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Existence of sign-changing solutions for $p(x)$-Laplacian Kirchhoff type problem in $\mathbb{R}^N$

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Abstract. The $p(x)$-Laplacian Kirchhoff type equation involving the nonlocal term $b\int_{\mathbb{R}^N}(1/p(x))|\nabla u|^{p(x)}dx$ is investigated. Based on the variational methods, deformation lemma and other technique of analysis, it is proved that the problem possesses one least energy sign-changing solution u_b which has precisely two nodal domains. Moreover, the convergence property of u_b as the parameter $b \searrow 0$ is also obtained.

1. Introduction.

In the past decades, the study of parabolic and elliptic equation involving $p(x)$-Laplacian has attracted great attention due to its extensive applications in the electrorheological fluids [**19**], elastic mechanics [**26**] and others various fields, and we refer to the results of the parabolic $p(x)$-Laplacian equation (see [**8**], [**15**]), the elliptic $p(x)$-Laplacian equation (see [**2**], [**6**]) and the reference therein. Recently, people are interested in study of the $p(x)$-Kirchhoff elliptic problems and obtain the existence of solution or multiple solutions for the equations, such as [**1**], [**4**], [**7**], [**17**] and so on. In fact, the initial model of $p(x)$-Kirchhoff type problems is firstly presented by Kirchhoff [**16**]:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where ρ, ρ_0, h, E, L are constants. This model is the extension of the classical D'Alembert wave equation, which considers the change in the length of the string when vibrating. The problem (1.1) can be studied under several physical and biological systems, and we refer readers to [**5**], [**14**]. The purpose of this paper is to consider the existence of sign-changing solutions for variable exponent Kirchhoff type problem in $\mathbb{R}^N$ with $N \geq 2$, namely,

$$-\left(a + b\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{P_b}$$

where $-\Delta_{p(x)}u \equiv \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $a > 0$, $b > 0$ and the function $f(x,u)$ is the external force. The variable exponent $p(x)$ is Lipschitz continuous

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and satisfies

$$1 < p^- := \inf_{x\in\mathbb{R}^N} p(x) \le p^+ := \sup_{x\in\mathbb{R}^N} p(x) < N.$$

The problem (P_b) is usually called a nonlocal problem due to $b\int_{\mathbb{R}^N}(1/p(x))|\nabla u|^{p(x)}dx$. If $b\to 0$, then the problem (P_b) reduces to the following problem

$$-a\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = f(x,u). \tag{P_0}$$

There are lots of notable results on the existence of sign-changing solutions for the nonlocal problem in case that the exponent function is just a constant function, such as $p(x)=2$ or $p(x)=p$. For $p(x)=2$, Shuai in [**21**] considered the following problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$, $N=1,2,3$, is a bounded domain with smooth boundary and obtained the sign-changing solutions with the external force $f(u)$ satisfying some monotonic condition (see condition (f_5) in Remark 2.6 for detail). This restriction is replaced by a weaker condition in Tang and Cheng [**23**] (see condition (f_6) in Remark 2.6 for detail). Later, Wang, Zhang and Cheng in [**25**] generalized the results of [**21**], [**23**] to the case of entire space and obtained the ground state sign-changing solution for (1.2) involving an additional term $V(x)u$ with a given weighted function $V(x)$. As to $p(x)=p$, Rasouli, Fani and Khademloo [**20**], Han, Yao [**13**], Han, Ma and He [**12**] generalized the results of Shuai [**21**], Tang and Cheng[**23**], Wang, Zhang and Cheng [**25**] to the one of p-Laplacian Kirchhoff problem respectively.

In view of the results mentioned above, there are several contributions in the present paper. Firstly, we transfer the Kirchhoff problem to the framework of $p(x)$-Laplacian. As far as we know, there is no result on the least energy sign-changing solution for the $p(x)$-Laplacian Kirchhoff problem. Furthermore, we consider the problem (P_b) in the whole space $\mathbb{R}^N$. Since the variational functional of the equation has totally different properties from the case of no nonlocal term, we not only depend on the constraint variational method and quantitative deformation lemma to establish the variational framework, but also use the term $V(x)|u|^{p(x)-2}u$ to solve the difficulty of losing compactness in $\mathbb{R}^N$. As to the conditions on the external force, we remove the differentiable condition $f\in C^1$ and the monotonic assumption $f(s)/|s|^\alpha$ on $\mathbb{R}\backslash\{0\}$ with some power $\alpha>0$ (see [**12**], [**20**], [**21**]). Motivated by [**23**], we also try to find a weaker condition than the one in [**12**], [**20**], [**21**] (see assumption (f_4) in next section for detail). One can see that our condition can be reduced to the one in [**13**], [**23**] when $p(x)=2$ and $p(x)=p$ respectively.

This paper is divided into four sections. In next section, we introduce the variational setting, some mathematical preliminaries and main results. Section 3 will give some crucial lemmas. In Section 4, we prove the existence result of (P_b), (P_0) and the convergence problem possess a least energy sign-changing solution.

2. Preliminaries and main results.

2.1. Mathematical preliminaries.

In order to deal with problem (P_b), we state the certain definitions and basic properties of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^N)$ and the variable exponent Lebesgue–Sobolev spaces $W^{1,p(\cdot)}(\mathbb{R}^N)$.

For any $p(x) \in C(\mathbb{R}^N)$, the variable exponents Lebesgue space is defined by:

$$L^{p(\cdot)}(\mathbb{R}^N) = \left\{ u : u \text{ is a measurable real-valued function } \int_{\mathbb{R}^N} |u|^{p(x)} dx < \infty \right\},$$

with the norm

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ is defined by:

$$W^{1,p(\cdot)}(\mathbb{R}^N) = \left\{ u \in L^{p(\cdot)}(\mathbb{R}^N) : |\nabla u| \in L^{p(\cdot)}(\mathbb{R}^N) \right\},$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} = \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)} + \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}.$$

We assume the weight function $V : \mathbb{R}^N \to (0, +\infty)$ is continuous and satisfies:

(V) There exist $\alpha > 0$ such that $V(x) \geq \alpha > 0$ for all $x \in \mathbb{R}^N$, and measurable $\{x \in \mathbb{R}^N : -\infty < V(x) \leq v_0\} < +\infty$ for all $v_0 \in \mathbb{R}$.

In this paper, we work in the following subspace :

$$E = \left\{ u \in W^{1,p(\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx < +\infty \right\},$$

with the norm

$$\|u\|_E = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left(\left| \frac{\nabla u}{\lambda} \right|^{p(x)} + V(x) \left| \frac{u}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

Proposition 2.1 ([**10**]). *Let*

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)} dx, \quad \forall\, u \in L^{p(\cdot)}(\mathbb{R}^N).$$

Then,

(1) $\rho(u) < 1 (= 1; > 1) \Longleftrightarrow \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} < 1 (= 1; > 1)$;

(2) *if* $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1$, *then* $\|u\|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\mathbb{R}^N)}$;

(3) *if* $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} < 1$, *then* $\|u\|^{p^+}_{L^{p(\cdot)}(\mathbb{R}^N)} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\mathbb{R}^N)}$.

PROPOSITION 2.2 ([**9**]). *Let*

$$\rho(u)=\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx,\quad \forall\ u\in E.$$

Then,

(1) $\rho(u)<1(=1;>1)\iff \|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)}<1(=1;>1)$;

(2) *if* $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)}>1$*, then* $\|u\|^{p^-}_{W^{1,p(\cdot)}(\mathbb{R}^N)}\le\rho(u)\le\|u\|^{p^+}_{W^{1,p(\cdot)}(\mathbb{R}^N)}$;

(3) *if* $\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)}<1$*, then* $\|u\|^{p^+}_{W^{1,p(\cdot)}(\mathbb{R}^N)}\le\rho(u)\le\|u\|^{p^-}_{W^{1,p(\cdot)}(\mathbb{R}^N)}$.

