P. MILNES
KODAI MATH. J.
25 (2002), 209–226

MINIMAL H_3 ACTIONS AND SIMPLE QUOTIENTS OF DISCRETE 7-DIMENSIONAL NILPOTENT GROUPS

Paul Milnes

Abstract

For connected nilpotent groups, 7 is the lowest dimension where there are infinitely many non-isomorphic groups, and also where some groups have no discrete cocompact subgroups. Here one infinite family of 7-dimensional connected groups is studied, discrete cocompact subgroups H are found for some of them, and then the faithful simple quotients A of $C^*(\mathrm{H})$ are identified. Such A are shown to be isomorphic to C^*-crossed products $C^*(\mathrm{H}_3, \mathscr{C}(\boldsymbol{T}^3))$ generated by some intriguing effective minimal distal flows $(\mathrm{H}_3, \boldsymbol{T}^3)$, where H_3 is the discrete 3-dimensional Heisenberg group.

1. Introduction

In 3 dimensions there is a unique (up to isomorphism) connected, simply connected, nilpotent Lie group, which we call G_3 (following Nielsen [N]); G_3 ($=\boldsymbol{R}^3$ as a set) is the Heisenberg group with multiplication

$$(k,m,n)(k',m',n') = (k+k'+nm', m+m', n+n').$$

The faithful irreducible representations of the lattice subgroup H_3 ($=\boldsymbol{Z}^3$ as a set) of G_3 generate the irrational rotation algebras A_θ. In 4 dimensions there is also a unique such connected group G_4, in 5 dimensions there are 6 of them, $\mathrm{G}_{5,i}$, $1 \le i \le 6$, and in 6 dimensions there are 24. The main thrust in [MW1, MW2] was to find cocompact subgroups $\mathrm{H}_4 \subset \mathrm{G}_4$ and $\mathrm{H}_{5,i} \subset \mathrm{G}_{5,i}$, that would be analogous to $\mathrm{H}_3 \subset \mathrm{G}_3$, and then for these H's to identify the infinite dimensional simple quotients of $C^*(\mathrm{H})$, both the faithful ones (generated by a faithful representation of H) and the non-faithful ones, and also to give matrix representations over lower dimensional algebras for as many of the non-faithful quotients as possible. In the course of this work, it was observed that all flow presentations of simple quotients that arose used actions of abelian groups, namely, $\boldsymbol{Z}$ or $\boldsymbol{Z}^2$, or subgroups of them; this situation changed for one of the 6-dimensional groups [M].

2000 *Mathematics Subject Classification*: 22D25, 46L55.

Keywords and phrases: Nilpotent group, cocompact subgroup, minimal effective flow, simple C^*-algebra, semidirect product.

This research was supported in part by NSERC grant A7857.

Received June 7, 2001; revised March 11, 2002.

In §2 of the present paper, following [SS1, SS2], we display an infinite family of connected, 7-dimensional, nilpotent groups G, and give a proof that they are pairwise non-isomorphic. In §3 presentations of these groups as semidirect products $\boldsymbol{R}^4 \times \mathrm{H}_3$ are given. In §4, after discrete cocompact subgroups H are identified for those G not involving an irrational parameter α or β, the semidirect product presentations are used to produce flows that identify, and give concrete representations of, the faithful simple quotients A of the group C^*-algebras $C^*(\mathrm{H})$; different presentations of G as $\boldsymbol{R}^4 \times \mathrm{H}_3$ give rise to unitarily inequivalent irreducible representations of H generating A.

Preliminaries.

To present the results and proofs of the paper, we need notation for semidirect products and C^*-crossed products; the discussion which follows is quite standard, appearing in [MW1], [Z-M] and many other places.

Suppose that N and K are discrete groups, the identity of each of them being denoted by e. Suppose that there is a homomorphism $s \mapsto \sigma_s$ from K into the automorphism group of N. Then $G = N \times K$ becomes a group, the *semidirect product* of N and K, with the multiplication formula

$$(t,s)(t',s') = (t\sigma_s(t'), ss').$$

We will usually write $s(t)$ instead of $\sigma_s(t)$.

Conversely, if N is a normal subgroup of G with quotient group $K = G/N$ suitably embedded as a subgroup in G, then G is canonically isomorphic to a semidirect product $N \times K$, whose automorphisms are determined by G, $\sigma_s(t) = sts^{-1}$ (product in G).

Now replace N by a C^*-algebra A with identity 1 and assume that we have a homomorphism $s \mapsto \sigma_s$ from K into the automorphism group of A. Then, for f and g in the Banach space $\ell^1(K, A)$, the convolution product $f * g$ and involution f^* are defined by

$$f * g(s') = \sum_{s \in K} f(s)\sigma_s(g(s^{-1}s')) \quad \text{and} \quad f^*(s) = \sigma_s(f(s^{-1})^*);$$

with these definitions, $\ell^1(K, A)$ becomes a Banach *-algebra. The *C^*-crossed product* $C^*(A, K)$ is defined to be the enveloping C^*-algebra of $\ell^1(K, A)$.

For $a \in A$ and $s \in K$, the δ-functions a_s and δ_s in $\ell^1(K, A) \subset C^*(A, K)$ are defined by $a_s(s) = a$, $a_s(s') = 0$ otherwise, and $\delta_s(s) = 1$ (the identity of A), $\delta_s(s') = 0$ for $s' \neq s$.

2. Uncountably many 7-dimensional nilpotent groups

The material in this section is adapted from [SS1, SS2], where the setting is that of Lie algebras. The groups themselves, as well as the classification theorem (Theorem 1), are given in some detail, partly because of the necessity to work closely with the groups, but also because it seems that [SS2] never appeared in print; we thank T. Sund for correspondence, including a copy of [SS2].

We start with a description of the connected groups that will concern us here.

I The groups $G_{7,a}$.

Let $e_1 = (\dots,0,0,1)$, $e_2 = (\dots,0,0,1,0)$, $e_3 = (\dots,0,1,0,0)$, $e_4 = (\dots,0,1,0,0,0)$, etc., give a basis for $\boldsymbol{R}^6$ or $\boldsymbol{R}^7$ and define multiplication as follows.

Let e_1, e_2 and e_3 have commutators

$$[e_1,e_2] = e_6, \quad [e_2,e_3] = e_4 \quad \text{and} \quad [e_3,e_1] = e_5. \tag{1}$$

The resulting multiplication for $\boldsymbol{R}^6$ is (isomorphic to) that of Nielsen's $G_{6,15}$ (see the last 6 coordinates of (m_a) below). Then add further commutators

$$[e_1,e_4] = C_{14}e_7, \quad [e_2,e_5] = C_{25}e_7 \quad \text{and} \quad [e_3,e_6] = C_{36}e_7. \tag{2}$$

The resulting operation for $\boldsymbol{R}^7$ is

$$(\mathrm{m}_a) \quad \begin{cases} (f,g,h,j,k,m,n)(f',g',h',j',k',m',n') \\ \quad = (f+f'+C_{14}nj' + C_{25}m(h'-nk') + C_{36}k(g'+nm') + C_{36}nm'k', \\ \qquad g+g'+nm', h+h'-nk', j+j'+mk', k+k', m+m', n+n'). \end{cases}$$

It is associative and a group multiplication if and only if

$$(\star) \qquad C_{14} + C_{25} + C_{36} = 0,$$

and then yields non-isomorphic groups $G_{7,a}$ for $0 < a \leq 1$ with

$$(\mathrm{C}_a) \qquad C_{25} = a, \quad C_{36} = 1 \quad \text{and} \quad C_{14} = -(C_{25}+C_{36}) = -(a+1)$$

(see Theorem 1 below). Thus $G_{7,a}$ is an extension of $G_{6,15}$ via the cocycle

$$\begin{aligned} &[(g,h,j,k,m,n),(g',h',j',k',m',n')] \\ &\qquad = -(a+1)nj' + am(h'-nk') + k(g'+nm') + nm'k'. \end{aligned}$$

Note. In this cocycle, and others in this paper, the quadratic terms are the important part, the cubic terms arising merely from the order of the coordinates.

