\partial/\partial\gamma_i$ is Fox derivation and $^{-} : \boldsymbol{Z}[\Gamma_0] \ni \sum a\gamma \mapsto \sum a\gamma^{-1} \in \boldsymbol{Z}[\Gamma_0]$.

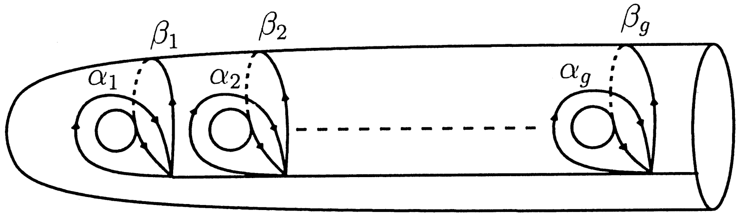

FIGURE 1. Generators of Γ_0

This mapping r is a crossed homomorphism. The product formula below follows from the chain rule of Fox derivation.

PROPOSITION 2.3 ([M1]). *For any two elements* $\varphi, \psi \in \mathcal{M}_{g,1}$, *we have*

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

where ${}^{\varphi}r(\psi)$ *denotes the matrix obtained from* $r(\psi)$ *by applying the automorphism* $\varphi : \boldsymbol{Z}[\Gamma_0] \to \boldsymbol{Z}[\Gamma_0]$ *on each entry.*

We denote by $r^{\mathfrak{a}}$ the composition of the mapping r by abelianizing $\mathfrak{a} : \boldsymbol{Z}[\Gamma_0] \to \boldsymbol{Z}[H]$ the coefficients. If we consider elements of the Torelli group, we write r_1 for $r^{\mathfrak{a}}$. That is to say, we get a genuine representation r_1 by restricting this mapping $r^{\mathfrak{a}}$ to the Torelli group:

$$r_1 : \mathcal{I}_{g,1} \to \mathrm{GL}(2g; \boldsymbol{Z}[H]).$$

3. Characteristic polynomials

In this section, we investigate characteristic polynomials of the Magnus matrices. Here the Magnus matrix means the image of r_1 for a mapping class. We define

$$R(\varphi) = \det(\lambda I_{2g} - r_1(\varphi))$$

for $\varphi \in \mathcal{I}_{g,1}$. In particular, for any elements $\varphi_1, \varphi_2 \in \mathcal{I}_{g,1}$ we have

$$R(\varphi_2\varphi_1\varphi_2^{-1}) = R(\varphi_1)$$

so that R is constant in the conjugacy classes of $\mathcal{I}_{g,1}$, that is, R is a class function on $\mathcal{I}_{g,1}$.

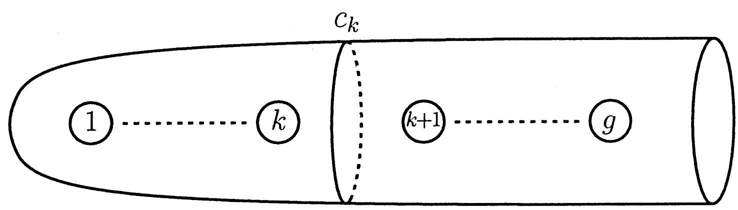

FIGURE 2. Bounding simple closed curve c_k

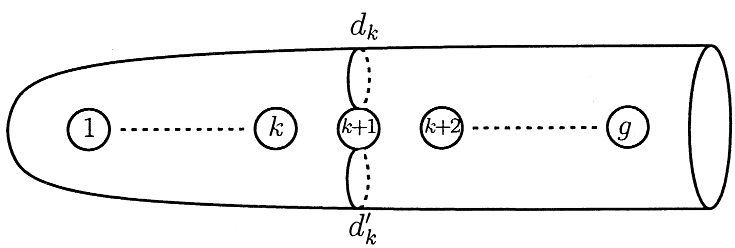

FIGURE 3. Bounding pair d_k, d'_k

The curve c_k shown in Figure 2 is a bounding simple closed curve, where bounding means 0-homologous. Let φ_k denote the BSCC map which is the Dehn twist along a bounding simple closed curve c_k. We denote by ψ_k the product of the right Dehn twist along a simple closed curve d_k and the left Dehn twist along a simple closed curve d_k' which is disjoint and homologous to d_k as shown in Figure 3. We call ψ_k to be the BP map. It is known that the Torelli group $\mathscr{I}_{g,1}$ is normally generated in $\mathscr{M}_{g,1}$ by ψ_1 (see [J] for details).