LEMMA 2.3 ([**3**]). *Assume that* $p:\mathbb{R}^N\to\mathbb{R}$ *is Lipschitz continuous with* $1<p^-\le p^+<N$*. If the weight function* V *satisfies assumption* (V)*, then*

(1) *there is a compact embedding* $E\hookrightarrow L^{p(\cdot)}(\mathbb{R}^N)$;

(2) *for any measurable function* $q:\mathbb{R}^N\to\mathbb{R}$ *with* $p(x)<q(x)$ *for all* $x\in\mathbb{R}^N$*, there is a compact embedding* $E\hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$ *if* $\inf_{x\in\mathbb{R}^N}(p^*(x)-q(x))>0$*, where* $p^*(x)=Np(x)/(N-p(x))$.

2.2. Variational setting.

We give the variational setting for (P_b) in the following way.

DEFINITION 2.4. We say that $u\in E$ is a weak solution of (P_b), if

$$a\left(\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx\right)+b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx\right)$$
$$+\int_{\mathbb{R}^N}V(x)|u|^{p(x)-2}u\varphi\,dx=\int_{\mathbb{R}^N}f(x,u)\varphi\,dx \tag{2.1}$$

for all $\varphi\in E$.

Define the energy functional $I_b:E\to\mathbb{R}$ to problem (P_b) as follows:

$$I_b(u)=a\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)+\frac{b}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2$$
$$+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u|^{p(x)}dx-\int_{\mathbb{R}^N}F(x,u)dx. \tag{2.2}$$

Moreover, for any $u,\varphi\in E$, we have

$$\langle I_b'(u),\varphi\rangle=a\left(\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx\right)$$
$$+b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx\right)$$
$$+\int_{\mathbb{R}^N}V(x)|u|^{p(x)-2}u\varphi\,dx-\int_{\mathbb{R}^N}f(x,u)\varphi\,dx. \tag{2.3}$$

Clearly, the critical points of above functional are the weak solutions of (P_b). Furthermore, if $u\in E$ is a solution of (P_b) and $u^\pm\neq0$, then u is a sign-changing solution of (P_b), where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

We say that $u \in E$ is the least energy nodal solution of problem (P_b) if u is a sign-changing solution of (P_b) and

$$I_b(u) = \inf\left\{I_b(v) : v^\pm \neq 0, I_b'(v) = 0\right\}.$$

For $u = u^+ + u^-$, we have

$$I_b(u) = I_b(u^+) + I_b(u^-) + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right), \quad (2.4)$$

$$\left\langle I_b'(u), u^+\right\rangle = \left\langle I_b'(u^+), u^+\right\rangle + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} |\nabla u^+|^{p(x)}dx\right), \quad (2.5)$$

$$\left\langle I_b'(u), u^-\right\rangle = \left\langle I_b'(u^-), u^-\right\rangle + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} |\nabla u^-|^{p(x)}dx\right). \quad (2.6)$$

When $b = 0$, problem (P_b) no longer depends on the nonlocal term $(\int_{\mathbb{R}^N}(1/p(x))|\nabla u|^{p(x)}dx)\Delta_{p(x)}u$. The energy functional $I_0 : E \to \mathbb{R}^N$ to problem (P_0) as follows:

$$I_0(u) = a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|u|^{p(x)}dx - \int_{\mathbb{R}^N} F(x, u)dx. \quad (2.7)$$

Moreover,

$$\langle I_0'(u), \varphi\rangle = a\left(\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\, dx\right) + \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2}u\varphi\, dx - \int_{\mathbb{R}^N} f(x, u)\varphi\, dx. \quad (2.8)$$

Motivated by the work in [**21**], in order to get the least energy sign-changing solutions, we introduce the constrained set:

$$\mathcal{M}_b := \left\{u \in E : u^\pm \neq 0, \left\langle I_b'(u), u^\pm\right\rangle = 0\right\}, \quad (2.9)$$

and

$$\mathcal{M}_0 := \left\{u \in E : u^\pm \neq 0, \left\langle I_0'(u), u^\pm\right\rangle = 0\right\}. \quad (2.10)$$

Whose minimizers correspond to the sign-changing solutions for equation (P_b) and (P_0).

We also define the following Nehari manifold for (P_b) and (P_0) as

$$\mathcal{N}_b := \{u \in E : u \neq 0, \langle I_b'(u), u\rangle = 0\}, \quad (2.11)$$

and

$$\mathcal{N}_0 := \{u \in E : u \neq 0, \langle I_0'(u), u\rangle = 0\}. \quad (2.12)$$

Obviously, $\mathcal{M}_b \subset \mathcal{N}_b$ contains all sign-changing solutions of (P_b). Then, we can define the critical level by

$$c_b = \inf_{u\in\mathcal{M}_b} I_b(u), \quad c_0 = \inf_{u\in\mathcal{M}_0} I_0(u).$$

2.3. Assumptions and main results.

Next, we will give the conditions for nonlinearity f and our main results.

We impose the following conditions on the nonlinearity f:

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and $f(x,s) = o(|s|^{p(x)-1})$ as $s \to 0$;

(f_2) there exists a nonnegative weight function $a(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|a(x)\|_{L^\infty} < \alpha$ such that

$$|f(x,s)| \le a(x)|s|^{q(x)-1}, \quad \forall\ s \in \mathbb{R},$$

where $q(\cdot) \in (p(\cdot), p^*(\cdot))$;

(f_3) $\lim_{|s|\to\infty} f(x,s)/s^{2p(x)-1} = \infty$;

(f_4) there exists a $\theta_0 \in (0,1)$ such that for every $\tau \in \mathbb{R}\backslash\{0\}$

$$f(x,s\tau)\tau - s^{2p^+-1}f(x,\tau)\tau - \min\{1,a\}\theta_0|\nabla\tau|^{p(x)}\big(1-s^{p^+}\big)s^{p^+-1} \ge 0, \quad \text{for } s \ge 1;$$

$$\frac{p^-\left(s^{2p^--1}\right)}{p^+}f(x,\tau)\tau - f(x,s\tau)\tau + \frac{p^-\min\{1,a\}\theta_0}{p^+}|\nabla\tau|^{p(x)}\big(1-s^{p^-}\big)s^{p^--1} \ge 0, \quad \text{for } 0 < s \le 1.$$

We are now in a position to give the main results of this paper.

THEOREM 2.5. *Assume that (f_1)–(f_4) hold. Then, problems (P_b) and (P_0) have a least energy sign-changing solution u_b and u_0 respectively, which has precisely two nodal domains. Moreover, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, there exists a subsequence of $\{u_{b_n}\}$, we still denote by $\{u_{b_n}\}$ such that $u_{b_n} \to u_0$ in E, where $u_0 \in \mathcal{M}_0$ is the least energy sign-changing solution of (P_0).*

REMARK 2.6. (1) We see that Shuai [**21**] studied the existence of sign-changing solutions for equation (1.2) with some strong conditions $f \in C^1$ and the Nehari type monotonicity condition:

(f_5) $f(t)/|t|^3$ is increasing on $(-\infty,0)\cup(0,+\infty)$.

Tang and Cheng [**23**] improved the results only need $f \in C$ and satisfy:

(f_6) there exists a $\theta_0 \in (0,1)$ such that for all $t > 0$ and $\tau \in \mathbb{R}\backslash\{0\}$

$$\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \frac{a\theta_0\lambda_1|1-t^2|}{(t\tau)^2} \ge 0,$$

where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. It is worth to mention that Tang and Cheng give an example to show (f_6) much weaker than (f_5) in paper [**23**]. Actually, if we choose the index $p^+ = p_- = p(x) = 2$ in our assumption (f_4), then condition (f_4) reduces to (f_6).

(2) We consider the problem in $\mathbb{R}^N$ cause the lack of compactness. To overcome this

difficulty, we define a subspace with weight function $V(x)$ of the generalized Lebesgue–Sobolev space with the equivalent norm, use the weight function condition (V) and the important Lemma 2.3.

(3) The difference is $p(x)$-Laplacian operator has more complex nonlinearity, such as it does not have the "first eigenvalue" [**11**]. When processing the article we can not scaled down the energy functional directly, our process is relatively simplified. Moreover, the degree theory, deformation lemma and Miranda's theorem are also can be used.

3. Some technical lemmas.

In this section, we give some preliminary lemmas which are crucial for proving our results.