II The rest of the groups, $G_{7,\beta}$ and $G_{7,*}$.

Now add commutators

$$[e_2,e_6] = C_{26}e_7 \quad \text{and} \quad [e_3,e_5] = C_{35}e_7 \tag{3}$$

to (1) and (2). The multiplication formula for $\boldsymbol{R}^7$ then has

$$C_{26}m(g'+nm') + C_{26}nm'(m'-1)/2 + C_{35}k(h'-nk') - C_{35}nk'(k'-1)/2$$

added to the e_7 coordinate, and gives an associative multiplication if

$$(\star) \qquad C_{14} + C_{25} + C_{36} = 0$$

(as before). Further non-isomorphic groups $G_{7,\beta}$ for $\beta > 0$ are arrived at with

$$(C_\beta) \qquad C_{14} = -2, \quad C_{25} = 1 = C_{36} \quad \text{and} \quad C_{35} = \beta = -C_{26}.$$

The cocycle on $G_{6,15}$ for $G_{7,\beta}$ is seen in the first coordinate of the multiplication formula

$$(m_\beta) \quad \begin{cases} (f,g,h,j,k,m,n)(f',g',h',j',k',m',n') \\ \quad = (f+f'-2nj'+m(h'-nk')+k(g'+nm')+nm'k' \\ \qquad -\beta m(g'+nm')-\beta nm'(m'-1)/2+\beta k(h'-nk')-\beta nk'(k'-1)/2, \\ \qquad g+g'+nm', h+h'-nk', j+j'+mk', k+k', m+m', n+n'). \end{cases}$$

There is one more group $G_{7,*}$; it has

$$(C_*) \qquad C_{14} = -2, \quad C_{25} = 1 = C_{36}, \quad C_{35} = 1 \quad \text{and} \quad C_{26} = 0.$$

For $G_{7,*}$ the cocycle on $G_{6,15}$ is seen in the first coordinate of the multiplication formula

$$(m_*) \quad \begin{cases} (f,g,h,j,k,m,n)(f',g',h',j',k',m',n') \\ \quad = (f+f'-2nj'+m(h'-nk')+k(g'+nm') \\ \qquad +nm'k'+k(h'-nk')-nk'(k'-1)/2, \\ \qquad g+g'+nm', h+h'-nk', j+j'+mk', k+k', m+m', n+n'). \end{cases}$$

THEOREM 1. *These groups,* $G_{7,a}$ $(0 < a \leq 1)$, $G_{7,\beta}$ $(\beta > 0)$, *and* $G_{7,*}$, *are pairwise non-isomorphic*; *any 7-dimensional extension* (*with one-dimensional centre*) *of* $G_{6,15}$ *is isomorphic to one of them.*

Proof. Let $\mathscr{G}$ be the set of extensions $G = \boldsymbol{R} \times G_{6,15}$ of $G_{6,15}$ with 1-dimensional centre. We want to divide these up into isomorphism classes and will do this by determining which cocycles $\alpha : G_{6,15} \times G_{6,15} \to \boldsymbol{R}$ give isomorphic groups, and then picking one group from each isomorphism class; these will be the groups in the theorem.

Now a cocycle $G_{6,15} \times G_{6,15} \to \boldsymbol{R}$ is a linear combination potentially of 36 (or more) terms, but a lot of these can be eliminated. We end up with the cohomology group $\mathscr{H}^2 = \mathscr{H}^2(G_{6,15}, \boldsymbol{R}) = \mathscr{C}\mathscr{C}^2(G_{6,15}, \boldsymbol{R})/\mathscr{B}^2(G_{6,15}, \boldsymbol{R})$ (the quotient of the cocycles by the boundaries) represented as cocycles of the form

$$\alpha = \sum_{i=1}^{3} \sum_{j=4}^{6} C_{ij}[e_i, e_j] \in \mathscr{H}^2, \quad \alpha(u,v) = \sum_{i=1}^{3} \sum_{j=4}^{6} C_{ij} u_i v_j,$$

for $u, v \in G_{6,15}$ ($=\boldsymbol{R}^6$ as a set. Here only the quadratic terms of the cocycles are being kept track of; see the note before subheading II). The only restriction on the C_{ij}'s is that

$$(\star) \qquad C_{14} + C_{25} + C_{36} = 0,$$

as above. (The other commutators $[e_i, e_j]$ give boundaries, or do not yield an associative operation, or differ from one of the commutators above by a boundary; see the Note below.) So, from now on we consider $\mathscr{G}$ to be the extensions of $G_{6,15}$ by these cocycles.

Now every $A = (A_{ij})_{i,j=1}^{3} \in GL(3, \boldsymbol{R})$ gives an automorphism of $G_{6,15}$ (also denoted by A), and hence an isomorphism of each $G \in \mathscr{G}$, by acting on the generating set,

$$e_i \mapsto A(e_i) = \sum_{j=1}^{3} A_{ij} e_j = e_i' \in G_{6,15}, \quad 1 \leq i \leq 3;$$

(essentially) every automorphism of $G_{6,15}$ arises in this way. We need to see how this action of $GL(3, \boldsymbol{R})$ on $G_{6,15}$ transfers to a cocycle

$$\alpha(u, v) = \sum_{i=1}^{3} \sum_{j=4}^{6} C_{ij} u_i v_j, \quad u, v \in G_{6,15}.$$

To do this, first note that, with

$$e_4' = [e_2', e_3'], \quad e_5' = [e_3', e_1'] \quad \text{and} \quad e_6' = [e_1', e_2']$$

(as for the e_i's), we have

$$A(e_{3+i}) = e_{3+i}' = \sum_{j=1}^{3} \mathscr{A}_{ij} e_{3+j}, \quad 1 \leq i \leq 3,$$

where $\mathscr{A} = (\mathscr{A}_{ij})_{i,j=1}^{3}$ is the matrix of cofactors of $A = (A_{ij})$. Then, since $A^{-1} = \mathscr{A}^{\mathrm{T}}/\det A$, one sees directly that a cocycle α on $G_{6,15}$ with basis $\{e_i\}$, α given by a matrix

$$C = C(\alpha) = (C_{ij}, 1 \leq i \leq 3, 4 \leq j \leq 6)$$

is transformed by A to a cocycle α' on $G_{6,15}$ with basis $\{e_i'\}$, α' given by a matrix $C(\alpha') = (\det A) ACA^{-1}$ (product of 3×3 matrices). But, by Jordan canonical form [LT; p. 243, for example], each $C(\alpha)$ can be written as AMA^{-1} for an $A \in GL(3, \boldsymbol{R})$ and a matrix M of one of the following forms.

$$(1)' \quad \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \quad \text{(3 real eigenvalues)}$$

$$(2)' \quad \begin{pmatrix} x & 0 & 0 \\ 0 & y & -z \\ 0 & z & y \end{pmatrix} \quad \text{(complex eigenvalues } y \pm iz \text{ with } z \neq 0)$$

$$(3)' \quad \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 1 & y \end{pmatrix} \quad \text{(Jordan block)}$$

Now the α's we are considering must give an extension $\boldsymbol{R} \times \mathrm{G}_{6,15}$ with 1-dimensional centre (i.e., α is non-degenerate); thus $x \neq 0$, also in $(1)'$ $y \neq 0$, $z \neq 0$; further, $y \neq 0$ in $(2)'$ and $(3)'$, because $\mathrm{Tr}\, C = C_{14} + C_{25} + C_{36} = 0$ and $x \neq 0$. Since the matrix A in AMA^{-1} can be multiplied by a scalar multiple of the identity without changing AMA^{-1}, we can see that the $\mathrm{GL}(3, \boldsymbol{R})$-orbit in $\mathscr{H}^2$ of $C = C(\alpha)$ contains exactly one of the following matrices.

$$(1) \quad \begin{pmatrix} -a-1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 < a \leq 1$$

$$(2) \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -\beta \\ 0 & \beta & 1 \end{pmatrix}, \quad \beta > 0$$

$$(3) \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(For example, in $(1)'$ we can scale with the $\det A$ from the action so that 2 of the eigenvalues are positive and the larger of these equals 1, maybe both equal 1; this gives (1).)

