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Diagram automorphisms and quantum groups

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Abstract. Let $\mathbf{U}_q^- = \mathbf{U}_q^-(\mathfrak{g})$ be the negative part of the quantum group associated to a finite dimensional simple Lie algebra $\mathfrak{g}$, and $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the automorphism obtained from the diagram automorphism. Let $\mathfrak{g}^\sigma$ be the fixed point subalgebra of $\mathfrak{g}$, and put $\underline{\mathbf{U}}_q^- = \mathbf{U}_q^-(\mathfrak{g}^\sigma)$. Let $\mathbf{B}$ be the canonical basis of $\mathbf{U}_q^-$ and $\underline{\mathbf{B}}$ the canonical basis of $\underline{\mathbf{U}}_q^-$. σ induces a natural action on $\mathbf{B}$, and we denote by $\mathbf{B}^\sigma$ the set of σ-fixed elements in $\mathbf{B}$. Lusztig proved that there exists a canonical bijection $\mathbf{B}^\sigma \simeq \underline{\mathbf{B}}$ by using geometric considerations. In this paper, we construct such a bijection in an elementary way. We also consider such a bijection in the case of certain affine quantum groups, by making use of PBW-bases constructed by Beck and Nakajima.

Introduction.

0.1. Let X be a Dynkin diagram with vertex set I, and $\mathfrak{g}$ the semisimple Lie algebra associated to X. We denote by $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ the quantum enveloping algebra of $\mathfrak{g}$, and by $\mathbf{U}_q^-$ its negative part, which are associative algebras over $\mathbf{Q}(q)$. Let W be the Weyl group of $\mathfrak{g}$, and w_0 the longest element of W. Let $\mathbf{h} = (i_1, \dots, i_\nu)$ be a sequence of $i_k \in I$ such that $w_0 = s_{i_1} \cdots s_{i_\nu}$ gives a reduced expression of w_0, where $s_i (i \in I)$ are simple reflections in W. For each $\mathbf{h}$ as above, there exists a basis $\mathscr{X}_{\mathbf{h}}$ of $\mathbf{U}_q^-$, called the PBW-basis of $\mathbf{U}_q^-$. Put $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$, and let ${}_{\mathbf{A}}\mathbf{U}_q^-$ be Lusztig's integral form of $\mathbf{U}_q^-$.

We consider the following statements.

(0.1.1)

(i) The $\mathbf{Z}[q]$-submodule of $\mathbf{U}_q^-$ generated by $\mathscr{X}_{\mathbf{h}}$ is independent of the choice of $\mathbf{h}$, which we denote by $\mathscr{L}_{\mathbf{Z}}(\infty)$.

(ii) The $\mathbf{Z}$-basis of $\mathscr{L}_{\mathbf{Z}}(\infty)/q\mathscr{L}_{\mathbf{Z}}(\infty)$ induced from $\mathscr{X}_{\mathbf{h}}$ is independent of the choice of $\mathbf{h}$.

(iii) For each $\mathbf{h}$, PBW-basis $\mathscr{X}_{\mathbf{h}}$ gives rise to an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\mathbf{U}_q^-$.

We also consider a weaker version of (iii),

(iii′) For each $\mathbf{h}$, any element of $\mathscr{X}_{\mathbf{h}}$ is contained in ${}_{\mathbf{A}}\mathbf{U}_q^-$.

The canonical basis $\mathbf{B}$ of $\mathbf{U}_q^-$ was constructed by Lusztig [**L2**], [**L3**] by using a geometric method. It is known that it coincides with the global crystal basis of Kashiwara [**K1**].

The statement (0.1.1) can be verified in general by making use of the canonical basis or Kashiwara's global crystal basis.