LEMMA 3.1. *Assume that (f_1)–(f_4) hold, then*

$$\begin{aligned} I_b(u) \geq I_b(su^+ + tu^-) &+ \frac{1-\mu^2}{2p^+}\left\langle I_b'(u), u^+\right\rangle + \frac{1-\eta^2}{2p^+}\left\langle I_b'(u), u^-\right\rangle \\ &+ \frac{\min\{1,a\}(1-\theta_0)(1-\mu)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u^+|^{p(x)} + V(x)|u^+|^{p(x)}\right]dx \\ &+ \frac{\min\{1,a\}(1-\theta_0)(1-\eta)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u^-|^{p(x)} + V(x)|u^-|^{p(x)}\right]dx \\ &+ \frac{b(\mu-\eta)^2}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right). \end{aligned} \tag{3.1}$$

where

$$\mu = \begin{cases} s^{p^-} & 0 < s \leq 1; \\ s^{p^+} & s \geq 1, \end{cases} \qquad \eta = \begin{cases} t^{p^-} & 0 < t \leq 1; \\ t^{p^+} & t \geq 1. \end{cases}$$

PROOF. According to the different situations of s and t mentioned above, we will divide the proof into four cases. We give the detail proof for the case of $0 < s \leq 1$, $0 < t \leq 1$ below and omit the similar proof for other cases.

Firstly, it follows from (f_4), we have for $0 < s \leq 1$

$$\begin{aligned} &\frac{1-t^{2p^-}}{2p^+}f(x,\tau)\tau + F(x,t\tau) - F(x,\tau) + \frac{\min\{1,a\}\theta_0}{2p^+}\left(1-t^{p^-}\right)^2|\nabla\tau|^{p(x)} \\ &= \int_t^1\left[\frac{p^-\left(s^{2p^- -1}\right)}{p^+}f(x,\tau)\tau - f(x,s\tau)\tau\right. \\ &\qquad\left. + \frac{p^-\min\{1,a\}\theta_0}{p^+}|\nabla\tau|^{p(x)}\left(1-s^{p^-}\right)s^{p^- -1}\right]ds \geq 0, \end{aligned} \tag{3.2}$$

and for $s \geq 1$

$$\frac{1-t^{2p^+}}{2p^+}f(x,\tau)\tau + F(x,t\tau) - F(x,\tau) + \frac{\min\{1,a\}\theta_0}{2p^+}\left(1-t^{p^+}\right)^2|\nabla\tau|^{p(x)}$$

$$= \int_1^t \left[f(x,s\tau)\tau - s^{2p^+-1}f(x,\tau)\tau - \min\{1,a\}\theta_0|\nabla\tau|^{p(x)}\big(1-s^{p^+}\big)s^{p^+-1}\right]ds \geq 0. \tag{3.3}$$

If $0 < s \leq 1,\ 0 < t \leq 1$, then from (2.2), (2.3), (3.2), one has

$$\begin{aligned}
&I_b(u) - I_b(su^+ + tu^-) = I_b(u^+ + u^-) - I_b(su^+ + tu^-)\\
&\geq \frac{a\big(1-s^{p^-}\big)}{p^+}\int_{\mathbb{R}^N}|\nabla u^+|^{p(x)}dx\\
&\quad + \frac{b\big(1-s^{2p^-}\big)}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^+|^{p(x)}dx\right)\\
&\quad + \frac{1-s^{p^-}}{p^+}\int_{\mathbb{R}^N}V(x)|u^+|^{p(x)}dx - \frac{1-s^{2p^-}}{2p^+}\int_{\mathbb{R}^N}f(x,u^+)u^+dx\\
&\quad + \frac{a\big(1-t^{p^-}\big)}{p^+}\int_{\mathbb{R}^N}|\nabla u^-|^{p(x)}dx\\
&\quad + \frac{b\big(1-t^{2p^-}\big)}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^-|^{p(x)}dx\right)\\
&\quad + \frac{1-t^{p^-}}{p^+}\int_{\mathbb{R}^N}V(x)|u^-|^{p(x)}dx - \frac{1-t^{2p^-}}{2p^+}\int_{\mathbb{R}^N}f(x,u^-)u^-dx\\
&\quad + b\big(1-s^{p^-}t^{p^-}\big)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\\
&\quad + \int_{\mathbb{R}^N}\left[\frac{1-s^{2p^-}}{2p^+}f(x,u^+)u^+ + F(x,su^+) - F(x,u^+)\right]dx\\
&\quad + \int_{\mathbb{R}^N}\left[\frac{1-t^{2p^-}}{2p^+}f(x,u^-)u^- + F(x,tu^-) - F(x,u^-)\right]dx\\
&= \frac{1-s^{2p^-}}{2p^+}\left\langle I_b'(u),u^+\right\rangle - \frac{b\big(1-s^{2p^-}\big)}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^+|^{p(x)}dx\right)\\
&\quad + \frac{1-t^{2p^-}}{2p^+}\left\langle I_b'(u),u^-\right\rangle - \frac{b\big(1-t^{2p^-}\big)}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^-|^{p(x)}dx\right)\\
&\quad + \frac{a\big(1-s^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}|\nabla u^+|^{p(x)}dx + \frac{\big(1-s^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}V(x)|u^+|^{p(x)}dx\\
&\quad + \frac{a\big(1-t^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}|\nabla u^-|^{p(x)}dx + \frac{\big(1-t^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}V(x)|u^-|^{p(x)}dx\\
&\quad + b\big(1-s^{p^-}t^{p^-}\big)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\\
&\quad + \int_{\mathbb{R}^N}\left[\frac{1-s^{2p^-}}{2p^+}f(x,u^+)u^+ + F(x,su^+) - F(x,u^+)\right]dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\left[\frac{1-t^{2p^-}}{2p^+}f(x,u^-)u^- + F(x,tu^-) - F(x,u^-)\right]dx\\
\geq\ & \frac{1-s^{2p^-}}{2p^+}\left\langle I_b'(u),u^+\right\rangle + \frac{1-t^{2p^-}}{2p^+}\left\langle I_b'(u),u^-\right\rangle + \frac{a\big(1-s^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}|\nabla u^+|^{p(x)}dx\\
&+\frac{\big(1-s^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}V(x)|u^+|^{p(x)}dx + \frac{a\big(1-t^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}|\nabla u^-|^{p(x)}dx\\
&+\frac{\big(1-t^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}V(x)|u^-|^{p(x)}dx\\
&+\frac{b\big(s^{p^-}-t^{p^-}\big)^2}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\\
&+\int_{\mathbb{R}^N}\left[\frac{1-s^{2p^-}}{2p^+}f(x,u^+)u^+ + F(x,su^+) - F(x,u^+)\right]dx\\
&+\int_{\mathbb{R}^N}\left[\frac{1-t^{2p^-}}{2p^+}f(x,u^-)u^- + F(x,tu^-) - F(x,u^-)\right]dx\\
\geq\ & \frac{1-s^{2p^-}}{2p^+}\left\langle I_b'(u),u^+\right\rangle + \frac{1-t^{2p^-}}{2p^+}\left\langle I_b'(u),u^-\right\rangle\\
&+\frac{\min\{1,a\}\,(1-\theta_0)\big(1-s^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u^+|^{p(x)} + V(x)|u^+|^{p(x)}\right]dx\\
&+\frac{\min\{1,a\}\,(1-\theta_0)\big(1-t^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u^-|^{p(x)} + V(x)|u^-|^{p(x)}\right]dx\\
&+\frac{b\big(s^{p^-}-t^{p^-}\big)^2}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\right)\\
&+\int_{\mathbb{R}^N}\Bigg[\frac{1-s^{2p^-}}{2p^+}f(x,u^+)u^+ + F(x,su^+) - F(x,u^+)\\
&\qquad\qquad + \frac{\min\{1,a\}\theta_0}{2^{p^+}}\big(1-s^{p^-}\big)^2|\nabla u^+|^{p(x)}\Bigg]dx\\
&+\int_{\mathbb{R}^N}\Bigg[\frac{1-t^{2p^-}}{2p^+}f(x,u^-)u^- + F(x,tu^-) - F(x,u^-)\\
&\qquad\qquad + \frac{\min\{1,a\}\theta_0}{2^{p^+}}\big(1-t^{p^-}\big)^2|\nabla u^-|^{p(x)}\Bigg]dx\\
\geq\ & \frac{1-s^{2p^-}}{2p^+}\left\langle I_b'(u),u^+\right\rangle + \frac{1-t^{2p^-}}{2p^+}\left\langle I_b'(u),u^-\right\rangle\\
&+\frac{\min\{1,a\}(1-\theta_0)\big(1-s^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u^+|^{p(x)} + V(x)|u^+|^{p(x)}\right]dx\\
&+\frac{\min\{1,a\}(1-\theta_0)\big(1-t^{p^-}\big)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u^-|^{p(x)} + V(x)|u^-|^{p(x)}\right]dx
\end{aligned}
$$

$$+\frac{b\big(s^{p^-}-t^{p^-}\big)^2}{2}\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\bigg)\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\bigg).$$