The proof is completed by observing that the matrices (C_{ij}, $1 \leq i \leq 3$, $4 \leq j \leq 6$) at (1), (2) and (3) give the cocycles determined at (C_a), (C_β) and (C_*) for the groups in the statement of the theorem, $\mathrm{G}_{7,a}$ $(0 < a \leq 1)$, $\mathrm{G}_{7,\beta}$ $(\beta > 0)$ and $\mathrm{G}_{7,*}$. ■

Note. To eliminate terms in $\mathscr{H}^2$, like
$[e_1, e_2] = Fe_7$: substitute $f \mapsto f + cg$ (gets rid of the term);
$[e_1, e_1] = Fe_7$: substitute $f \mapsto f + cn^2$ (gets rid of the term).
(These last 2 terms are boundaries.)
$[e_4, e_5] = Fe_7$: doesn't give an associative operation;
$[e_5, e_1] = Fe_7$: substitute $f \mapsto f + cnh$ (converts to $-F[e_1, e_5]$).

3. Presentations as a semidirect product $\boldsymbol{R}^4 \times \mathrm{G}_3$

All of these groups G have normal subgroups $N_1 = (\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0, 0, 0, \boldsymbol{R}) \cong \boldsymbol{R}^4$ with quotient $K_1 = \mathrm{G}/N_1 = \mathrm{G}_3$ embedded in G as the subgroup $(0, 0, 0, \boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0)$, so that G is a semidirect product $\boldsymbol{R}^4 \times \mathrm{G}_3$. The groups $\mathrm{G}_{7,a}$ have 2 more presentations as a semidirect product $\boldsymbol{R}^4 \times \mathrm{G}_3$, and the group $\mathrm{G}_{7,*}$ has one more.

Presentation 1 for $\mathrm{G}_{7,a}$.

The relevant action of $\mathrm{G}_3 = (0, 0, 0, \boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0) \subset \mathrm{G}_{7,a}$ on $\boldsymbol{R}^4 = (\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0, 0, 0, \boldsymbol{R})$ is given by

$$(\mathrm{act}_{a,1})\quad \begin{cases} (j,k,m):(f',g',h',n') \\ \qquad \mapsto (f'+(a+1)jn'+amh'+k(g'-mn'),g'-mn',h'+kn',n'), \end{cases}$$

and yields multiplication for $\boldsymbol{R}^4 \times \mathrm{G}_3 \cong \mathrm{G}_{7,a}$,

$$(\mathrm{m}_{a,1})\quad \begin{cases} (f,g,h,n,j,k,m)(f',g',h',n',j',k',m') \\ \quad = (f+f'+(a+1)jn'+amh'+k(g'-mn'), \\ \qquad g+g'-mn',h+h'+kn',n+n',j+j'+mk',k+k',m+m'), \end{cases}$$

with inverse $(f,g,h,n,j,k,m)^{-1} =$

$$(-f+(a+1)jn+kg+am(h-kn),-g-mn,-h+kn,-n,-j+mk,-k,-m).$$

We call this group $G_{a,1}$. An isomorphism $\varphi_1 : \mathrm{G}_{7,a} \to G_{a,1}$ is given by

$$\varphi_1 : (f,g,h,j,k,m,n) \mapsto (f+(a+1)nj-nkm,g-nm,h+nk,n,j,k,m).$$

Presentation 2 for $\mathrm{G}_{7,a}$.
Here $N_2 = (\boldsymbol{R},0,\boldsymbol{R},\boldsymbol{R},\boldsymbol{R},0,0) \cong \boldsymbol{R}^4$ is normal in $\mathrm{G}_{7,a}$ with action of

$$\mathrm{G}_3 = \mathrm{G}_{7,a}/N_2 = (0,\boldsymbol{R},0,0,0,\boldsymbol{R},\boldsymbol{R})$$

on $\boldsymbol{R}^4$ given by

$$(\mathrm{act}_{a,2})\quad \begin{cases} (g,m,n):(f',h',j',k') \\ \qquad \mapsto (f'-(a+1)nj'+am(h'-nk')-gk',h'-nk',j'+mk',k'), \end{cases}$$

so that $\boldsymbol{R}^4 \times \mathrm{G}_3 \cong \mathrm{G}_{7,a}$ has multiplication

$$(\mathrm{m}_{a,2})\quad \begin{cases} (f,h,j,k,g,m,n)(f',h',j',k',g',m',n') \\ \quad = (f+f'-(a+1)nj'+am(h'-nk')-gk', \\ \qquad h+h'-nk',j+j'+mk',k+k',g+g'+nm',m+m',n+n'). \end{cases}$$

We call this group $G_{a,2}$. An isomorphism $\mathrm{G}_{7,a} \to G_{a,2}$ is given by

$$\varphi_2 : (f,g,h,j,k,m,n) \mapsto (f-kg,h,j,k,g,m,n).$$

Presentation 3 for $\mathrm{G}_{7,a}$.
Here $N_3 = (\boldsymbol{R},\boldsymbol{R},0,\boldsymbol{R},0,\boldsymbol{R},0) \cong \boldsymbol{R}^4$ is normal in $\mathrm{G}_{7,a}$ with

$$\mathrm{G}_{7,a}/N_3 = (0,0,\boldsymbol{R},0,\boldsymbol{R},0,\boldsymbol{R}) \cong \mathrm{G}_3$$

(note the quotient is only isomorphic to G_3 for this presentation). The action of G_3 on $\boldsymbol{R}^4$ is given by

$$(\mathrm{act}_{a,3})\quad \begin{cases} (h,k,n):(f',g',j',m') \\ \qquad \mapsto (f'-(a+1)nj'+k(g'+nm')-ahm',g'+nm',j'-km',m'), \end{cases}$$

so that $\boldsymbol{R}^4 \times \mathrm{G}_3 \cong \mathrm{G}_{7,a}$ has multiplication

$$(\mathrm{m}_{a,3}) \quad \begin{cases} (f,g,j,m,h,k,n)(f',g',j',m',h',k',n') \\ \quad = (f+f'-(a+1)nj'+k(g'+nm')-ahm', \\ \qquad g+g'+nm', j+j'-km', m+m', h+h'-nk', k+k', n+n'). \end{cases}$$

We call this group $G_{a,3}$. An isomorphism $\varphi : \mathrm{G}_{7,a} \to G_{a,3}$ is given by

$$\varphi_3 : (f,g,h,j,k,m,n) \mapsto (f-amh, g, j-mk, m, h, k, n).$$

Of course, these presentations of $\mathrm{G}_{7,a}$ are isomorphic, e.g., the map

$$\begin{aligned} &\varphi_1 \circ \varphi_2^{-1} : (f,h,j,k,g,m,n) \\ &\qquad \mapsto (f+gk+(a+1)nj-nkm, g-nm, h+nk, n, j, k, m) \end{aligned}$$

is an isomorphism of $G_{a,2}$ onto $G_{a,1}$. One might expect also to get automorphisms of $G_{a,1}$ with these isomorphisms, much as in [M], where 2 presentations of a 6-dimensional group $\mathrm{G}_{6,4}$ as a semidirect product $\boldsymbol{R}^3 \times \mathrm{G}_3$ do give rise to an automorphism of $\mathrm{G}_{6,4}$ (and also of the simple quotients of the lattice subgroup $\mathrm{H}_{6,4} = \boldsymbol{Z}^6 \subset \mathrm{G}_{6,4}$). However, here we do not seem to get an isomorphism of $G_{a,1}$; the problem is the asymmetry of multiplication in the f-coordinate (of $(\mathrm{m}_{a,1})$, for example).