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0.2. We are interested in an elementary construction of canonical bases, in the sense that we don't appeal to Lusztig's geometric theory of canonical bases nor Kashiwara's theory of crystal bases. We shall construct canonical bases (as discussed in [**L2**]), by making use of PBW-basis, based on the properties (0.1.1). Actually, in their theories, canonical bases or crystal bases are constructed independently from PBW-bases. However those constructions look like a huge black box, and it is not easy to trace the construction even in the small rank cases. On the other hand, the construction of PBW-bases is more explicit, the parametrization is easy, and they fit to direct computations. So it is important to express canonical basis in terms of PBW-bases, which is the problem closely related to the elementary construction of canonical basis.

In the case where X is simply laced, the verification of (0.1.1) is rather easy. In the non-simply laced case, the problem is reduced to the case of type B_2 or G_2. In the case of B_2, the properties (i) and (iii) were verified by [**L1**], by computing the commutation relations of root vectors in the case of type B_2, and furthermore by applying the method of Kostant on the $\mathbf{Z}$-form of Chevalley groups in the case of type G_2. Later [**X1**] gave a proof of (iii) similar to the case of B_2. But in any case, it requires a hard computation. In [**X2**], Xi computed, in the case of B_2, the canonical basis of $\mathbf{U}_q^-$ explicitly in terms of PBW-basis. The property (ii) follows from his result. But the property (ii) for G_2 is not yet verified (in an elementary method).

If we assume (i) and (iii) in (0.1.1), one can construct the "canonical basis", which is only independent of $\mathbf{h}$, up to ± 1. We call them the signed basis of $\mathbf{U}_q^-$. Thus in the non-simply laced case, one can construct the signed basis.

0.3. Assume that X is simply laced, and let σ be a graph automorphism of X. We denote by $\underline{I}$ the set of orbits in I under the action of $\sigma : I \to I$. Then σ determines a Dynkin diagram $\underline{X}$ whose vertex set is given by $\underline{I}$. $\underline{X}$ corresponds to the σ-fixed point subalgebra $\mathfrak{g}^\sigma$ of $\mathfrak{g}$, and we denote by $\underline{\mathbf{U}}_q = \mathbf{U}_q(\mathfrak{g}^\sigma)$ the corresponding quantum enveloping algebra, and $\underline{\mathbf{U}}_q^-$ its negative part. Let $\mathbf{B}$ be the canonical basis of $\mathbf{U}_q^-$. Then σ permutes $\mathbf{B}$, and we denote by $\mathbf{B}^\sigma$ the set of σ-fixed elements in $\mathbf{B}$. We also denote by $\underline{\mathbf{B}}$ the set of canonical basis of $\underline{\mathbf{U}}_q^-$. In [**L4**] (and in [**L3**]), Lusztig proved that there exists a canonical bijection between $\mathbf{B}^\sigma$ and $\underline{\mathbf{B}}$, based on geometric considerations of canonical basis.

In this paper, we construct the bijection $\mathbf{B}^\sigma \xrightarrow{\sim} \underline{\mathbf{B}}$ in an elementary way. We assume that σ is admissible, namely for $\eta \in \underline{I}$, if $i, j \in \eta$ with $i \neq j$, then i and j are not joined in X. Let ε be the order of σ. We assume that $\varepsilon = 2$ or 3 (note that if X is irreducible, then $\varepsilon = 2$ or 3). Let $\mathbf{F}$ be the finite field $\mathbf{Z}/\varepsilon\mathbf{Z}$, and put $\mathbf{A}' = \mathbf{F}[q, q^{-1}] = \mathbf{A}/\varepsilon\mathbf{A}$. Let ${}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$ be the subalgebra of ${}_{\mathbf{A}}\mathbf{U}_q^-$ consisting of σ-fixed elements, and consider the $\mathbf{A}'$-algebra ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} = {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma} \otimes_{\mathbf{A}} \mathbf{A}'$. Let J be the $\mathbf{A}'$-submodule of ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$ consisting of elements of the form $\sum_{0 \leq i < \varepsilon} \sigma^i(x)$ for $x \in {}_{\mathbf{A}'}\mathbf{U}_q^-$. Then J is a two-sided ideal of ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$, and we denote by $\mathbf{V}_q$ the quotient algebra ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}/J$. We define ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ similarly to ${}_{\mathbf{A}'}\mathbf{U}_q^-$. We can prove the following result (Proposition 1.20 and Corollary 1.21).