The above inequality shows that (3.1) holds. □

COROLLARY 3.2. *Under the assumption of (f_1)–(f_4), let $u=u^++u^-\in\mathcal{M}_b$, then*

$$\begin{aligned}I_b(u)\geq I_b(su^++tu^-)&+\frac{\min\{1,a\}(1-\theta_0)(1-\mu)^2}{2p^+}\int_{\mathbb{R}^N}\Big[|\nabla u^+|^{p(x)}+V(x)|u^+|^{p(x)}\Big]dx\\&+\frac{\min\{1,a\}(1-\theta_0)(1-\eta)^2}{2p^+}\int_{\mathbb{R}^N}\Big[|\nabla u^-|^{p(x)}+V(x)|u^-|^{p(x)}\Big]dx\\&+\frac{b(\mu-\eta)^2}{2}\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^+|^{p(x)}dx\bigg)\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^-|^{p(x)}dx\bigg).\end{aligned}\tag{3.4}$$

COROLLARY 3.3. *Under the assumption of (f_1)–(f_4), let $u=u^++u^-\in\mathcal{M}_b$, then*

$$I_b(u^++u^-)=\max_{s,t\geq 0}I_b(su^++tu^-).\tag{3.5}$$

LEMMA 3.4. *Suppose $u\in E$ with $u^\pm\neq 0$ and f satisfies (f_1)–(f_4), then there exists a unique pair (s_u,t_u) with $s_u>0$, $t_u>0$ such that $s_uu^++t_uu^-\in\mathcal{M}_b$.*

PROOF. For any $u\in E$ with $u^\pm\neq 0$, let

$$\begin{aligned}g_1(s,t)=&\int_{\mathbb{R}^N}\Big[a|\nabla(su^+)|^{p(x)}+V(x)|su^+|^{p(x)}\Big]dx\\&+b\bigg(\int_{\mathbb{R}^N}|\nabla(su^+)|^{p(x)}dx\bigg)\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(su^+)|^{p(x)}dx\bigg)\\&+b\bigg(\int_{\mathbb{R}^N}|\nabla(su^+)|^{p(x)}dx\bigg)\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tu^-)|^{p(x)}dx\bigg)\\&-\int_{\mathbb{R}^N}f(x,su^+)su^+\,dx\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}g_2(s,t)=&\int_{\mathbb{R}^N}\Big[a|\nabla(tu^-)|^{p(x)}+V(x)|tu^-|^{p(x)}\Big]dx\\&+b\bigg(\int_{\mathbb{R}^N}|\nabla(tu^-)|^{p(x)}dx\bigg)\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tu^-)|^{p(x)}dx\bigg)\\&+b\bigg(\int_{\mathbb{R}^N}|\nabla(tu^-)|^{p(x)}dx\bigg)\bigg(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(su^+)|^{p(x)}dx\bigg)\\&-\int_{\mathbb{R}^N}f(x,tu^-)tu^-\,dx.\end{aligned}\tag{3.7}$$

According to (f_1) and (f_3), we can get that $g_1(s,s)>0$ and $g_2(s,s)>0$ when $s>0$ small, $g_1(t,t)<0$ and $g_2(t,t)<0$ when $t>0$ large. Thus, there exist $0<r<R$ such that

$$g_1(r,r) > 0,\ g_2(r,r) > 0;\ g_1(R,R) < 0,\ g_2(R,R) < 0. \tag{3.8}$$

From (3.6), (3.7), (3.8), we deduce that

$$g_1(r,t) > 0, \quad g_1(R,t) < 0 \quad \forall\, t \in [r,R] \tag{3.9}$$

and

$$g_2(s,r) > 0, \quad g_2(s,R) < 0 \quad \forall\, s \in [r,R]. \tag{3.10}$$

By Miranda's theorem in [**18**], there exists some points (s_u,t_u) with $r < s_u, t_u < R$ such that $g_1(s_u,t_u) = g_2(s_u,t_u) = 0$. Thus, $s_u u^+ + t_u u^- \in \mathcal{M}_b$.

Next, we try to give the uniqueness of (s_u,t_u). Suppose there exist (s_1,t_1) and (s_2,t_2) such that $s_1u^+ + t_1u^- \in \mathcal{M}_b$ and $s_2u^+ + t_2u^- \in \mathcal{M}_b$. We may wish to assume $s_1 < s_2$ and $t_1 < t_2$. From Corollary 3.2, we have

$$\begin{aligned} I_b(s_1u^+ + t_1u^-) &\ge I_b(s_2u^+ + t_2u^-) \\ &+ \frac{\min\{1,a\}(1-\theta_0)\big(s_1^{p^+} - s_2^{p^+}\big)^2}{2p^+ s_1^{p^+}} \int_{\mathbb{R}^N} \Big[|\nabla u^+|^{p(x)} + V(x)|u^+|^{p(x)}\Big]\,dx \\ &+ \frac{\min\{1,a\}(1-\theta_0)\big(t_1^{p^+} - t_2^{p^+}\big)^2}{2p^+ t_1^{p^+}} \int_{\mathbb{R}^N} \Big[|\nabla u^-|^{p(x)} + V(x)|u^-|^{p(x)}\Big]\,dx \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} I_b(s_2u^+ + t_2u^-) &\ge I_b(s_1u^+ + t_1u^-) \\ &+ \frac{\min\{1,a\}(1-\theta_0)\big(s_2^{p^-} - s_1^{p^-}\big)^2}{2p^+ s_2^{p^-}} \int_{\mathbb{R}^N} \Big[|\nabla u^+|^{p(x)} + V(x)|u^+|^{p(x)}\Big]\,dx \\ &+ \frac{\min\{1,a\}(1-\theta_0)\big(t_2^{p^-} - t_1^{p^-}\big)^2}{2p^+ t_2^{p^-}} \int_{\mathbb{R}^N} \Big[|\nabla u^-|^{p(x)} + V(x)|u^-|^{p(x)}\Big]\,dx. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) implies $(s_1,t_1) = (s_2,t_2)$. □

LEMMA 3.5. *Assume that (f_1)–(f_4) hold. Then*

$$c_b = \inf_{u\in\mathcal{M}_b} I_b(u) = \inf_{u\in E,\, u^\pm \ne 0}\ \max_{s,t\ge 0} I_b(su^+ + tu^-).$$

PROOF. On the one hand, through Corollary 3.3 and the fact $\mathcal{M}_b \subset \{u | u \in E, u^\pm \ne 0\}$, we deduce that

$$\inf_{u\in E,\, u^\pm\ne 0}\ \max_{s,t\ge 0} I_b(su^+ + tu^-) \le \inf_{u\in\mathcal{M}_b}\ \max_{s,t\ge 0} I_b(su^+ + tu^-) = \inf_{u\in\mathcal{M}_b} I_b(u) = c_b. \tag{3.13}$$

On the other hand, for any $u \in E$ with $u^\pm \ne 0$, by Lemma 3.4 we have

$$\max_{s,t\ge 0} I_b(su^+ + tu^-) \ge I_b(s_u u^+ + t_u u^-) \ge \inf_{v\in\mathcal{M}_b} I_b(v) = c_b.$$

Also, we get the fact

$$\inf_{u\in E, u^\pm\neq 0}\max_{s,t\geq 0} I_b(su^+ + tu^-) \geq \inf_{u\in\mathcal{M}_b} I_b(u) = c_b. \tag{3.14}$$

Combining (3.13) and (3.14) we get the conclusion. □

LEMMA 3.6. *If (f_4) holds, then*

$$\frac{1}{2p^+} f(x,\tau)\tau - F(x,\tau) + \frac{\min\{1,a\}\theta_0}{2p^+}|\nabla\tau|^{p(x)} \geq 0, \quad \forall\, \tau\in\mathbb{R}. \tag{3.15}$$

PROOF. We can take $t = 0$ in (3.2) and (3.3) to get the conclusion. □

LEMMA 3.7. *There exists $\beta > 0$ such that*
(1) $\min\Big\{\|u^\pm\|_E^{p^+}, \|u^\pm\|_E^{p^-}\Big\} \geq \beta$ *for all* $u\in\mathcal{M}_b$;
(2) $I_b(u) \geq \min\Big\{\dfrac{a(1-\theta_0)}{2p^+}, \dfrac{1-\theta_0}{2p^+}\Big\}\beta$ *and* $\min\Big\{\|u\|_E^{p^+}, \|u\|_E^{p^-}\Big\} \geq \beta$ *for all* $u\in\mathcal{N}_b$.