Presentations for $\mathrm{G}_{7,*}$.

The relevant action of $\mathrm{G}_3 = \mathrm{G}_{7,*}/N_1 \cong (0,0,0,\boldsymbol{R},\boldsymbol{R},\boldsymbol{R},0) \subset \mathrm{G}_{7,*}$ on $N_1 = \boldsymbol{R}^4 = (\boldsymbol{R},\boldsymbol{R},\boldsymbol{R},0,0,0,\boldsymbol{R})$ is given by

$$(\mathrm{act}_{*,1}) \quad \begin{cases} (j,k,m) : (f',g',h',n') \mapsto (f'+2jn'+mh'+k(g'-mn') \\ \quad + kh' + n'k(k-1)/2, g'-mn', h'+kn', n'), \end{cases}$$

and yields multiplication for $\boldsymbol{R}^4 \times \mathrm{G}_3 \cong \mathrm{G}_{7,*}$,

$$(\mathrm{m}_{*,1}) \quad \begin{cases} (f,g,h,n,j,k,m)(f',g',h',n',j',k',m') \\ \quad = (f+f'+2jn'+mh'+k(g'-mn')+kh'+n'k(k-1)/2, \\ \qquad g+g'-mn', h+h'+kn', n+n', j+j'+mk', k+k', m+m'). \end{cases}$$

with inverse

$$\begin{cases} (f,g,h,j,k,m,n)^{-1} \\ \quad = (-f+2jn+kg+m(h-kn)+kh-nk(k+1)/2, \\ \qquad -g-mn, -h+kn, -n, -j+mk, -k, -m). \end{cases}$$

We call this group $G_{*,1}$. An isomorphism $\varphi : \mathrm{G}_{7,*} \to G_{*,1}$ is given by

$$(f,g,h,j,k,m,n) \mapsto (f+2nj-nkm+nk(k-1)/2, g-nm, h+nk, n, j, k, m).$$

The second presentation for $\mathrm{G}_{7,*}$ has normal subgroup

$$N_2 = (\boldsymbol{R}, 0, \boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0, 0) \cong \boldsymbol{R}^4 \subset \mathrm{G}_{7,*}$$

with $\mathrm{G}_3 = \mathrm{G}_{7,a}/N_2 \cong (0, \boldsymbol{R}, 0, 0, 0, \boldsymbol{R}, \boldsymbol{R})$. The reason there is no third presentation for $\mathrm{G}_{7,*}$ as $\boldsymbol{R}^4 \times \mathrm{H}_3$ is simply that $(0, 0, \boldsymbol{R}, 0, \boldsymbol{R}, 0, \boldsymbol{R})$ is not a subgroup of $\mathrm{G}_{7,*}$, as it was for $\mathrm{G}_{7,a}$.

Presentation for $\mathrm{G}_{7,\beta}$.

The relevant action of $\mathrm{G}_3 = \mathrm{G}_{7,\beta}/N = (0, 0, 0, \boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0) \subset \mathrm{G}_{7,\beta}$ on $N = \boldsymbol{R}^4 = (\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}, 0, 0, 0, \boldsymbol{R})$ is given by

$$(\mathrm{act}_{\beta,1}) \quad \begin{cases} (j,k,m) : (f', g', h', n') \mapsto (f' + 2jn' + mh' + k(g' - mn') \\ \quad + \beta(kh' + n'k(k-1)/2 - mg' + n'm(m-1)/2), \\ \quad g' - mn', h' + kn', n'), \end{cases}$$

and yields multiplication for $\boldsymbol{R}^4 \times \mathrm{G}_3 \cong \mathrm{G}_{7,a}$,

$$(\mathrm{m}_{\beta,1}) \quad \begin{cases} (f,g,h,n,j,k,m)(f',g',h',n',j',k',m') \\ \quad = (f + f' + 2jn' + mh' + k(g' - mn') \\ \quad\quad + \beta(kh' + n'k(k-1)/2 - mg' + n'm(m-1)/2, \\ \quad\quad g + g' - mn', h + h' + kn', n + n', j + j' + mk', k + k', m + m'). \end{cases}$$

We call this group $G_{\beta,1}$. An isomorphism $\varphi : \mathrm{G}_{7,\beta} \to G_{\beta,1}$ is given by

$$\begin{aligned} \varphi : &(f,g,h,j,k,m,n) \\ &\mapsto (f + 2jn - nkm + \beta n(m(m-1)/2 + k(k-1)/2), g - nm, h + nk, n, j, k, m). \end{aligned}$$

4. Lattice subgroups H and faithful simple quotients of $C^*(\mathrm{H})$

The groups $\mathrm{H}_{7,a}$, $0 < a \leq 1$

The objective is to identify a cocompact lattice subgroup of $\mathrm{G}_{7,a}$; such a subgroup H clearly exists if a is rational and this is the case we will deal with in detail. However, such an H also exists for some ‘mildly’ irrational a. What this means is there are matrices C, as in the proof of Theorem 1, even with entries in $\boldsymbol{Z}$, for which the corresponding (connected) group is shown by the classification process in Theorem 1 to be isomorphic to $\mathrm{G}_{7,a}$ for an irrational a; e.g., the group with

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

is isomorphic to $\mathrm{G}_{7,a}$ with $a = \sqrt{2} - 1/2$. (We thank Dave Witte for correspondence about these matters.)

So, let us work with $G_{a,1}$ (as above), and suppose that $0 < a = p/q \leq 1$ with $(p,q) = 1$. Then $(\boldsymbol{Z}/q) \times \boldsymbol{Z}^6$ is a cocompact subgroup of $G_{a,1}$ (whose inverse image under the isomorphism $\varphi_1 : \mathrm{G}_{7,a} \to G_{a,1}$ is a cocompact subgroup of $\mathrm{G}_{7,a}$); we multiply the first coordinate of $G_{a,1}$ by q and define $\mathrm{H}_{7,a}$ to be the set $\boldsymbol{Z}^7$ with multiplication

$$(\mathrm{m}'_{a,1}) \quad \begin{cases} (f,g,h,n,j,k,m)(f',g',h',n',j',k',m') \\ \quad = (f+f'+(p+q)jn'+pmh'+qk(g'-mn'), \\ \qquad g+g'-mn', h+h'+kn', n+n', j+j'+mk', k+k', m+m'). \end{cases}$$

Of course, $\mathrm{H}_{7,a}$ is the semidirect product $\boldsymbol{Z}^4 \times \mathrm{H}_3$, with action of H_3 on $\boldsymbol{Z}^4$ given by

$$(\mathrm{act}'_{a,1}) \quad \begin{cases} (j,k,m) : (f',g',h',n') \\ \quad \mapsto (f'+(p+q)jn'+pmh'+qk(g'-mn'), g'-mn', h'+kn', n'). \end{cases}$$

To identify the simple quotients of $C^*(\mathrm{H}_{7,a})$ this action must be transferred to the generators x, w and v of $\mathscr{C}(\boldsymbol{T}^3)$ (and of $L^2(\boldsymbol{T}^3)$, i.e.,

$$x, w \text{ and } v : (x,w,v) \mapsto x, w \text{ and } v).$$

For $\lambda = e^{2\pi i\theta}$ with irrational θ, we get

$$(\mathrm{act}''_{a,1}) \quad (j,k,m) : x \mapsto \lambda^{qk}x, \quad w \mapsto \lambda^{pm}w \quad \text{and} \quad v \mapsto \lambda^{(p+q)j-qkm}x^{-m}w^k v.$$