THEOREM 0.4. *Assume that* (iii) *in* (0.1.1) *holds for* ${}_{\mathbf{A}}\mathbf{U}_q^-$, *and* (iii′) *holds for* ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$. *Then we have an isomorphism of* $\mathbf{A}'$*-algebras*

$$ {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \simeq \mathbf{V}_q. \tag{0.4.1} $$

Moreover (iii) *holds for* ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$.

By Theorem 0.4, one can define the signed basis for $\underline{\mathbf{U}}_q^-$ by assuming (iii′). But in the case of G_2, we have a more precise result (Proposition 1.23), namely

PROPOSITION 0.5. *Let* $\underline{\mathbf{U}}_q^-$ *be of type* G_2. *Then the ambiguity of the sign can be removed in the signed basis, hence* (ii) *of* (0.1.1) *holds for* $\underline{\mathbf{U}}_q^-$.

(0.4.1) gives a surjective map ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ combined with the natural surjection ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to \mathbf{V}_q$. This map is compatible with PBW-bases, hence induces a natural map $\mathbf{B}^\sigma \to \underline{\mathbf{B}}$, which is shown to be bijective (see Remark 1.24). Thus we can recover Lusztig's bijection $\mathbf{B}^\sigma \xrightarrow{\sim} \underline{\mathbf{B}}$ by an elementary method.

0.6. In Beck and Nakajima [**BN**], PBW-bases were constructed for the affine quantum enveloping algebras $\mathbf{U}_q^-$. They showed that an analogous property of (iii′) holds for those PBW-basis, and that of (iii) holds if the corresponding diagram X is simply laced. We apply the previous discussion to the case where X is simply laced of type $A_{2n+1}^{(1)}$ $(n \geq 1)$, $D_n^{(1)}$ $(n \geq 4)$, $E_6^{(1)}$ with $\varepsilon = 2$, and $D_4^{(1)}$ with $\varepsilon = 3$. Then $\underline{X}$ is twisted affine of type $D_{n+2}^{(2)}$, $A_{2n-3}^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$, respectively (under the notation in [**Ka**, 4.8]). We have (Corollary 2.17)

THEOREM 0.7. *Assume that* $\underline{X}$ *is twisted of type* $D_n^{(2)}$, $A_{2n-1}^{(2)}$, $E_6^{(2)}$ *or* $D_4^{(3)}$. *Then* (iii) *holds for* $\underline{\mathbf{U}}_q^-$. *Moreover the surjective map* ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ *gives a natural bijection* $\mathbf{B}^\sigma \xrightarrow{\sim} \underline{\mathbf{B}}$.

REMARK 0.8. Assume that $\mathfrak{g}$ is an affine Lie algebra, and $\mathfrak{g}_0$ the associated finite dimensional subalgebra of $\mathfrak{g}$. We consider the automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$. In order to apply the construction of PBW-basis in [**BN**] to our σ-setting, we need to assume that σ leaves $\mathfrak{g}_0$ invariant. Then $\mathfrak{g}^\sigma$ is necessarily twisted affine type. Our discussion can not cover the case where $\mathfrak{g}^\sigma$ is untwisted type.

0.9. As mentioned in 0.3, Lusztig has given a canonical bijection between the set of σ-stable canonical bases of $\mathbf{U}_q^-$ and the set of canonical bases of $\underline{\mathbf{U}}_q^-$. A closely related problem for crystal bases was also studied by many researchers, such as Naito and Sagaki [**NS**], Savage [**S**]. However those results are concerned with the level of the parametrization, since there exists no direct relationship between $\mathbf{U}_q^\sigma$ and $\underline{\mathbf{U}}_q^-$. The main observation in our work is that if we replace $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$ by $\mathbf{A}' = (\mathbf{Z}/\varepsilon\mathbf{Z})[q, q^{-1}]$, we obtain a natural surjective map from ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$ to ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ as $\mathbf{A}'$-algebras. This has an advantage that we can compare directly the algebra structure of $\underline{\mathbf{U}}_q^-$ and of $\mathbf{U}_q^{-,\sigma}$, not only the correspondence of bases. For example, the following is an easy consequence of our results. (Notations are as in Section 1 for the Dynkin case. A similar result also holds for the affine case.)