First, we compute the Magnus matrices of BSCC map φ_k and BP map ψ_k directly. Since

$$\varphi_k(\alpha_j) = \begin{cases} [\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\alpha_j[\alpha_1,\beta_1]\cdots[\alpha_k,\beta_k] & 1 \le j \le k \\ \alpha_j & k < j \end{cases}$$

and

$$\varphi_k(\beta_j) = \begin{cases} [\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\beta_j[\alpha_1,\beta_1]\cdots[\alpha_k,\beta_k] & 1 \le j \le k \\ \beta_j & k < j \end{cases},$$

these free differential calculuses are

$$\frac{\partial(\varphi_k(\alpha_j))}{\partial\alpha_i} = \begin{cases} [\beta_k,\alpha_k]\cdots[\beta_{i+1},\alpha_{i+1}]\beta_i \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_i,\alpha_i] \\ \quad +\delta_{i,j}[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1] \\ \quad +[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\alpha_j[\alpha_1,\beta_1]\cdots[\alpha_{i-1},\beta_{i-1}] \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\alpha_j[\alpha_1,\beta_1]\cdots[\alpha_{i-1},\beta_{i-1}]\alpha_i\beta_i\bar{\alpha}_i \\ \hfill 1 \le j \le k,\ i \le k \\ 0 \hfill 1 \le j \le k,\ k < i \le g \\ \delta_{i,j} \hfill k < j \end{cases}$$

$$\frac{\partial(\varphi_k(\beta_j))}{\partial\alpha_i} = \begin{cases} [\beta_k,\alpha_k]\cdots[\beta_{i+1},\alpha_{i+1}]\beta_i \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_i,\alpha_i] \\ \quad +[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\beta_j[\alpha_1,\beta_1]\cdots[\alpha_{i-1},\beta_{i-1}] \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\beta_j[\alpha_1,\beta_1]\cdots[\alpha_{i-1},\beta_{i-1}]\alpha_i\beta_i\bar{\alpha}_i \\ \hfill 1 \le j \le k,\ i \le k \\ 0 \hfill 1 \le j \le k,\ k < i \le g \\ 0 \hfill k < j \end{cases}$$

$$\frac{\partial(\varphi_k(\alpha_j))}{\partial\beta_i} = \begin{cases} [\beta_k,\alpha_k]\cdots[\beta_{i+1},\alpha_{i+1}] \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_{i+1},\alpha_{i+1}]\beta_i\alpha_i\bar{\beta}_i \\ \quad +[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\alpha_j[\alpha_1,\beta_1]\cdots[\alpha_{i-1},\beta_{i-1}]\alpha_i \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\alpha_j[\alpha_1,\beta_1]\cdots[\alpha_i,\beta_i] \\ \hfill 1 \le j \le k,\ i \le k \\ 0 \hfill 1 \le j \le k,\ k < i \le g \\ 0 \hfill k < j \end{cases}$$

$$\frac{\partial(\varphi_k(\beta_j))}{\partial\beta_i} = \begin{cases} [\beta_k,\alpha_k]\cdots[\beta_{i+1},\alpha_{i+1}] \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_{i+1},\alpha_{i+1}]\beta_i\alpha_i\bar{\beta}_i \\ \quad +\delta_{i,j}[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1] \\ \quad +[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\beta_j[\alpha_1,\beta_1]\cdots[\alpha_{i-1},\beta_{i-1}]\alpha_i \\ \quad -[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1]\beta_j[\alpha_1,\beta_1]\cdots[\alpha_i,\beta_i] \\ \hfill 1\le j\le k,\ i\le k \\ 0 \qquad\qquad 1\le j\le k,\ k<i\le g \\ \delta_{i,j} \qquad\qquad k<j \end{cases}$$