Some operator equations on $L^2(\boldsymbol{T}^3)$ are relevant here. The generators are

$$(0,0,1) \sim U : w \mapsto \lambda^p w \quad \text{and} \quad v \mapsto x^{-1}v,$$
$$(0,1,0) \sim V : x \mapsto \lambda^q x \quad \text{and} \quad v \mapsto wv, \quad \text{and}$$
$$W : f \mapsto vf \quad (f \in L^2(\boldsymbol{T}^3))$$

(a generator u, v or w in $L^2(\boldsymbol{T}^3)$ not being mentioned when it is left fixed). The operator equations and subsidiary operators are given by

$$(\mathrm{CR}_{a,1}) \quad \begin{cases} (1,0,0) \sim [U,V] = X : v \mapsto \lambda^{p+q}v, \\ [U,W] = Z^{-1} : f \mapsto x^{-1}f, \quad [V,W] = Y : f \mapsto wf, \\ [X,W] = \lambda^{p+q}, \quad [U,Y] = \lambda^p \quad \text{and} \quad [V,Z] = \lambda^q \end{cases}$$

(the other commutators from U, V, W, X, Y and Z being trivial, i.e., the operators commute). The point about the equations $(\mathrm{CR}_{a,1})$ is that the map

$$(\pi) \qquad (f,g,h,n,j,k,m) \mapsto \lambda^f Z^g Y^h W^n X^j V^k U^m$$

is a representation of $\mathrm{H}_{7,a}$, generating a C^*-algebra $\mathfrak{A}^{7,a}_{\theta,1}$, say.

The next theorem also involves the (minimal distal) flow $\mathscr{F}_{a,1} = (\mathrm{H}_3, \boldsymbol{T}^3)$ from $(\mathrm{act}''_{a,1})$; to get it, just apply inversion to H_3 at $(\mathrm{act}''_{a,1})$ and arrive at

$$(\mathscr{F}_{a,1}) \quad (j,k,m) : (x,w,v) \mapsto (\lambda^{-qk}x, \lambda^{-pm}w, \lambda^{-(p+q)j+pkm}x^m w^{-k} v), \quad \boldsymbol{T}^3 \to \boldsymbol{T}^3.$$

THEOREM 2. *Let* $\lambda = e^{2\pi i\theta}$ *for an irrational* θ.

(a) *There is a unique* (*up to isomorphism*) C^**-algebra* $\mathrm{A}_\theta^{7,a}$ *generated* (*via* π) *by* (*any*) *unitaries* Z, Y, W, X, V *and* U *satisfying*

$$(\mathrm{CR}_{a,1}) \qquad \begin{cases} [U,V] = X, \quad [U,W] = Z^{-1}, \quad [V,W] = Y, \\ [X,W] = \lambda^{p+q}, \quad [U,Y] = \lambda^p \quad \text{and} \quad [V,Z] = \lambda^q. \end{cases}$$

$\mathrm{A}_\theta^{7,a}$ *is simple and is universal for the equations* $(\mathrm{CR}_{a,1})$. *Let* H_3 *act on* $\mathscr{C}(\boldsymbol{T}^3)$ *as indicated at* $(\mathrm{act}''_{a,1})$; *then*

$$\mathrm{A}_\theta^{7,a} \cong C^*(\mathscr{C}(\boldsymbol{T}^3), \mathrm{H}_3) \cong \mathfrak{A}_{\theta,1}^{7,a}.$$

(b) *Let* π' *be a representation of* $\mathrm{H}_{7,a}$ *such that* $\pi = \pi'$ (*as scalars*) *on the center* $(\boldsymbol{Z}, 0, 0, 0, 0, 0, 0)$ *of* $\mathrm{H}_{7,a}$, *and let* A *be the* C^**-algebra generated by* π'. *Then* $\mathrm{A} \cong \mathrm{A}_\theta^{7,a}$ *via a unique isomorphism* ω *such that the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{H}_{7,a} & \xrightarrow{\pi} & \mathrm{A}_\theta^{7,a} \\ \pi' \searrow & & \swarrow \omega \\ & \mathrm{A} & \end{array}$$

(c) *The* C^**-algebra* $\mathrm{A}_\theta^{7,a}$ *has a unique tracial state.*

Proof. The proof can be much as for Theorem 1.1 in [MW2]; we give some details.

One must note first that the flow $\mathscr{F}_{a,1} = (\mathrm{H}_3, \boldsymbol{T}^3)$ above is minimal and effective; so the generated C^*-crossed $C^*(\mathscr{C}(\boldsymbol{T}^3), \mathrm{H}_3)$ product is simple, by Corollary 5.16 of Effros and Hahn [EH].

Once the simplicity of $C^*(\mathscr{C}(\boldsymbol{T}^3), \mathrm{H}_3)$ is established, it is straightforward to prove the rest of (a) using the correspondence of

$$\delta_{(0,0,1)}, \delta_{(0,1,0)}, \delta_{(1,0,0)}, v_{(0,0,0)}, w_{(0,0,0)}, x_{(0,0,0)}$$
$$\in \ell^1(\mathrm{H}_3, \mathscr{C}(\boldsymbol{T}^3)) \subset C^*(\mathscr{C}(\boldsymbol{T}^3), \mathrm{H}_3)$$

to U, V, X, W, Y, Z, respectively; see [MW2; proof of Theorem 1.1], for example, also for (b) and (c). ■

Since the C^*-algebra $\mathrm{A}_\theta^{7,a}$ is generated by a faithful representation of $\mathrm{H}_{7,a}$, we refer to it as a *faithful simple quotient* of the group C^*-algebra $C^*(\mathrm{H}_{7,a})$.

Concrete representations of $A_\theta^{7,a}$.

The first 3 are derived from the 3 semidirect product presentations of $G_{7,a}$ as $\boldsymbol{R}^4 \times H_3$; they are interesting in part because they generate unitarily inequivalent representations of $A_\theta^{7,a}$ on $L^2(\boldsymbol{T}^3)$.

1. The first representation is the one generating $\mathfrak{A}_{\theta,1}^{7,a}$ above.

2. The second representation of $A_\theta^{7,a}$ is derived in similar fashion from $G_{a,2} = \boldsymbol{R}^4 \times H_3 \cong G_{7,a}$, which yields multiplication

$$(\mathrm{m}'_{a,2}) \quad \begin{cases} (f,h,j,k,g,m,n)(f',h',j',k',g',m',n') \\ \quad = (f+f'-(p+q)nj'+pm(h'-nk')-qgk', \\ \qquad h+h'-nk', j+j'+mk', k+k', g+g'+nm', m+m', n+n') \end{cases}$$

for $H'_{7,a} = \boldsymbol{Z}^7$; $H'_{7,a}$ is a semidirect product $\boldsymbol{Z}^4 \times H_3$ with action of H_3 on $\boldsymbol{Z}^4$ given by

$$(\mathrm{act}'_{a,2}) \quad \begin{cases} (g,m,n) : (f',h',j',k') \\ \quad \mapsto (f'-(p+q)nj'+pm(h'-nk')-qgk', h'-nk', j'+mk', k'). \end{cases}$$