THEOREM 0.10. *Let* $b(\underline{\mathbf{c}}, \underline{\mathbf{h}})$ *be a canonical basis of* $\underline{\mathbf{U}}_q^-$, *and* $b(\mathbf{c}, \mathbf{h})$ *the corresponding* σ*-stable canonical basis of* $\mathbf{U}_q^-$. *We write them as*

$$b(\underline{\mathbf{c}},\underline{\mathbf{h}}) = L(\underline{\mathbf{c}},\underline{\mathbf{h}}) + \sum_{\underline{\mathbf{d}}>\underline{\mathbf{c}}} a_{\underline{\mathbf{d}}} L(\underline{\mathbf{d}},\underline{\mathbf{h}}),$$
$$b(\mathbf{c},\mathbf{h}) = L(\mathbf{c},\mathbf{h}) + \sum_{\mathbf{d}'>\mathbf{c}} a'_{\mathbf{d}'} L(\mathbf{d}',\mathbf{h}),$$

with $a'_{\mathbf{d}'}, a_{\underline{\mathbf{d}}} \in q\mathbf{Z}[q]$. *If* $L(\mathbf{d}',\mathbf{h})$ *is the* σ*-stable PBW-basis corresponding to* $L(\underline{\mathbf{d}},\underline{\mathbf{h}})$, *then we have* $a'_{\mathbf{d}'} \equiv a_{\underline{\mathbf{d}}} \pmod{\varepsilon}$.

Some examples of Theorem 0.10 for small rank cases were computed in [**MNZ**].

This research has grown up from the question, concerning the elementary construction of canonical bases, posed by Nakajima in his lecture note [**N**] on the lectures at Sophia University, 2006. The authors are grateful to him for his helpful suggestions.

1. PBW-bases and canonical bases.

1.1. In this paper, we understand that a Cartan datum is a pair $X = (I, (\ ,\ ))$, where $(\ ,\ )$ is a symmetric bilinear form on $\bigoplus_{i\in I} \mathbf{Q}\alpha_i$ (a finite dimensional vector space over $\mathbf{Q}$ with the basis $\{\alpha_i\}$ indexed by I) such that $(\alpha_i,\alpha_j) \in \mathbf{Z}$, satisfying the property

- $(\alpha_i,\alpha_i) \in 2\mathbf{Z}_{>0}$ for any $i \in I$,
- $2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i) \in \mathbf{Z}_{\leq 0}$ for any $i \neq j$ in I.

The Cartan datum X is called simply laced if $(\alpha_i,\alpha_j) \in \{0,-1\}$ for any $i \neq j$ in I, and $(\alpha_i,\alpha_i) = 2$ for any $i \in I$. The Cartan datum X determines a graph with the vertex set I. If the associated graph is connected, X is said to be irreducible. Put $a_{ij} = 2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$ for any $i,j \in I$. The matrix (a_{ij}) is called the Cartan matrix.

In the case where the bilinear form is positive definite, X is called finite type. In that case, the associated graph is a Dynkin diagram. In the case where the bilinear form is positive semi-definite, X is called affine type. In that case, the associated graph is a Euclidean diagram. In this paper, we are concerned with X of finite type or affine type.

1.2. Let $X = (I, (\ ,\ ))$ be a simply laced Cartan datum, and let $\sigma : I \to I$ be a permutation such that $(\sigma(\alpha_i),\sigma(\alpha_j)) = (\alpha_i,\alpha_j)$ for any $i,j \in I$. Let $\underline{I}$ be the set of orbits of σ on I. We assume that σ is admissible, namely for each orbit $\eta \in \underline{I}$, $(\alpha_i,\alpha_j) = 0$ for any $i \neq j$ in η.