Then the Magnus matrix of genus k BSCC map φ_k is

$$r_1(\varphi_k) = I_{2g} + a_k b_k$$

where

$$a_k = {}^t(\bar{y}_1 - 1 \cdots \bar{y}_k - 1 \underbrace{0\cdots0}_{g-k \text{ times}} 1-\bar{x}_1 \cdots 1-\bar{x}_k \underbrace{0\cdots0}_{g-k \text{ times}})$$

$$b_k = (1-\bar{x}_1 \cdots 1-\bar{x}_k \underbrace{0\cdots0}_{g-k \text{ times}} 1-\bar{y}_1 \cdots 1-\bar{y}_k \underbrace{0\cdots0}_{g-k \text{ times}}).$$

Similarly, since

$$\psi_k(\alpha_j) = \begin{cases} [\alpha_1,\beta_1]\cdots[\alpha_k,\beta_k]\alpha_{k+1}\beta_{k+1}\bar{\alpha}_{k+1}\alpha_j\alpha_{k+1}\bar{\beta}_{k+1}\bar{\alpha}_{k+1}[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1] & 1\le j\le k \\ [\alpha_1,\beta_1]\cdots[\alpha_k,\beta_k]\alpha_{k+1} & j=k+1 \\ \alpha_j & k+1<j \end{cases}$$

and

$$\psi_k(\beta_j) = \begin{cases} [\alpha_1,\beta_1]\cdots[\alpha_k,\beta_k]\alpha_{k+1}\beta_{k+1}\bar{\alpha}_{k+1}\beta_j\alpha_{k+1}\bar{\beta}_{k+1}\bar{\alpha}_{k+1}[\beta_k,\alpha_k]\cdots[\beta_1,\alpha_1] & 1\le j\le k \\ \beta_j & k<j \end{cases},$$

the Magnus matrix of genus k BP map ψ_k is

$$r_1(\psi_k) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} \bar{y}_{k+1} + X_1 Y_1 & \cdots & X_k Y_1 & Y_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 Y_k & \cdots & \bar{y}_{k+1} + X_k Y_k & Y_k & 0 & \cdots & 0 \\ X_1 Y_{k+1} & \cdots & X_k Y_{k+1} & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} Y_1 Y_1 & \cdots & Y_k Y_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_1 Y_k & \cdots & Y_k Y_k & 0 & \cdots & 0 \\ Y_1 Y_{k+1} & \cdots & Y_k Y_{k+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} -X_1 X_1 & \cdots & -X_k X_1 & -X_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_1 X_k & \cdots & -X_k X_k & -X_k & 0 & \cdots & 0 \\ \bar{x}_{k+1} X_1 & \cdots & \bar{x}_{k+1} X_k & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} \bar{y}_{k+1} - X_1 Y_1 & \cdots & -X_1 Y_k & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_k Y_1 & \cdots & \bar{y}_{k+1} - X_k Y_k & 0 & 0 & \cdots & 0 \\ \bar{x}_{k+1} Y_1 & \cdots & \bar{x}_{k+1} Y_k & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Here $X_i = 1 - \bar{x}_i = 1 - x_i^{-1}$, $Y_i = 1 - \bar{y}_i = 1 - y_i^{-1}$.

Then straightforward calculations show the following results about the characteristic polynomials of them.

Lemma 3.1. *Let φ_k, ψ_k be as above. Then we have*
1. $\det(\lambda I_{2g} - r_1(\varphi_k)) = (\lambda - 1)^{2g}$
2. $\det(\lambda I_{2g} - r_1(\psi_k)) = (\lambda - 1)^{2g-2k}(\lambda - y_{k+1}^{-1})^{2k}$.

We remark that the characteristic polynomial R is a class function not on $\mathcal{M}_{g,1}$ but on $\mathcal{I}_{g,1}$. More precisely,

Proposition 3.2. *For any* $\varphi \in \mathcal{I}_{g,1}$ *and* $f \in \mathcal{M}_{g,1}$,

$$R(f\varphi f^{-1}) = f(R(\varphi))$$

where we also denote by f *the mapping* $\boldsymbol{Z}[\lambda, x_1^{\pm 1}, \dots, x_g^{\pm 1}, y_1^{\pm 1}, \dots, y_g^{\pm 1}] \to \boldsymbol{Z}[\lambda, f(x_1)^{\pm 1}, \dots, f(x_g)^{\pm 1}, f(y_1)^{\pm 1}, \dots, f(y_g)^{\pm 1}]$.