Transferring this action of H_3 to $\mathscr{C}(\boldsymbol{T}^3)$ (and $L^2(\boldsymbol{T}^3)$) gives

$$(\mathrm{act}''_{a,2}) \quad (g,m,n) : x \mapsto \lambda^{pm}x, \quad w \mapsto \lambda^{-(p+q)n}w \quad \text{and} \quad v \mapsto \lambda^{-pmn-qg}x^{-n}w^m v,$$

and leads to operators and operator equations on $L^2(\boldsymbol{T}^3)$

$$(0,0,1) \sim U' : w \mapsto \lambda^{-(p+q)}w \quad \text{and} \quad v \mapsto x^{-1}v,$$
$$(0,1,0) \sim V' : x \mapsto \lambda^p x \quad \text{and} \quad v \mapsto wv,$$
$$W' : f \mapsto vf \quad (f \in L^2(\boldsymbol{T}^3)), \quad \text{and}$$

$$(\mathrm{CR}_{a,2}) \quad \begin{cases} (1,0,0) \sim [U', V'] = X' : v \mapsto \lambda^{-q}v, \\ [U', W'] = Z'^{-1} : f \mapsto x^{-1}f, \quad [V', W'] = Y' : f \mapsto wf, \\ [X', W'] = \lambda^{-q}, \quad [U', Y'] = \lambda^{-(p+q)} \quad \text{and} \quad [V', Z'] = \lambda^p. \end{cases}$$

Then, much as for presentation 1, the map

$$(\pi') \qquad (f,h,j,k,g,m,n) \mapsto \lambda^f Z'^h Y'^j W'^k X'^g V'^m U'^n$$

is a representation of $H'_{7,a}$, generating a C^*-algebra $\mathfrak{A}_{\theta,2}^{7,a}$, say. Of course, the isomorphism $\varphi_1 \circ \varphi_2^{-1} : G_{a,2} \to G_{a,1}$ yields isomorphisms $H'_{7,a} \to H_{7,a}$ and $\psi : \mathfrak{A}_{\theta,2}^{7,a} \to \mathfrak{A}_{\theta,1}^{7,a}$. To implement the last isomorphism, note that the unitaries X', Z', U', Y', W', V' satisfy $(\mathrm{CR}_{a,1})$, so that, by Theorem 2, $\psi^{-1} : \mathfrak{A}_{\theta,1}^{7,a} \to \mathfrak{A}_{\theta,2}^{7,a}$ is generated by

$$Z^g Y^h W^n X^j V^k U^m \mapsto X'^g Z'^h U'^n Y'^j W'^k V'^m.$$

3. The third representation of $A_\theta^{7,a}$ is derived from $G_{a,3} = \boldsymbol{R}^4 \times H_3 \cong G_{7,a}$, which yields multiplication

$$(\mathrm{m}'_{a,3}) \quad \begin{cases} (f,g,j,m,h,k,n)(f',g',j',m',h',k',n') \\ \quad = (f+f'-(p+q)nj'+qk(g'+nm')-phm', \\ \qquad g+g'+nm', j+j'-km', m+m', h+h'-nk', k+k', n+n'), \end{cases}$$

for $H''_{7,a} = \boldsymbol{Z}^7$, which is a semidirect product $\boldsymbol{Z}^4 \times H_3$ with the action of H_3 on $\boldsymbol{Z}^4$ given by

$$(\mathrm{act}'_{a,3}) \quad \begin{cases} (h,k,n) : (f',g',j',m') \\ \quad \mapsto (f'-(p+q)nj'+qk(g'+nm')-phm', g'+nm', j'-km', m'), \end{cases}$$

Transferring this action of H_3 to $\mathscr{C}(\boldsymbol{T}^3)$ (and $L^2(\boldsymbol{T}^3)$) gives

$$(\mathrm{act}''_{a,3}) \quad (h,k,n) : x \mapsto \lambda^{qk}x, \quad w \mapsto \lambda^{-(p+q)n}w \quad \text{and} \quad v \mapsto \lambda^{qkn-ph}x^n w^{-k} v,$$

and leads to operators and operator equations on $L^2(\boldsymbol{T}^3)$,

$$(0,0,1) \sim U'' : w \mapsto \lambda^{-(p+q)}w \quad \text{and} \quad v \mapsto xv,$$

$$(0,1,0) \sim V'' : x \mapsto \lambda^q x \quad \text{and} \quad v \mapsto w^{-1}v,$$

$$W'' : f \mapsto vf \quad (f \in L^2(\boldsymbol{T}^3)), \quad \text{and}$$

$$(\mathrm{CR}_{a,3}) \qquad \begin{cases} (-1,0,0) \sim [U'', V''] = X''^{-1} : v \mapsto \lambda^p v, \\ [U'', W''] = Z'' : f \mapsto xf, \quad [V'', W''] = Y''^{-1} : f \mapsto w^{-1}f, \\ [X'', W''] = \lambda^{-p}, \quad [U'', Y''] = \lambda^{-(p+q)} \quad \text{and} \quad [V'', Z''] = \lambda^q. \end{cases}$$

The map

$$(\pi'') \qquad (f,g,j,m,h,k,n) \mapsto \lambda^f Z''^g Y''^j W''^m X''^h V''^k U''^n$$

is a representation of $H''_{7,a}$, generating a C^*-algebra $\mathfrak{A}_{\theta,3}^{7,a}$. There are isomorphisms $H''_{7,a} \to H_{7,a}$ and $\psi_1 : \mathfrak{A}_{\theta,3}^{7,a} \to \mathfrak{A}_{\theta,1}^{7,a}$, the latter of which is implemented by noting that the unitaries Z'', X'', U'', Y'', V'', W'' satisfy $(\mathrm{CR}_{a,1})$, so that ψ_1^{-1} is generated by

$$Z^g Y^h W^n X^j V^k U^m \mapsto Z''^g X''^h U''^n Y''^j V''^k W''^m.$$

4. There is a fourth representation. It is generated by a representation ρ of $H_{7,a}$ on $\ell^2(\boldsymbol{Z}^6)$,

$$\rho(f,g,h,k,j,m,n) : \delta_{(g',h',j',k',m',n')} \mapsto \lambda^{f+(p+q)jn'+pmh'+qk(g'-mn')}\delta_s,$$

where

$$s = (g+g'-mn', h+h'+kn', n+n', j+j'+mk', k+k', m+m').$$

The unitaries $U = \rho(0,0,0,0,0,0,1)$, $V = \rho(0,0,0,0,0,1,0)$ and $W = \rho(0,0,0,1,0,0,0)$ satisfy $(\mathrm{CR}_{a,1})$, so the C^*-algebra generated by ρ is isomorphic to $A_\theta^{7,a}$.

THEOREM 3. *The representations* π, π' *and* π'' *of* $\mathrm{H}_{7,a}$ *generating* $\mathfrak{A}^{7,a}_{\theta,i}$ $(\cong \mathrm{A}^{7,a}_{\theta})$, $1 \le i \le 3$, *are irreducible, and are not unitarily equivalent.*

Proof. We show that the representation π of $\mathrm{H}_{7,a}$ is irreducible, and start by noting that, in our notation, $\{x^g w^h v^n : g,h,n \in \boldsymbol{Z}\}$ is the usual basis for $\mathscr{H} = L^2(\boldsymbol{T}^3)$. Then we have

$$U : x^g w^h v^n \mapsto \lambda^{ph} x^{g-n} w^h v^n, \quad V : x^g w^h v^n \mapsto x^{qg} x^g w^{h+n} v^n \quad \text{and}$$
$$W : x^g w^m v^n \mapsto x^g w^h v^{n+1}.$$

Suppose that $T \in B(\mathscr{H})$ commutes with U, V and W; we must show that T is a multiple of the identity. Let matrix coefficients for T be given by

$$Tx^g w^h v^n = \sum_{g',h',n' \in \boldsymbol{Z}} t^{g,h,n}_{g',h',n'} x^{g'} w^{h'} v^{n'},$$

with $\sum_{g',h',n' \in \boldsymbol{Z}} |t^{g,h,n}_{g',h',n'}|^2 < \infty$ and, in fact, uniformly bounded in g,h,n.