We define a symmetric bilinear form $(\ ,\ )_1$ on $\bigoplus_{\eta\in\underline{I}} \mathbf{Q}\alpha_\eta$ by

$$(\alpha_\eta,\alpha_{\eta'})_1 = \begin{cases} 2|\eta| & \text{if } \eta = \eta', \\ -|\{(i,j) \in \eta\times\eta' \mid (\alpha_i,\alpha_j) \neq 0\}| & \text{if } \eta \neq \eta'. \end{cases}$$

It is easy to see that $\underline{X} = (\underline{I}, (\ ,\ )_1)$ defines a Cartan datum.

1.3. Let $I = \{1,2,\dots,2n-1\}$ for $n \geq 1$. For $i,j \in I$, we put $(\alpha_i,\alpha_j) = 2$ if $i = j$, $(\alpha_i,\alpha_j) = -1$ if $i - j = \pm 1$, and $(\alpha_i,\alpha_j) = 0$ otherwise. Then $(I, (\ ,\ ))$ is a simply laced irreducible Cartan datum of type A_{2n-1}. We define a permutation $\sigma : I \to I$ by $\sigma(i) = 2n - i$ for all i. Then σ satisfies the condition in 1.2. We can identify $\underline{I}$ with the

set $\{\underline{1},\dots,\underline{n}\}$, where $\underline{i}=\{i,2n-i\}$ for $1\le i\le n-1$ and $\underline{n}=\{n\}$. Then $(\underline{I},(\ ,\ )_1)$ is the Cartan datum of type B_n.

1.4. Let $I=\{1,2,2',2''\}$. We define a permutation $\sigma: I\to I$ of order 3 by $\sigma(1)=1$ and $\sigma: 2\mapsto 2'\mapsto 2''\mapsto 2$. The set $\underline{I}$ of orbits of σ in I is given by $\underline{I}=\{\underline{1},\underline{2}\}$, where $\underline{1}=\{1\}$ and $\underline{2}=\{2,2',2''\}$. We define a symmetric bilinear form on $\bigoplus_{i\in I}\mathbf{Q}\alpha_i$ by

$$(\alpha_i,\alpha_j)=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } i\in\underline{1}, j\in\underline{2} \text{ or } i\in\underline{2}, j\in\underline{1},\\ 0 & \text{if } i,j\in\underline{2}, i\ne j.\end{cases}$$

Then $(I,(\ ,\ ))$ gives the Cartan datum of type D_4. $\sigma: I\to I$ satisfies the condition in 1.2, and $(\underline{I},(\ ,\ )_1)$ gives the Cartan datum of type G_2.

1.5. Let q be indeterminate, and for an integer n, a positive integer m, put

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}},\quad [m]_q^!=\prod_{i=1}^m[i]_q,\quad [0]!_q=1.$$

For each $i\in I$, put $q_i=q^{(\alpha_i,\alpha_i)/2}$, and consider $[n]_{q_i}$, etc. by replacing q by q_i in the above formulas. Let $\mathbf{U}_q^-$ be the negative part of the quantum enveloping algebra $\mathbf{U}_q$ associated to a Cartan datum $X=(I,(\ ,\ ))$. Hence $\mathbf{U}_q^-$ is an associative algebra over $\mathbf{Q}(q)$ with generators f_i $(i\in I)$ satisfying the fundamental relations

$$\sum_{k=0}^{1-a_{ij}}(-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)}=0 \tag{1.5.1}$$

for any $i\ne j\in I$, where $f_i^{(n)}=f_i^n/[n]_{q_i}^!$ for a non-negative integer n.