PROOF. (1) From (f_2) we can deduce that

$$f(x,u)u \leq a(x)|u|^{q(x)}. \tag{3.16}$$

For every $u\in\mathcal{M}_b$, we have $\langle I_b'(u^\pm), u^\pm\rangle < 0$. It implies that

$$\begin{aligned}\langle I_b'(u^\pm), u^\pm\rangle &= \int_{\mathbb{R}^N}\Big[a|\nabla u^\pm|^{p(x)} + V(x)|u^\pm|^{p(x)}\Big]dx \\ &\quad + b\left(\int_{\mathbb{R}^N}|\nabla u^\pm|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^\pm|^{p(x)}dx\right) \\ &\quad - \int_{\mathbb{R}^N} f(x,u^\pm)u^\pm dx < 0,\end{aligned}$$

that is

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[a|\nabla u^\pm|^{p(x)} + V(x)|u^\pm|^{p(x)}\Big]dx \\ &\quad\leq \int_{\mathbb{R}^N}\Big[a|\nabla u^\pm|^{p(x)} + V(x)|u^\pm|^{p(x)}\Big]dx \\ &\qquad + b\left(\int_{\mathbb{R}^N}|\nabla u^\pm|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u^\pm|^{p(x)}dx\right) \\ &\quad< \int_{\mathbb{R}^N} f(x,u^\pm)u^\pm dx.\end{aligned}$$

Under the assumption of (V_1) and (3.16), it follows that

$$\begin{aligned}\min\{a,1\}\|u^\pm\|_E^{p^+} &\leq \int_{\mathbb{R}^N}\Big[a|\nabla u^\pm|^{p(x)} + V(x)|u^\pm|^{p(x)}\Big]dx \\ &\leq \int_{\mathbb{R}^N} a(x)|u^\pm|^{q(x)}dx\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\|a(x)\|_{L_\infty}}{\alpha}\int_{\mathbb{R}^N} V(x)|u^\pm|^{q(x)}dx \\
&\le \frac{\|a(x)\|_{L_\infty}}{\alpha}\|u^\pm\|_E^{q^-} \qquad \text{for} \quad 0<\|u^\pm\|_E\le 1, \\
\min\{a,1\}\|u^\pm\|_E^{p^-} &\le \frac{\|a(x)\|_{L_\infty}}{\alpha}\|u^\pm\|_E^{q^+} \qquad \text{for} \quad \|u^\pm\|_E\ge 1.
\end{aligned}
$$

Then, we can find suitable β such that $\beta \le \min\left\{\|u^\pm\|_E^{p^+}, \|u^\pm\|_E^{p^-}\right\}$.

(2) We have for function $u\in\mathcal{N}_b$,

$$
\begin{aligned}
I_b(u) &= I_b(u) - \frac{1}{2p^+}\langle I_b'(u),u\rangle \\
&= a\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) + \frac{b}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u|^{p(x)}dx - \int_{\mathbb{R}^N}F(x,u)dx \\
&\quad - \left[\frac{a}{2p^+}\left(\int_{\mathbb{R}^N}|\nabla u|^{p(x)}dx\right) + \frac{b}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u|^{p(x)}dx\right)\right. \\
&\qquad \left. + \frac{1}{2p^+}\int_{\mathbb{R}^N}V(x)|u|^{p(x)}dx - \frac{1}{2p^+}\int_{\mathbb{R}^N}f(x,u)udx\right] \\
&\ge \frac{a}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) + \frac{1}{2}\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u|^{p(x)}dx \\
&\quad + \frac{1}{2p^+}\int_{\mathbb{R}^N}f(x,u)udx - \int_{\mathbb{R}^N}F(x,u)dx \\
&\ge \frac{a}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) + \frac{1}{2}\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u|^{p(x)}dx \\
&\quad - \frac{\min\{1,a\}\theta_0}{2p^+}\int_{\mathbb{R}^N}|\nabla u|^{p(x)}dx \\
&\ge \frac{a}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) + \frac{1}{2}\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u|^{p(x)}dx \\
&\quad - \frac{\min\{1,a\}\theta_0}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right]dx \\
&\ge \min\left\{\frac{a(1-\theta_0)}{2p^+}, \frac{1-\theta_0}{2p^+}\right\}\int_{\mathbb{R}^N}\left[a|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right]dx \\
&\ge \min\left\{\frac{a(1-\theta_0)}{2p^+}, \frac{1-\theta_0}{2p^+}\right\}\min\left\{\|u\|_E^{p^+}, \|u\|_E^{p^-}\right\} \\
&\ge \min\left\{\frac{a(1-\theta_0)}{2p^+}, \frac{1-\theta_0}{2p^+}\right\}\beta.
\end{aligned}
$$

The proof is thus completed. □

LEMMA 3.8. *Assume that (f_1)–(f_4) hold. Then $c_b>0$ can be achieved.*

PROOF. From Lemma 3.7, we know that $c_b \ge \min\{a(1-\theta_0)/2p^+, (1-\theta_0)/2p^+\}\beta >$

0 due to $0 < \theta_0 < 1$. Let $u_n \in \mathcal{M}_b$ be such that $I_b(u_n) \to c_b$. We first show that $\{u_n\}$ is bounded in E. In fact, for large $n \in \mathbb{N}$, we have

$$c_b + 1 \geq I_b(u_n) - \frac{1}{2p^+}\langle I_b'(u_n), u_n\rangle \geq \min\left\{\frac{a(1-\theta_0)}{2p^+}, \frac{1-\theta_0}{2p^+}\right\} \min\left\{\|u_n^\pm\|_E^{p^+}, \|u_n^\pm\|_E^{p^-}\right\},$$

thus we get the conclusion due to $0 < \theta_0 < 1$, and then exist a $u_b \in E$ such that $u_n^\pm \rightharpoonup u_b^\pm$ in E. Since $u_n \in \mathcal{M}_b$, we have $\langle I_b'(u_n), u_n^\pm\rangle = 0$, that is

$$\begin{aligned}\int_{\mathbb{R}^N}\left[a|\nabla u_n^\pm|^p(x) + V(x)|u_n^\pm|^{p(x)}\right]dx + b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_n^\pm|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u_n^\pm|^{p(x)}dx\right)\\ = \int_{\mathbb{R}^N} f(x, u_n^\pm)u_n^\pm\,dx.\end{aligned}$$