Proof. First, we note that ${}^f r(f^{-1}) = r(f)^{-1}$, because we have

$$I_{2g} = r(ff^{-1}) = r(f) \cdot {}^f r(f^{-1}).$$

Then we get

$$\begin{aligned} R(f\varphi f^{-1}) &= \det(I_{2g} - r_1(f\varphi f^{-1})) \\ &= \det(I_{2g} - r^{\mathfrak{a}}(f) \cdot {}^f r_1(\varphi) \cdot {}^{f\varphi} r^{\mathfrak{a}}(f^{-1})) \\ &= \det(I_{2g} - r^{\mathfrak{a}}(f) \cdot {}^f r_1(\varphi) \cdot r^{\mathfrak{a}}(f)^{-1}) \\ &= \det(I_{2g} - {}^f r_1(\varphi)) \\ &= f(\det(I_{2g} - r_1(\varphi))) \\ &= f(R(\varphi)) \end{aligned}$$

∎

Any BSCC map φ can be written as $\varphi = f\varphi_k f^{-1}$, where $f \in \mathcal{M}_{g,1}$ and φ_k is the Dehn twist along a simple closed curve c_k as before. According to Lemma 3.1, we deduce the following corollary.

Corollary 3.3. *For any BSCC map* φ, *we have*

$$R(\varphi) = (\lambda - 1)^{2g}.$$

For any BSCC map φ, the characteristic polynomial of $r_1(\varphi)$ is trivial. However, the characteristic polynomial of a product of two BSCC maps is not always trivial. For example, we can show that

$$R(\varphi_1 v_1 \varphi_1 v_1^{-1}) = (\lambda - 1)^{2g} + \lambda(\lambda - 1)^{2g-2}(y_1 - 2 + \bar{y}_1)(y_2 - 2 + \bar{y}_2).$$

Here v_1 is the Dehn twist along n_1 as shown in Figure 4. This means that the restriction of R to $\mathcal{K}_{g,1}$ is non-trivial, where $\mathcal{K}_{g,1}$ is the normal subgroup of $\mathcal{I}_{g,1}$ generated by all the BSCC maps.

Proposition 3.4. *For any* $\psi \in \mathcal{I}_{g,1}$, *$R$ has a common factor* $(\lambda - 1)^2$.

Proof. From our previous paper [S2], there exsists a non-singular matrix P such that for any element $\psi \in \mathcal{I}_{g,1}$

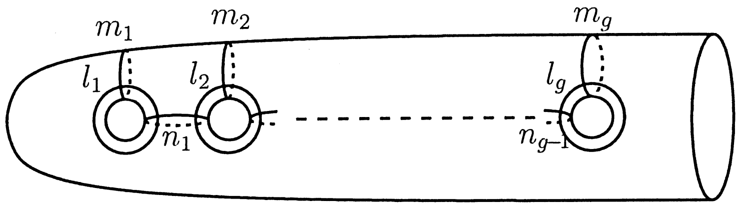

FIGURE 4. Lickorish generators

$$P^{-1}r_1(\psi)P = \left(\begin{array}{c|ccc|c} 1 & & * & & * \\ \hline 0 & & & & \\ \vdots & & \rho_B(\psi) & & * \\ & & & & \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Here ρ_B is a $(2g-2)$-dimensional irreducible representation of $\mathscr{I}_{g,1}$ (see [S2] for details). This means that the assertion holds. ■

Moreover, from our previous paper [S2], we have

$$\rho_B(\tau_\zeta) = I_{2g-2},$$

where τ_ζ be the Dehn twist along a simple closed curve on $\Sigma_{g,1}$ which is parallel to the boundary. This equality says that R factors through $\mathscr{I}_{g,*}$. Here $\mathscr{I}_{g,*}$ is the Torelli group of Σ_g relative to the base point $* \in \Sigma_g$.

4. The relation between $R(\psi)$ and $R(\psi^{-1})$

The relation between $R(\psi)$ and $R(\psi^{-1})$ is given by the following formula.

PROPOSITION 4.1. *For $\psi \in \mathscr{I}_{g,1}$, we have*

$$R(\psi) = \overline{R(\psi^{-1})},$$

where $\bar{\cdot} : x_i \mapsto x_i^{-1},\ y_i \mapsto y_i^{-1}$.