We now assume that the Cartan datum X is simply laced. Then $[n]_{q_i}=[n]_q$ for any $i\in I$. Let $\sigma: I\to I$ be the automorphism as in 1.2. Then σ induces an algebra automorphism $\sigma:\mathbf{U}_q^-\xrightarrow{\sim}\mathbf{U}_q^-$ by $f_i\to f_{\sigma(i)}$. We denote by $\mathbf{U}_q^{-,\sigma}$ the subalgebra of $\mathbf{U}_q^-$ consisting of σ-fixed elements. Let $\mathbf{A}=\mathbf{Z}[q,q^{-1}]$, and ${}_{\mathbf{A}}\mathbf{U}_q^-$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}_q^-$ generated by $f_i^{(a)}$ for $i\in I$ and $a\in\mathbf{N}$ ($\mathbf{N}$ is the set of non-negative integers). Then σ stabilizes ${}_{\mathbf{A}}\mathbf{U}_q^-$, and we can define ${}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$ the subalgebra of ${}_{\mathbf{A}}\mathbf{U}_q^-$ consisting of σ-fixed elements.

Let $\underline{X}=(\underline{I},(\ ,\ )_1)$ be the Cartan datum obtained from σ as in 1.2. We denote by $\underline{\mathbf{U}}_q^-$ the negative part of the quantum enveloping algebra associated to $\underline{X}$, namely, $\underline{\mathbf{U}}_q^-$ is the $\mathbf{Q}(q)$-algebra generated by $\underline{f}_\eta$ with $\eta\in\underline{I}$ satisfying a similar relation as in (1.5.1).

Let ε be the order of σ (here we assume that $\varepsilon=2$ or 3), and let $\mathbf{F}=\mathbf{Z}/\varepsilon\mathbf{Z}$ be the finite field of ε-elements. Put $\mathbf{A}'=\mathbf{F}[q,q^{-1}]$, and consider the $\mathbf{A}'$-algebra

$${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}={}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}\otimes_{\mathbf{A}}\mathbf{A}'\simeq {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}/\varepsilon({}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}). \tag{1.5.2}$$

Let J be the $\mathbf{A}'$-submodule of ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$ consisting of elements of the form $\sum_{0\le i<\varepsilon}\sigma^i(x)$ for $x\in{}_{\mathbf{A}'}\mathbf{U}_q^-$. Then J is a two-sided ideal of ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$, and we denote by $\mathbf{V}_q$ the quotient algebra ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}/J$. Let $\pi:{}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}\to\mathbf{V}_q$ be the natural map.

Let $\underline{\mathbf{U}}_q^-$ be as before. We can define ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$ and ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ similarly to ${}_{\mathbf{A}}\mathbf{U}_q^-$ and ${}_{\mathbf{A}'}\mathbf{U}_q^-$.

1.6. In the rest of this section, we assume that X is of finite type. Let W be the Weyl group associated to the Cartan datum X, with simple reflections $\{s_i \mid i \in I\}$. Let $l : W \to \mathbf{N}$ be the standard length function of W relative to the generators s_i ($i \in I$). Let w_0 be the unique longest element in W with respect to l, and put $\nu = l(w_0)$. Let $\underline{W}$ be the Weyl group associated to the Cartan datum $\underline{X}$, with simple reflections $\{\underline{s}_\eta \mid \eta \in \underline{I}\}$. Then $\underline{l}, \underline{w}_0, \underline{\nu}$ with respect to $\underline{X}$ are defined similarly to l, w_0, ν. For any $\eta \in \underline{I}$, let w_η be the product of s_i for $i \in \eta$ (note, by our assumption, that such s_i are mutually commuting). Then $\underline{W}$ can be identified with the subgroup of W generated by $\{w_\eta \mid \eta \in \underline{I}\}$ under the correspondence $\underline{s}_\eta \leftrightarrow w_\eta$. The map $s_i \mapsto s_{\sigma(i)}$ defines an automorphism $\sigma : W \to W$, and $\underline{W}$ coincides with the subgroup $W^\sigma = \{w \in W \mid \sigma(w) = w\}$ of W under the above identification. We have $w_0 = \underline{w}_0$, and if $\underline{w}_0 = \underline{s}_{\eta_1} \cdots \underline{s}_{\eta_{\underline{\nu}}}$ is a reduced expression of $\underline{w}_0$, then $w_0 = w_{\eta_1} \cdots w_{\eta_{\underline{\nu}}}$, which satisfies the relation $\sum_{k=1}^{\underline{\nu}} l(w_{\eta_k}) = \nu$. Thus if we write $w_\eta = \prod_{i\in\eta} s_i$ for any $\eta \in \underline{I}$, $w_0 = w_{\eta_1} \cdots w_{\eta_{\underline{\nu}}}$ induces a reduced expression of w_0,