From Lemma 3.7 we know that $\min\{\|u_n^\pm\|_E^{p^+}, \|u_n^\pm\|_E^{p^-}\} \geq \beta$ for $n \in \mathbb{N}$. With (f_2) and the boundedness of u_n we have

$$\begin{aligned}\min\{1,a\}\beta &\leq \min\{1,a\}\min\left\{\|u_n^\pm\|_E^{p^+}, \|u_n^\pm\|_E^{p^-}\right\}\\ &\leq \int_{\mathbb{R}^N}\left[a|\nabla u_n^\pm|^p(x) + V(x)|u_n^\pm|^{p(x)}\right]dx\\ &< \int_{\mathbb{R}^N} f(x, u_n^\pm)u_n^\pm\,dx\\ &\leq \int_{\mathbb{R}^N} a(x)|u_n^\pm|^{q(x)}dx\\ &\leq \frac{\|a(x)\|_{L^\infty}}{\alpha}\int_{\mathbb{R}^N} V(x)|u_n^\pm|^{q(x)}dx\\ &\leq \frac{\|a(x)\|_{L^\infty}}{\alpha}\max\left\{\|u_n^\pm\|_E^{q^+}, \|u_n^\pm\|_E^{q^-}\right\},\end{aligned}$$

which means

$$\beta\min\{1,a\} \times \frac{\alpha}{\|a(x)\|_{L^\infty}} \leq \max\left\{\|u_n^\pm\|_E^{q^+}, \|u_n^\pm\|_E^{q^-}\right\}.$$

By Lemma 2.3 the compactness of the embedding $E \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$ for $\inf_{x\in\mathbb{R}^N}(p^*(x) - q(x)) > 0$, we can get

$$\beta\min\{1,a\} \times \frac{\alpha}{\|a(x)\|_{L^\infty}} \leq \max\left\{\|u_b^\pm\|_E^{q^+}, \|u_b^\pm\|_E^{q^-}\right\},$$

which implies that $u_b^\pm \neq 0$. With conditions (f_1)–(f_2) and the compactness lemma of Strauss [**22**], we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x, u_n^\pm)u_n^\pm\,dx &= \int_{\mathbb{R}^N} f(x, u_b^\pm)u_b^\pm\,dx,\\ \lim_{n\to\infty}\int_{\mathbb{R}^N} F(x, u_n^\pm)dx &= \int_{\mathbb{R}^N} F(x, u_b^\pm)dx.\end{aligned} \tag{3.17}$$

By the Fatou's lemma, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[a|\nabla u_b^{\pm}|^{p(x)} + V(x)|u_b^{\pm}|^{p(x)}\right] dx + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_b|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} |\nabla u_b^{\pm}|^{p(x)}dx\right) \\
&\quad \le \liminf \left\{\int_{\mathbb{R}^N} \left[a|\nabla u_n^{\pm}|^{p(x)} + V(x)|u_n^{\pm}|^{p(x)}\right] dx \right. \\
&\qquad \left. + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} |\nabla u_n^{\pm}|^{p(x)}dx\right)\right\}.
\end{aligned}$$

Then we can get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[a|\nabla u_b^{\pm}|^{p(x)} + V(x)|u_b^{\pm}|^{p(x)}\right] dx + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_b|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} |\nabla u_b^{\pm}|^{p(x)}dx\right) \\
&\quad \le \int_{\mathbb{R}^N} f(x, u_b^{\pm})u_b^{\pm}\, dx,
\end{aligned}$$

which means that $\langle I_b'(u_b), u_b^{\pm}\rangle \le 0$. Applying Fatou's lemma, we obtain

$$\begin{aligned}
c_b &= \lim_{n\to\infty} \left[I_b(u_n) - \frac{1}{2p^+}\langle I_b'(u_n), u_n\rangle\right] \\
&= a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right) + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|u_n|^{p(x)}dx - \int_{\mathbb{R}^N} F(x, u_n)dx \\
&\quad - \frac{1}{2p^+}\left[a\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}dx\right) + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}dx\right)\right. \\
&\qquad \left. + \int_{\mathbb{R}^N} V(x)|u_n|^{p(x)}dx - \int_{\mathbb{R}^N} f(x, u_n)u_n\, dx\right] \\
&\ge \liminf_{n\to\infty} \left\{a\left(\int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)|\nabla u_n|^{p(x)}dx\right)\right\} \\
&\quad + \liminf_{n\to\infty} \left\{\frac{b}{2}\left[\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla u_n|^{p(x)}dx\right)\right]\right\} \\
&\quad + \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)V(x)|u_n|^{p(x)}dx\right) \\
&\quad + \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2p^+}f(x, u_n)u_n - F(x, u_n)\right] dx \\
&\ge a\left(\int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)|\nabla u_b|^{p(x)}dx\right) + \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)V(x)|u_b|^{p(x)}dx \\
&\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2p^+}f(x, u_b)u_b - F(x, u_b)\right] dx \\
&\quad + \frac{b}{2}\left[\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u_b|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla u_b|^{p(x)}dx\right)\right] \\
&= I_b(u_b) - \frac{1}{2p^+}\langle I'(u_b), u_b\rangle
\end{aligned}$$

$$\geq \sup_{s,t\geq 0}\left[I_b(su_b^+ + tu_b^-) + \frac{1-s^{2p^-}}{2p^+}\left\langle I_b'(u_b), u_b^+\right\rangle + \frac{1-t^{2p^-}}{2p^+}\left\langle I_b'(u_b), u_b^-\right\rangle\right]$$
$$- \frac{1}{2p^+}\langle I_b'(u_b), u_b\rangle$$
$$= \sup_{s,t\geq 0}\left[I_b(su_b^+ + tu_b^-) - \frac{s^{2p^-}}{2p^+}\left\langle I_b'(u_b), u_b^+\right\rangle - \frac{t^{2p^-}}{2p^+}\left\langle I_b'(u_b), u_b^-\right\rangle\right]$$
$$\geq \max_{s,t\geq 0} I_b(su_b^+ + tu_b^-)$$
$$\geq c_b.$$

Therefore, we conclude that $I_b(u_b) = c_b$ and $u_b \in \mathcal{M}_b$. □

Lemma 3.9. *Under the assumption (f_1)–(f_4) with $u_0 \in \mathcal{M}_b$ and $I_b(u_0) = c_b$, then u_0 is the critical point of I_b.*

Proof. We prove that $I_b'(u_b) = 0$ by using the deformation lemma. Suppose by contradiction $I'(u_b) \neq 0$, then there exist $\omega > 0$ and $\lambda > 0$ such that

$$u \in E, \quad \|I_b'(u)\|_{E'} \geq \lambda \quad \text{for all} \quad \|u - u_0\|_E \leq 3\omega. \tag{3.18}$$

For any $s, t \geq 0$

$$\begin{aligned} I_b(su_0^+ + tu_0^-) &\leq I_b(u_0) \\ &- \frac{\min\{1,a\}(1-\theta_0)(1-\mu)^2}{2p^+}\int_{\mathbb{R}^N}\left[a|\nabla u_0^+|^{p(x)} + V(x)|u_0^+|^{p(x)}\right]dx \\ &- \frac{\min\{1,a\}(1-\theta_0)(1-\eta)^2}{2p^+}\int_{\mathbb{R}^N}\left[a|\nabla u_0^-|^{p(x)} + V(x)|u_0^-|^{p(x)}\right]dx \quad \forall s,t\geq 0. \end{aligned} \tag{3.19}$$

Let $D = (1/2, 3/2) \times (1/2, 3/2)$. From (3.19), we see that

$$\kappa := \max_{(s,t)\in\partial D} I_b(su_0^+ + tu_0^-) < c_b. \tag{3.20}$$

For $\varepsilon := \min\{(c_b-\kappa)/3, 1, \omega\lambda/8\}$, $S := B(U_0, \omega)$. Lemma 2.3 in [**24**] yields a deformation such that

i) $\eta(1,u) = u$ if $u \notin I_b^{-1}([c_b - 2\varepsilon, c_b + 2\varepsilon]) \cap S_{2\omega}$;
ii) $\eta(1, I_b^{c_b+\varepsilon} \cap B(u_0,\omega)) \subset I_b^{c_b-\varepsilon}$;
iii) $I_b(\eta(1,u)) \leq I_b(u), \forall\, u \in E$.