To prove Proposition 4.1, we recall that the Magnus representation of the Torelli group is symplectic.

PROPOSITION 4.2 (Morita [M1]). *There exists a matrix $J \in \mathrm{GL}(2g; \boldsymbol{Z}[H])$ such that for any $\psi \in \mathscr{I}_{g,1}$ we have the equality*

$$\overline{{}^t r_1(\psi)} J r_1(\psi) = J.$$

Here J is defined as follows:

$$J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}$$

where

$$J_1 = \begin{pmatrix} 1-x_1 & & & \\ (1-x_2)(1-\bar{x}_1) & 1-x_2 & & \\ (1-x_3)(1-\bar{x}_1) & (1-x_3)(1-\bar{x}_2) & 1-x_3 & \\ \vdots & \vdots & \ddots & \\ (1-x_g)(1-\bar{x}_1) & (1-x_g)(1-\bar{x}_2) & \cdots & 1-x_g \end{pmatrix}$$

$$J_2 = \begin{pmatrix} x_1\bar{y}_1 & & & \\ (1-x_2)(1-\bar{y}_1) & x_2\bar{y}_2 & & \\ (1-x_3)(1-\bar{y}_1) & (1-x_3)(1-\bar{y}_2) & x_3\bar{y}_3 & \\ \vdots & \vdots & \ddots & \\ (1-x_g)(1-\bar{y}_1) & (1-x_g)(1-\bar{y}_2) & \cdots & x_g\bar{y}_g \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 1-\bar{x}_1-y_1 & & & \\ (1-y_2)(1-\bar{x}_1) & 1-\bar{x}_2-y_2 & & \\ (1-y_3)(1-\bar{x}_1) & (1-y_3)(1-\bar{x}_2) & 1-\bar{x}_3-y_3 & \\ \vdots & \vdots & \ddots & \\ (1-y_g)(1-\bar{x}_1) & (1-y_g)(1-\bar{x}_2) & \cdots & 1-\bar{x}_g-y_g \end{pmatrix}$$

$$J_4 = \begin{pmatrix} 1-\bar{y}_1 & & & \\ (1-y_2)(1-\bar{y}_1) & 1-\bar{y}_2 & & \\ (1-y_3)(1-\bar{y}_1) & (1-y_3)(1-\bar{y}_2) & 1-\bar{y}_3 & \\ \vdots & \vdots & \ddots & \\ (1-y_g)(1-\bar{y}_1) & (1-y_g)(1-\bar{y}_2) & \cdots & 1-\bar{y}_g \end{pmatrix}.$$

Proof of Proposition 4.1. By Proposition 4.2, we get

$$\overline{{}^t r_1(\psi)} = J r_1(\psi)^{-1} J^{-1} = J r_1(\psi^{-1}) J^{-1}.$$

Hence we conclude

$$\begin{aligned} R(\psi) &= \det(\lambda I - r_1(\psi)) \\ &= \det(\lambda I - J r_1(\psi) J^{-1}) \\ &= \det(\lambda I - \overline{{}^t r_1(\psi^{-1})}) \\ &= \overline{\det(\lambda I - r_1(\psi^{-1}))} \\ &= \overline{R(\psi^{-1})}. \end{aligned}$$

∎

Corollary 3.3 states that the determinant of the Magnus matrix for any BSCC map is one. Because the group $\mathcal{K}_{g,1}$ is generated by BSCC maps, $\det r_1(\varphi) = 1$ for any $\varphi \in \mathcal{K}_{g,1}$. Then we deduce the following.

COROLLARY 4.3. *Let φ be an element of $\mathcal{K}_{g,1}$. Suppose that the characteristic polynomial is written as*

$$R(\varphi) = \lambda^{2g} + p_1\lambda^{2g-1} + p_2\lambda^{2g-2} + \cdots + p_{2g-1}\lambda + 1,$$

where $p_k \in \boldsymbol{Z}[H]$, then $p_k = \overline{p_{2g-k}}$. In particular, we have $p_g = \overline{p_g}$.