$$w_0 = \Big(\prod_{k_1\in\eta_1} s_{k_1}\Big) \cdots \Big(\prod_{k_{\underline{\nu}}\in\eta_{\underline{\nu}}} s_{k_{\underline{\nu}}}\Big) = s_{i_1} \cdots s_{i_\nu}. \tag{1.6.1}$$

We write $\underline{\mathbf{h}} = (\eta_1, \dots, \eta_{\underline{\nu}})$ and $\mathbf{h} = (i_1, \dots, i_\nu)$. Note that $\mathbf{h}$ is determined from $\underline{\mathbf{h}}$ by choosing the expression $w_\eta = s_{k_1} \cdots s_{k_{|\eta|}}$ for each η.

1.7. For any $i \in I$ the braid group action $T_i : \mathbf{U}_q \to \mathbf{U}_q$ is defined as in [**L4**, Chapter 39] (denoted by $T''_{s_i,1}$ there). Let $\mathbf{h} = (i_1, \dots, i_\nu)$ be a sequence such that $w_0 = s_{i_1} \cdots s_{i_\nu}$ is a reduced expression. For $\mathbf{c} = (c_1, \dots, c_\nu) \in \mathbf{N}^\nu$, put

$$L(\mathbf{c}, \mathbf{h}) = f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots (T_{i_1} \cdots T_{i_{\nu-1}})(f_\nu^{(c_\nu)}). \tag{1.7.1}$$

Then $\{L(\mathbf{c}, \mathbf{h}) \mid \mathbf{c} \in \mathbf{N}^\nu\}$ gives a PBW-basis of $\mathbf{U}_q^-$, which we denote by $\mathscr{X}_{\mathbf{h}}$. Now assume given $\sigma : I \to I$ as in 1.2. Then $\sigma \circ T_i \circ \sigma^{-1} = T_{\sigma(i)}$ and $T_iT_j = T_jT_i$ if $i, j \in \eta$. Hence one can define $R_\eta = \prod_{i\in\eta} T_i$ for each $\eta \in \underline{I}$, and R_η commutes with σ.

We consider the braid group action $\underline{T}_\eta : \underline{\mathbf{U}}_q \to \underline{\mathbf{U}}_q$. Let $\underline{\mathbf{h}} = (\eta_1, \dots, \eta_{\underline{\nu}})$ be a sequence for $\underline{w}_0$. For any $\underline{\mathbf{c}} = (\gamma_1, \dots, \gamma_{\underline{\nu}}) \in \mathbf{N}^{\underline{\nu}}$, $L(\underline{\mathbf{c}}, \underline{\mathbf{h}})$ is defined in a similar way as in (1.7.1),

$$L(\underline{\mathbf{c}}, \underline{\mathbf{h}}) = \underline{f}_{\eta_1}^{(\gamma_1)} \underline{T}_{\eta_1}(\underline{f}_{\eta_2}^{(\gamma_2)}) \cdots (\underline{T}_{\eta_1} \cdots \underline{T}_{\eta_{\underline{\nu}-1}})(\underline{f}_{\eta_{\underline{\nu}}}^{(\gamma_{\underline{\nu}})}). \tag{1.7.2}$$

Then $\{L(\underline{\mathbf{c}}, \underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$ gives a PBW-basis of $\underline{\mathbf{U}}_q^-$, which we denote by $\underline{\mathscr{X}}_{\underline{\mathbf{h}}}$.