Now $TW = WT$, $TV = VT$ and $TU = UT$ imply that

$$(1) \quad t^{g,h,n+1}_{g',h',n'+1} = t^{g,h,n}_{g',h',n'}, \quad (2) \quad \lambda^{gq} t^{g,h+n,n}_{g',h',n'} = \lambda^{qg'} t^{g,h,n}_{g',h'-n',n'}$$
$$\text{and} \quad (3) \quad \lambda^{ph} t^{g-n,h,n}_{g',h',n'} = \lambda^{ph'} t^{g,h,n}_{g'+n',h',n'}$$

respectively. Then (3), with $n = 0$, implies that (4) $t^{g,h,0}_{g',h',n'} = 0$ if $n' \ne 0$, because of the convergence condition. Also (1) and (4) imply that $t^{g,h,n}_{g',h',n'} = 0$ if $n \ne n'$, (2) and (1) imply that $t^{g,h,n}_{g',h',n} = 0$ if $g \ne g'$, and then (3) and (1) imply that $t^{g,h,n}_{g,h',n} = 0$ if $h \ne h'$; it follows that $t^{g,h,n}_{g,h,n}$ is constant for all g,h,n, T is a multiple of the identity.

The proof that the other representations are irreducible is similar. To see that the representations π and π' of $\mathrm{A}^{7,a}_{\theta}$ are not unitarily equivalent, suppose that T is a linear isometry of $\mathscr{H}$ onto itself, intertwines U, V, W and V', W', U', and is given by

$$Tx^g w^h v^n = \sum_{h',j,k \in \boldsymbol{Z}} t^{g,h,n}_{h',j,k} x^{h'} w^j v^k.$$

Then $TU = V'T$, $TV = W'T$ and $TW = U'T$ imply that

$$(1) \quad \lambda^{ph} t^{g-n,h,n}_{h',j,k} = \lambda^{ph'} t^{g,h,n}_{h',j-k,k} \quad (2) \quad \lambda^{gq} t^{g,h+n,n}_{h',j,k} = t^{g,h,n}_{h',j,k-1}$$
$$\text{and} \quad (3) \quad t^{g,h,n+1}_{h',j,k} = \lambda^{-(p+q)j} t^{g,h,n}_{h'+k,j,k},$$

respectively. Then (1), with $n = 0$, implies that (4) $t^{g,h,0}_{h',j,k} = 0$ if $k \ne 0$, (2) implies that $t^{g,h,0}_{h',j,0} = 0$, and then (3) implies that $t^{g,h,n}_{h',j,k} = 0$ always, which is a contradiction. The other proofs of inequivalence are similar. ■

The group $H_{7,*}$

Here there is only one group $G_{7,*} \cong G_{*,1}$, and $H_{7,*}$ is defined to be the lattice subgroup $\boldsymbol{Z}^7 \subset G_{*,1}$; so $H_{7,*}$ has the same multiplication formula as $G_{*,1}$

$$(\mathrm{m}'_{*,1}) \quad \begin{cases} (f,g,h,n,j,k,m)(f',g',h',n',j',k',m') \\ \quad = (f+f'+2jn'+mh'+k(g'-mn')+kh'+n'k(k-1)/2, \\ \qquad g+g'-mn', h+h'+kn', n+n', j+j'+mk', k+k', m+m'). \end{cases}$$

Of course, $H_{7,*}$ is the semidirect product $\boldsymbol{Z}^4 \times H_3$, with action of H_3 on $\boldsymbol{Z}^4$ given by

$$(\mathrm{act}'_{*,1}) \quad \begin{cases} (j,k,m) : (f',g',h',n') \mapsto (f'+2jn'+mh'+k(g'-mn') \\ \quad + kh'+n'k(k-1)/2, g'-mn', h'+kn', n'). \end{cases}$$

For $\lambda = e^{2\pi i\theta}$ with irrational θ, transferring this action of H_3 to $\mathscr{C}(\boldsymbol{T}^3)$ (and $L^2(\boldsymbol{T}^2)$) gives

$$(\mathrm{act}''_{*,1}) \quad (j,k,m) : x \mapsto \lambda^k x, \quad w \mapsto \lambda^{m+k} w \quad \text{and} \quad v \mapsto \lambda^{2j-km+k(k-1)/2} x^{-m} w^k v,$$

and leads to operators and operator equations on $L^2(\boldsymbol{T}^3)$.

$$(0,0,1) \sim U : w \to \lambda w \quad \text{and} \quad v \mapsto x^{-1} v,$$

$$(0,1,0) \sim V : x \mapsto \lambda x, \quad w \mapsto \lambda w \quad \text{and} \quad v \mapsto wv,$$

$$W : f \mapsto vf \quad (f \in L^2(\boldsymbol{T}^3)) \quad \text{and}$$

$$(\mathrm{CR}_{*,1}) \quad \begin{cases} (1,0,0) \sim [U,V] = X : v \mapsto \lambda^2 v, \\ [U,W] = Z^{-1} : f \mapsto x^{-1} f, \quad [V,W] = Y : f \mapsto wf, \\ [X,W] = \lambda^2, \quad [U,Y] = \lambda = [V,Z] = [V,Y]. \end{cases}$$

The map

$$(\rho) \qquad (f,g,h,n,j,k,m) \mapsto \lambda^f Z^g Y^h W^n X^j V^k U^m$$

is a representation of $H_{7,*}$, generating a C^*-algebra $\mathfrak{A}_{\theta,1}^{7,*}$.

To get a (minimal distal) flow $(H_3, \boldsymbol{T}^3)$, apply inversion to H_3 at $(\mathrm{act}''_{*,1})$ and arrive at

$$(\mathscr{F}_{*,1}) \quad (j,k,m) : (x,w,v) \mapsto (\lambda^{-m} x, \lambda^{k-m} w, \lambda^{-2j+km+m(m+1)/2} x^{-k} w^{-m} v).$$

Of course, there is an analogue of Theorem 2 for $H_{7,*}$.

THEOREM 4. *Let* $\lambda = e^{2\pi i\theta}$ *for an irrational* θ.

(a) *There is a unique (up to isomorphism)* C^**-algebra* $\mathrm{A}_\theta^{7,*}$ *generated (via* ρ*) by (any) unitaries* Z, Y, W, X, V *and* U *satisfying*

$$(\mathrm{CR}_{*,1}) \quad \begin{cases} [U,V] = X, \quad [U,W] = Z^{-1}, \quad [V,W] = Y, \\ [X,W] = \lambda^2, \quad [U,Y] = \lambda \quad \text{and} \quad [V,Z] = \lambda^{-1} \end{cases}$$

$A_\theta^{7,*}$ *is simple and is universal for the equations* $(CR_{*,1})$. *Let* H_3 *act on* $\mathscr{C}(\boldsymbol{T}^3)$ *as indicated at* $(act''_{*,1})$; *then*

$$A_\theta^{7,*} \cong C^*(\mathscr{C}(\boldsymbol{T}^3), H_3) \cong \mathfrak{A}_{\theta,1}^{7,*}.$$

(b) *Let* ρ' *be a representation of* $H_{7,*}$ *such that* $\rho = \rho'$ *(as scalars) on the center* $(\boldsymbol{Z}, 0, 0, 0, 0, 0, 0)$ *of* $H_{7,*}$, *and let* A *be the* C^**-algebra generated by* ρ'. *Then* $A \cong A_\theta^{7,*}$ *via a unique isomorphism* ω *such that the following diagram commutes.*

$$\begin{array}{ccc} H_{7,*} & \xrightarrow{\rho} & A_\theta^{7,*} \\ \rho' \searrow & & \swarrow \omega \\ & A & \end{array}$$

(c) *The* C^**-algebra* $A_\theta^{7,*}$ *has a unique tracial state.*

The simple quotient $A_\theta^{7,*}$ of $C^*(H_{7,*})$ has 3 concrete representations, the first of which is $\mathfrak{A}_{\theta,1}^{7,*}$ above. The second presentation $G_{*,2}$ for $G_{7,*}$ gives another concrete representation $\mathfrak{A}_{\theta,2}^{7,*}$ for $A_\theta^{7,*}$. These C^*-algebras $\mathfrak{A}_{\theta,1}^{7,*}$ and $\mathfrak{A}_{\theta,2}^{7,*}$ arise from irreducible representations of $H_{7,*}$ that are unitarily inequivalent (much as in Theorem 3). The third representation of $A_\theta^{7,*}$ is analogous to representation 4 of $A_\theta^{7,a}$.