Moreover, in the case of genus 2, for any $\varphi \in \mathcal{K}_{2,1}$ the variables p_k can be reduced to just one. That is, we have

$$R(\varphi) = (\lambda - 1)^2(\lambda^2 + p\lambda + 1)$$

by Proposition 3.4. The above equation and Corollary 4.3 yield the following statement.

COROLLARY 4.4. *For any $\varphi \in \mathcal{K}_{2,1}$ we have*

$$R(\varphi) = R(\varphi^{-1}).$$

For higher genera, this statement does not hold. For example, an explicit calculation shows that

$$R(\varphi_1\lambda_2\nu_1\varphi_1\nu_1^{-1}\lambda_2^{-1}\nu_2\lambda_2\nu_1\varphi_1'\nu_1^{-1}\lambda_2^{-1}\nu_2^{-1}) \neq R((\varphi_1\lambda_2\nu_1\varphi_1\nu_1^{-1}\lambda_2^{-1}\nu_2\lambda_2\nu_1\varphi_1'\nu_1^{-1}\lambda_2^{-1}\nu_2^{-1})^{-1}).$$

Here $\varphi_1', \lambda_2, \nu_2$ are the Dehn twists along c_1', l_2, ν_2 as shown in Figure 5 and Figure 4. However, for a product of two BSCC maps, we arrive at the following.

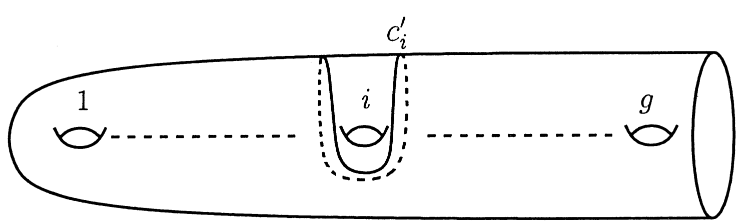

FIGURE 5. bounding simple closed curve c_i'

THEOREM 4.5. *Let φ_1, φ_2 be BSCC maps. Then we have*

$$R(\varphi_1\varphi_2) = R((\varphi_1\varphi_2)^{-1}).$$

We provide the following to prove Theorem 4.5.

Lemma 4.6. *Let u_1, u_2 be n-dimensional column vectors and v_1, v_2 n-dimensional row vectors. Then we get*

$$\det(tI_n + u_1v_1 + u_2v_2 - u_1v_1u_2v_2)$$
$$= \det(tI_n - u_1v_1 - u_2v_2 - u_1v_1u_2v_2) + 2(\operatorname{tr} u_1v_1 + \operatorname{tr} u_2v_2)t^{n-1}.$$

If we write M_1, M_2 for $-u_1v_1 - u_2v_2 + u_1v_1u_2v_2$, $u_1v_1 + u_2v_2 + u_1v_1u_2v_2$ respectively, then the above equality can be stated as

$$\det(tI_n - M_1) = \det(tI_n - M_2) + 2(\operatorname{tr} u_1v_1 + \operatorname{tr} u_1v_1)t^{n-1}.$$

Thus we will prove the above equation.

Proof. The characteristic polynomial of matrix $M_l = (m^l_{i,j})$ is given as

$$\det(tI_n - M_l) = t^n + c^l_1 t^{n-1} + c^l_2 t^{n-2} + \cdots + c^l_{n-1}t + c^l_n \quad (l = 1, 2)$$

where

$$c^l_k = (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \begin{vmatrix} m^l_{i_1,i_1} & m^l_{i_1,i_2} & \cdots & m^l_{i_1,i_k} \\ m^l_{i_2,i_1} & m^l_{i_2,i_2} & \cdots & m^l_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ m^l_{i_k,i_1} & m^l_{i_k,i_2} & \cdots & m^l_{i_k,i_k} \end{vmatrix}. \tag{4.1}$$

Since the rank of M_l is less than 3, then

$$c^l_3 = c^l_4 = \cdots = c^l_n = 0.$$

This means that the difference between the characteristic polynomial of M_1 and that of M_2 appears only in terms of t^{n-1} and t^{n-2}. First, the coefficient of t^{n-1} is the difference between $\operatorname{tr} M_1$ and $\operatorname{tr} M_2$:

$$(c^1_1 - c^2_1)t^{n-1} = (-\operatorname{tr} M_1 + \operatorname{tr} M_2)t^{n-1} = 2(\operatorname{tr} u_1v_1 + \operatorname{tr} u_2v_2)t^{n-1}.$$