Corollary 3.3 shows that $I_b(su_0^+ + tu_0^-) \leq I_b(u_0) = c_b$ for $s, t > 0$. With ii) we can get

$$I_b\left(\eta(1, su_0^+ + tu_0^-)\right) \leq c_b - \varepsilon, \quad \forall s,t \geq 0,\ |s-1|^2 + |t-1|^2 < \omega^2/\|u_0\|^2. \tag{3.21}$$

Also, combine iii) and (3.19), we have

$$\begin{aligned} I_b\left(\eta(1, su_0^+ + tu_0^-)\right) &\leq I_b(su_0^+ + tu_0^-) \\ &\leq c_b - \frac{\min\{1,a\}(1-\theta_0)(1-\mu)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u_0^+|^{p(x)} + V(x)|u_0^+|^{p(x)}\right]dx \end{aligned}$$

$$
\begin{aligned}
&- \frac{\min\{1,a\}(1-\theta_0)(1-\eta)^2}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u_0^-|^{p(x)}+V(x)|u_0^-|^{p(x)}\right]dx\\
&\leq c_b - \frac{\min\{1,a\}(1-\theta_0)\omega^2}{2p^+\|u_0\|^2}\\
&\times \min\left\{\int_{\mathbb{R}^N}\left[|\nabla u_0^+|^{p(x)}+V(x)|u_0^+|^{p(x)}\right]dx, \int_{\mathbb{R}^N}\left[|\nabla u_0^-|^{p(x)}+V(x)|u_0^-|^{p(x)}\right]dx\right\}\\
&\forall\ s,t\geq 0, |s-1|^2+|t-1|^2\geq \omega^2/\|u_0\|^2. \qquad (3.22)
\end{aligned}
$$

According to (3.21) and (3.22), we have

$$
\max_{(s,t)\in \overline{D}} I_b\left(\eta(1, su_0^+ + tu_o^-)\right) < c_b. \qquad (3.23)
$$

Next, we prove that $\eta(1, su_0^+ + tu_0^-)\cap \mathcal{M}_b \neq \emptyset$, contradicting to the definition of c_b. Define

$$
\Psi_0(s,t) = \left(\left\langle I_b'(su_0^+ + tu_0^-), u_0^+\right\rangle, \left\langle I_b'(su_0^+ + tu_0^-), u_0^-\right\rangle\right)
$$

and

$$
\begin{aligned}
&\Psi_1(s,t)\\
&= \left(\frac{\left\langle I_b'\left(\eta(1, su_0^+ + tu_0^-)\right), \left(\eta(1, su_0^+ + tu_0^-)\right)^+\right\rangle}{s},\right.\\
&\qquad\left.\frac{\left\langle I_b'\left(\eta(1, su_0^+ + tu_0^-)\right), \left(\eta(1, su_0^+ + tu_0^-)\right)^-\right\rangle}{t}\right).
\end{aligned}
$$

According to Lemma 3.4 and degree theory we can deduce $\deg(\Psi_0, D, 0) = 1$. From (3.20) and i) it follows that $\eta(1, su_0^+ + tu_0^-) = su_0^+ + tu_0^-$. Therefore $\Psi_1(s_0,t_0) = 0$ for some $(s_0,t_0)\in D$ that is $\eta(1, s_0u_0^+ + t_0u_0^-)\in \mathcal{M}_b$, which contradicts (3.23). From this, we conclude that u_0 is a critical of I_b. □

4. Proof of main results.

This section is divided into two subsections to prove the main result Theorem 2.5.

4.1. Sign-changing solutions for (P_b).

Proof of Theorem 2.5. By Lemma 3.8 and Lemma 3.9, we can get a $u_b \in \mathcal{M}_b$ such that $I_b(u_b) = c_b$ and $I_b'(u_b) = 0$. By definition, we know that u_b is a least energy sign-changing solutions of (P_b).

Next, we prove that u_b has precisely two nodal domains. We assume by contradiction that $u_b = u_1 + u_2 + u_3$, where

$$
u_i \neq 0,\ u_1 \geq 0,\ u_2 \leq 0 \ \text{ and } \ \text{suppt}(u_i)\cap \text{suppt}(u_j) = \emptyset, \ \text{ for } \ i\neq j,\ i,j = 1,2,3.
$$

Setting $v = u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Using the fact that $I_b'(u_b)=0$, we can get $\langle I_b'(u_b),v^+\rangle=0$ and $\langle I_b'(u_b),v^-\rangle=0$, and it follows that

$$\begin{aligned}\left\langle I_b'(v),v^+\right\rangle &= \left\langle I_b'(u_b),v^+\right\rangle - b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^+|^{p(x)}dx\right)\\ &= -b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^+|^{p(x)}dx\right) \qquad (4.1)\end{aligned}$$

and

$$\begin{aligned}\left\langle I_b'(v),v^-\right\rangle &= \left\langle I_b'(u_b),v^-\right\rangle - b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^-|^{p(x)}dx\right)\\ &= -b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^-|^{p(x)}dx\right). \qquad (4.2)\end{aligned}$$

Using above equality and Lemma 3.1, Lemma 3.6, we have

$$\begin{aligned}c_b &= I_b(u_b) = I_b(u_b) - \frac{1}{2p^+}\langle I_b'(u_b),u_b\rangle\\ &= I_b(v)+I_b(u_3)+b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\\ &\quad -\frac{1}{2p^+}\Bigg[\langle I_b'(v),v\rangle+\langle I_b'(u_3),u_3\rangle+b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla u_3|^{p(x)}dx\right)\\ &\qquad\qquad + b\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v|^{p(x)}dx\right)\Bigg]\\ &\geq \sup_{s,t\geq 0}\left[I_b(sv^++tv^-)+\frac{1-\mu^2}{2p^+}\left\langle I_b'(v),v^+\right\rangle+\frac{1-\eta^2}{2p^+}\left\langle I_b'(v),v^-\right\rangle\right]\\ &\quad -\frac{1}{2p^+}\langle I_b'(v),v\rangle+I_b(u_3)-\frac{1}{2p^+}\langle I_b'(u_3),u_3\rangle\\ &\geq \sup_{s,t\geq 0}\left[I_b(sv^++tv^-)-\frac{\mu^2}{2p^+}\left\langle I_b'(v),v^+\right\rangle-\frac{\eta^2}{2p^+}\left\langle I_b'(v),v^-\right\rangle\right]\\ &\quad +\frac{a}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)+\frac{1}{2}\left(\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u_3|^{p(x)}dx\right)\\ &\quad +\int_{\mathbb{R}^N}\left[\frac{1}{2p^+}f(x,u_3)u_3-F(x,u_3)\right]dx\\ &\geq \sup_{s,t\geq 0}\Bigg[I_b(sv^++tv^-)+\frac{b\mu^2}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^+|^{p(x)}dx\right)\\ &\quad +\frac{b\eta^2}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^-|^{p(x)}dx\right)\Bigg]+\frac{a}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_3|^{p(x)}dx\right)\\ &\quad +\frac{1}{2}\left(\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|u_3|^{p(x)}dx\right)+\int_{\mathbb{R}^N}\left[\frac{1}{2p^+}f(x,u_3)u_3-F(x,u_3)\right]dx\\ &\geq \max_{s,t\geq 0}I_b(sv^++tv^-)+\frac{\min\{1,a\}(1-\theta_0)}{2p^+}\int_{\mathbb{R}^N}\left[|\nabla u_3|^{p(x)}+V(x)|u_3|^{p(x)}\right]dx\\ &\geq c_b+\frac{\min\{1,a\}(1-\theta_0)}{2p^+}\min\left\{\|u_3\|_E^{p^+},\|u_3\|_E^{p^-}\right\}.\end{aligned}$$

This way, $u_3 = 0$ due to Lemma 3.5 and $0 < \theta_0 < 1$. Thus, u_b has exactly two nodal domains.