The groups $H_{7,\beta}$, $0 < \beta$

Here $G_{7,\beta} \cong G_{\beta,1}$, and as for $G_{a,1}$, we deal only with the rational case $0 < \beta = p/q$ with $(p, q) = 1$. Then $(\boldsymbol{Z}/q) \times \boldsymbol{Z}^6$ is a cocompact subgroup of $G_{\beta,1}$, so multiply the first coordinate of $G_{\beta,1}$ by q and define $H_{7,\beta}$ to be the set $\boldsymbol{Z}^7$ with multiplication

$$(m'_{\beta,1}) \quad \begin{cases} (f,g,h,n,j,k,m)(f',g',h',n',j',k',m') \\ \quad = (f+f'+q(2jn'+mh'+k(g'-mn')) \\ \qquad + p(kh'+n'k(k-1)/2 - mg' + n'm(m-1)/2), \\ \qquad g+g'-mn', h+h'+kn', n+n', j+j'+mk', k+k', m+m'). \end{cases}$$

Of course, $H_{7,a}$ is the semidirect product $\boldsymbol{Z}^4 \times H_3$, with action of H_3 on $\boldsymbol{Z}^4$ given by

$$(act'_{\beta,1}) \quad \begin{cases} (j,k,m) : (f',g',h',n') \mapsto (f' + q(2jn' + mh' + k(g'-mn')) \\ \qquad + p(kh' + n'k(k-1)/2 - mg' + n'm(m-1)/2), g'-mn', h'+kn', n'). \end{cases}$$

For $\lambda = e^{2\pi i\theta}$ with irrational θ, transferring the action of H_3 to $\mathscr{C}(\boldsymbol{T}^3)$ (and $L^2(\boldsymbol{T}^2)$) gives

$$(act''_{\beta,1}) \quad \begin{cases} (j,k,m) : x \mapsto \lambda^{qk-pm}x, \quad w \mapsto \lambda^{qm+pk}w \quad \text{and} \\ v \mapsto \lambda^{q(2j-km)+p(m(m-1)/2+k(k-1)/2)}x^{-m}w^k v, \end{cases}$$

and leads to operators and operator equations on $L^2(\boldsymbol{T}^3)$

$$(0,0,1) \sim U : x \mapsto \lambda^{-p}x, \quad w \mapsto \lambda^q w \quad \text{and} \quad v \mapsto x^{-1}v,$$
$$(0,1,0) \sim V : x \mapsto \lambda^q x, \quad w \mapsto \lambda^p w \quad \text{and} \quad v \mapsto wv,$$
$$W : f \mapsto vf \quad (f \in L^2(\boldsymbol{T}^3)), \quad \text{and}$$

$$(\mathrm{CR}_\beta) \qquad \begin{cases} (1,0,0) \sim [U,V] = X : v \mapsto \lambda^{2q}v, \\ [U,W] = Z^{-1} : f \mapsto x^{-1}f, \quad [V,W] = Y : f \mapsto wf, \\ [X,W] = \lambda^{2q}, \quad [U,Y] = \lambda^q, \quad [U,Z] = \lambda^{-p}, \\ [V,Y] = \lambda^p \quad \text{and} \quad [V,Z] = \lambda^q. \end{cases}$$

To get a (minimal distal) flow $(\mathrm{H}_3, \boldsymbol{T}^3)$, apply inversion to H_3 at $(\mathrm{act}'_{\beta,1})$ and arrive at

$$(\mathscr{F}_\beta) \qquad \begin{cases} (j,k,m) : (x,w,v) \mapsto (\lambda^{pm-qk}x, \lambda^{-qm-pk}w, \\ \qquad \lambda^{q(-2j+km)+p(m(m+1)/2+k(k+1)/2)}x^m w^{-k} v). \end{cases}$$

The reader can devise analogues of Theorems 2 and 3 for $\mathrm{H}_{7,\beta}$.

Notes. As for A^3_θ, we have here $\mathrm{A}^{7,a}_\theta \cong \mathrm{A}^{7,a}_{-\theta}$; this is because of the isomorphism of flows effected by the homeomorphism

$$(x,w,v) \mapsto (\bar{x}, \bar{w}, \bar{v})$$

of $\boldsymbol{T}^3$. The same observation holds for $\mathrm{A}^{7,*}_\theta$ and $\mathrm{A}^{7,\beta}_\theta$. However, the broad task of classifying these C^*-algebras must wait until a later time; it seems that the presence of different powers of the same λ in the commutation equations, e.g., $(\mathrm{CR}_{a,1})$, is an obstacle. It may be worthwhile to point out that the faithful simple quotients here, e.g., $\mathrm{A}^{7,a}_\theta$, are non-faithful simple quotients of an 8-dimensional group H_8 ($=Z^8$ as a set) that does not involve such exponents; H_8 has 2-dimensional centre, and multiplication

$$(\mathrm{m}_8) \quad \begin{cases} (e,f,g,h,n,j,k,m)(e',f',g',h',n',j',k',m') \\ \quad = (e+e'+jn'+mh', f+f'+jn'+k(g'-mn'), \\ \qquad g+g'-mn', h+h'+kn', n+n', j+j'+mk', k+k', m+m'). \end{cases}$$

The faithful simple quotients of this group are generated by representations of H_8, $(e,f,\ldots) \mapsto \lambda^e \mu^f \ldots$ for linearly independent $\lambda, \mu \in \boldsymbol{T}$. The algebras $\mathrm{A}^{7,a}_\theta$ are generated by (non-faithful) representations of H_8, $(e,f,\ldots) \mapsto \lambda^{pe}\lambda^{qf}\ldots$.

References

[EH] E. G. Effros and F. Hahn, Locally compact transformation groups and C^*-algebras, Memoirs of the American Mathematical Society 75, American Mathematical Society, Providence, 1967.

[LT] P. Lancaster and M. Tismenetsky, The Theory of Matrices, Second Edition, Computer Science and Applied Mathematics, Academic Press, Orlando, 1985.

[M] P. Milnes, Minimal H_3 actions and simple quotients of a discrete 6-dimensional nilpotent group, Kodai Math. J., **21** (1998), 306–317.

[MW1] P. Milnes and S. Walters, Simple quotients of the group C^*-algebra of a discrete 4-dimensional nilpotent group, Houston J. Math., **19** (1993), 615–636.

[MW2] P. Milnes and S. Walters, Simple infinite dimensional quotients of $C^*(G)$ for discrete 5-dimensional nilpotent groups G, Illinois J. Math., **41** (1997), 315–340.

[N] O. A. Nielsen, Unitary Representations and Coadjoint Orbits of Low-Dimensional Nilpotent Lie Groups, Queen's Papers in Pure and Applied Mathematics 63, Queen's University, Kingston, 1983.

[SS1] T. Skjelbred and T. Sund, Sur la classification des algèbres de Lie nilpotentes, C. R. Acad. Sci. Paris Sér. A-B, **286** (1978), A241–A242.

[SS2] T. Skjelbred and T. Sund, On the classification of nilpotent Lie algebras, preprint, 1977.

[Z-M] G. Zeller-Meier, Produits croisés d'une C^*-algèbre par un groupe d'automorphismes, J. Math. Pures et Appl., **47** (1968), 101–239.

Department of Mathematics
University of Western Ontario
London, Ontario
Canada
e-mail: milnes@uwo.ca