Second, we compute the term of t^{n-2}. We set

$$u_1 = {}^t(c_1 \cdots c_n), \quad u_2 = {}^t(d_1 \cdots d_n), \quad v_1 = (e_1 \cdots e_n), \quad v_2 = (f_1 \cdots f_n).$$

The (i,j)-components of M_1 and M_2 are

$$m^1_{i,j} = -c_ie_j - d_if_j + Ac_if_j, \quad m^2_{i,j} = c_ie_j + d_if_j + Ac_if_j$$

where $A = \sum_{k=1}^n e_kd_k$. Because of the equation (4.1), we get the following.

$$\begin{aligned}
c_2^1 - c_2^2 &= \sum_{1\le i<j\le n} \begin{vmatrix} m_{i,i}^1 & m_{i,j}^1 \\ m_{j,i}^1 & m_{j,j}^1 \end{vmatrix} - \sum_{1\le i<j\le n} \begin{vmatrix} m_{i,i}^2 & m_{i,j}^2 \\ m_{j,i}^2 & m_{j,j}^2 \end{vmatrix} \\
&= \sum_{1\le i<j\le n} (m_{i,i}^1 m_{j,j}^1 - m_{i,j}^1 m_{j,i}^1 - m_{i,i}^2 m_{j,j}^2 + m_{i,j}^2 m_{j,i}^2) \\
&= \sum_{1\le i<j\le n} \{(-c_i e_i - d_i f_i + A c_i f_i)(-c_j e_j - d_j f_j + A c_j f_j) \\
&\qquad - (-c_i e_j - d_i f_j + A c_i f_j)(-c_j e_i - d_j f_i + A c_j f_i) \\
&\qquad - (c_i e_i + d_i f_i + A c_i f_i)(c_j e_j + d_j f_j + A c_j f_j) \\
&\qquad + (c_i e_j + d_i f_j + A c_i f_j)(c_j e_i + d_j f_i + A c_j f_i)\} \\
&= 0
\end{aligned}$$

This means that the coefficient of t^{n-2} is zero and completes the proof. ■

Proof of Theorem 4.5. The Magnus matrix of φ_i which is any BSCC map can be written as

$$r_1(\varphi_i) = I_{2g} + u_i v_i \quad i = 1, 2$$

where u_i is a n-dimensional column vector and v_i is a n-dimensional row vector. Corollary 3.3 states that $\operatorname{tr} u_i v_i = v_i u_i$ equals zero. This deduces $r_1(\varphi_i^{-1}) = I_{2g} - u_i v_i$. Therefore we have

$$\begin{aligned}
R(\varphi_2^{-1}\varphi_1^{-1}) &= R(\varphi_1^{-1}\varphi_2^{-1}) \\
&= \det(\lambda I_{2g} - (I_{2g} - u_1 v_1)(I_{2g} - u_2 v_2)) \\
&= \det((\lambda - 1) I_{2g} + u_1 v_1 + u_2 v_2 - u_1 v_1 u_2 v_2) \\
&= \det((\lambda - 1) I_{2g} - u_1 v_1 - u_2 v_2 - u_1 v_1 u_2 v_2) \quad \text{Because of Lemma 4.6} \\
&= \det(\lambda I_{2g} - (I_{2g} + u_1 v_1)(I_{2g} + u_2 v_2)) \\
&= R(\varphi_1 \varphi_2)
\end{aligned}$$

This completes the proof. ■

In general, we can not decide how many BSCC maps are producted for a given element of $\mathcal{K}_{g,1}$. However, Corollary 3.3 and Theorem 4.5 help to determine the number. More precisely, we have the following criterion.

Corollary 4.7. *First, for an element φ of $\mathcal{K}_{g,1}$, if the characteristic polynomial is not trivial, then the element φ can not be written as just one BSCC map. Second, if the characteristic polynomial of $r_1(\varphi)$ and that of $r_1(\varphi)^{-1}$ are not the same, then the element φ can neither be written as one BSCC map nor a product of two BSCC maps.*

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