4.2. Sign-changing solutions for (P_0) and convergence property.

In the proof above, we know that $b = 0$ is allowed. Therefore (P_0) has the least energy sign-changing solution, which changes only once under the assumptions of Theorem 2.5. For convergence problem, we put any $b > 0$, and let $u_b \in \mathcal{M}_b$ be the least sign-changing solution of (P_b), which changes sign only once. We first prove that $\{u_{b_n}\}$ is bounded. We can let $\omega_0 \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ such that $\omega_0^\pm \neq 0$.
From (f_2) and (f_3), we can deduce that there exists $C_1 > 0$ and $C_2 > 0$ such that

$$F(x,s) \geq C_1 a(x) s^{2p(x)} - C_2 a(x), \quad \forall s \in \mathbb{R}. \tag{4.3}$$

Hence, for any $b \in [0,1]$, by Lemma 3.5 we see that

$$\begin{aligned}
I_b(u_b) &= c_b \leq \max_{s,t\geq 0} I_b(s\omega_0^+ + t\omega_0^-) \\
&= \max_{s,t\geq 0} \Bigg[a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right) + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |s\omega_0^+|^{p(x)} dx - \int_{\mathbb{R}^N} F(x, s\omega_0^+) dx \\
&\quad + a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right) + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |t\omega_0^-|^{p(x)} dx - \int_{\mathbb{R}^N} F(x, t\omega_0^-) dx \\
&\quad + b\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right)\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right) \Bigg] \\
&\leq \max_{s,t\geq 0} \Bigg[a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right) + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |s\omega_0^+|^{p(x)} dx - C_1 \int_{\mathbb{R}^N} a(x) |s\omega_0^+|^{2p(x)} dx + C_2 \int_{\mathbb{R}^N} a(x) dx \\
&\quad + a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right) + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^- |^{p(x)} dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |t\omega_0^-|^{p(x)} dx - C_1 \int_{\mathbb{R}^N} a(x) |t\omega_0^-|^{2p(x)} dx + C_2 \int_{\mathbb{R}^N} a(x) dx \\
&\quad + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right)^2 + \frac{b}{2}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right)^2 \Bigg] \\
&\leq \max_{s,t\geq 0} \Bigg[a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right) + \left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(s\omega_0^+)|^{p(x)} dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |s\omega_0^+|\, dx + a\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right) \\
&\quad + \left(\int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla(t\omega_0^-)|^{p(x)} dx\right)^2 + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |t\omega_0^-|\, dx \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2C_2\|a(x)\|_{L^\infty} \\
&:= \Lambda_0 \in (0,+\infty). \qquad (4.4)
\end{aligned}
$$

Then for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, from Lemma 3.6 and (4.4), we have for large $n \in \mathbb{N}$

$$
\begin{aligned}
\Lambda_0 + 1 &\geq I_{b_n}(u_{b_n}) - \frac{1}{2p^+}\left\langle I'_{b_n}(u_{b_n}), u_{b_n}\right\rangle \\
&\geq \min\left\{\frac{a(1-\theta_0)}{2p^+}, \frac{1-\theta_0}{2p^+}\right\} \min\left\{\|u_{b_n}\|_E^{p^+}, \|u_{b_n}\|_E^{p^-}\right\},
\end{aligned}
$$

which shows that $\{u_{b_n}\}$ is bounded in E due to $0 < \theta_0 < 1$. Then, we prove that $I'_0(u_0) = 0$. There exists a subsequence of $\{u_{b_n}\}$, still denote by $\{u_{b_n}\}$ and $u_0 \in E$ such that $u_{b_n} \rightharpoonup u_0$ in E. Then u_0 is a weak solution of (P_b). By Lemma 2.3 the compactness of the embedding $E \hookrightarrow L^{q(\cdot)}$ for $p(\cdot) < q(\cdot) < p^*(x)$, we deduce that $u_{b_n} \to u_0 \neq 0$ strongly in E as $n \to \infty$.

In fact,

$$
\begin{aligned}
\langle I'_0(u_0), \varphi\rangle &= a\left(\int_{\mathbb{R}^N} |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla\varphi\, dx\right) \\
&\quad + \int_{\mathbb{R}^N} V(x)|u_0|^{p(x)-2}u_0\varphi\, dx - \int_{\mathbb{R}^N} f(x,u_0)\varphi\, dx \\
&= \lim_{n\to\infty}\Bigg[\left(a + b_n\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u_{b_n}|^{p(x)}dx\right)\right)\left(\int_{\mathbb{R}^N}|\nabla u_{b_n}|^{p(x)-2}\nabla u_{b_n}\nabla\varphi\, dx\right) \\
&\qquad + \int_{\mathbb{R}^N} V(x)|u_{b_n}|^{p(x)-2}u_{b_n}\varphi\, dx - \int_{\mathbb{R}^N} f(x,u_{b_n})\varphi\, dx\Bigg] \\
&= \lim_{n\to\infty}\left\langle I'_{b_n}(u_{b_n}), \varphi\right\rangle = 0, \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^{\mathbb{N}}).
\end{aligned}
$$

Obviously, it follows that $I'_0(u_0) = 0$, thus $u_0 \in \mathcal{M}_0$ and $I_0(u_0) \geq c_0$. Next, we give the proof of $I_0(u_0) = c_0$. Let $b_n \in [0,1]$, due to (f_3) that there exists $K_0 > 0$ such that

$$
\begin{aligned}
I_{b_n}\left(sv_0^+ + tv_0^-\right) &= a\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(sv_0^+)|^{p(x)}dx\right) + \frac{b_n}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(sv_0^+)|^{p(x)}dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|sv_0^+|^{p(x)}dx - \int_{\mathbb{R}^N} F(x,sv_0^+)dx \\
&\quad + a\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tv_0^-)|^{p(x)}dx\right) + \frac{b_n}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tv_0^-)|^{p(x)}dx\right)^2 \\
&\quad + \int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|tv_0^-|^{p(x)}dx - \int_{\mathbb{R}^N} F(x,tv_0^-)dx \\
&\quad + b_n\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(sv_0^+)|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tv_0^-)|^{p(x)}dx\right) \\
&\leq a\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(sv_0^+)|^{p(x)}dx\right) + \left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(sv_0^+)|^{p(x)}dx\right)^2
\end{aligned}
$$

$$\begin{aligned}
&+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|sv_0^+|^{p(x)}dx-\int_{\mathbb{R}^N}F(x,sv_0^+)dx\\
&+a\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tv_0^-)|^{p(x)}dx\right)+\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla(tv_0^-)|^{p(x)}dx\right)^2\\
&+\int_{\mathbb{R}^N}\frac{V(x)}{p(x)}|tv_0^-|^{p(x)}dx-\int_{\mathbb{R}^N}F(x,tv_0^-)dx<0,\qquad \forall s,t\geq K_0. \quad (4.5)
\end{aligned}$$

According to Lemma 3.4, there exists (s_n,t_n) such that $s_nv_0^++t_nv_0^-\in\mathcal{M}_{b_n}$. Combining with Lemma 3.8 and (4.5), we have $0<s_n,t_n<K_0$. Because $I_0'(v_0)=0$, then from (2.7) and Lemma 3.1, we can get

$$\begin{aligned}
c_0=I_0(v_0)&=I_{b_n}-\frac{b_n}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\right)^2\\
&\geq I_{b_n}(s_nv_0^++t_nv_0^-)+\frac{1-s_n^{2p^-}}{2p^+}\left\langle I_{b_n}'(v_0),v_0^+\right\rangle+\frac{1-t_n^{2p^-}}{2p^+}\left\langle I_{b_n}'(v_0),v_0^-\right\rangle\\
&\quad-\frac{b_n}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\right)^2\\
&\geq c_{b_n}-\frac{1+K_0^{2p^-}}{2p^+}\left|\left\langle I_{b_n}'(v_0),v_0^+\right\rangle\right|-\frac{1+K_0^{2p^-}}{2p^+}\left|\left\langle I_{b_n}'(v_0),v_0^-\right\rangle\right|\\
&\quad-\frac{b_n}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\right)^2\\
&=c_{b_n}-\frac{(1+K_0^{2p^-})b_n}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v_0^+|^{p(x)}dx\right)\\
&\quad-\frac{(1+K_0^{2p^-})b_n}{2p^+}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\right)\left(\int_{\mathbb{R}^N}|\nabla v_0^-|^{p(x)}dx\right)\\
&\quad-\frac{b_n}{2}\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\right)^2, \qquad (4.6)
\end{aligned}$$

which implies

$$\limsup_{n\to\infty}c_{b_n}\leq c_0. \qquad (4.7)$$

From (2.2), (2.7) and (4.7), we have

$$c_0\leq I_0(u_0)=\limsup_{n\to\infty}I_{b_n}(u_{b_n})=\limsup_{n\to\infty}c_{b_n}\leq c_0.$$

This suggests that $I_0(u_0)=c_0$. □

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