Now assume that $\mathbf{h}$ is obtained from $\underline{\mathbf{h}}$ as in 1.6. Then $L(\mathbf{c}, \mathbf{h})$ can be written as follows. For $k = 1, \dots, \underline{\nu}$, let I_k be the interval in $[1, \nu]$ corresponding to η_k so that $w_{\eta_k} = \prod_{j\in I_k} s_{i_j}$ in the expression of w_0 in (1.6.1). Put $F_{\eta_k}(\mathbf{c}) = \prod_{j\in I_k} f_{i_j}^{(c_j)}$ for each k. Then we have

$$L(\mathbf{c}, \mathbf{h}) = F_{\eta_1}(\mathbf{c}) R_{\eta_1}(F_{\eta_2}(\mathbf{c})) \cdots (R_{\eta_1} \cdots R_{\eta_{\underline{\nu}-1}})(F_{\eta_{\underline{\nu}}}(\mathbf{c})). \tag{1.7.3}$$

In particular, the following holds.

LEMMA 1.8. *Under the notation as above,*

(i) *σ gives a permutation of the PBW-basis $\mathscr{X}_{\mathbf{h}}$, namely $\sigma(L(\mathbf{c},\mathbf{h})) = L(\mathbf{c}',\mathbf{h})$ for some $\mathbf{c}' \in \mathbf{N}^{\nu}$. $L(\mathbf{c},\mathbf{h})$ is σ-invariant if and only if c_j is constant for $j \in I_k$ for $k = 1,\dots,\underline{\nu}$.*

(ii) *For each $\underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}$, let $\mathbf{c} \in \mathbf{N}^{\nu}$ be the unique element such that $c_j = \gamma_k$ for each $j \in I_k$. Then $L(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mapsto L(\mathbf{c},\mathbf{h})$ gives a bijection*

$$\underline{\mathscr{X}}_{\underline{\mathbf{h}}} \xrightarrow{\sim} \mathscr{X}_{\mathbf{h}}^{\sigma},$$

where $\mathscr{X}_{\mathbf{h}}^{\sigma}$ is the set of σ-stable PBW-basis in $\mathscr{X}_{\mathbf{h}}$.

1.9. For each $\eta \in \underline{I}$ and $a \in \mathbf{N}$, put $\widetilde{f}_{\eta}^{(a)} = \prod_{i\in\eta} f_i^{(a)}$. Since $f_i^{(a)}$ and $f_j^{(a)}$ commute each other for $i,j \in \eta$, we have $\widetilde{f}_{\eta}^{(a)} \in {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$. We denote its image in ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$ also by $\widetilde{f}_{\eta}^{(a)}$. Thus we can define $g_{\eta}^{(a)} \in \mathbf{V}_q$ by

$$g_{\eta}^{(a)} = \pi(\widetilde{f}_{\eta}^{(a)}). \tag{1.9.1}$$

In the case where $a = 1$, we put $\widetilde{f}_{\eta}^{(1)} = \widetilde{f}_{\eta} = \prod_{i\in\eta} f_i$ and $g_{\eta}^{(1)} = g_{\eta}$. Recall that ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^{-}$ is generated by $\underline{f}_{\eta}^{(a)}$ for $\eta \in \underline{I}$ and $a \in \mathbf{N}$. We have the following result.

PROPOSITION 1.10. *The correspondence $\underline{f}_{\eta}^{(a)} \mapsto g_{\eta}^{(a)}$ gives rise to a homomorphism $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^{-} \to \mathbf{V}_q$ of $\mathbf{A}'$-algebras.*

1.11. Proposition 1.10 will be proved in Section 3. Here assuming the proposition, we continue the discussion. Let $\mathscr{X}_{\mathbf{h}}$ be as in Lemma 1.8. It is known that the PBW-basis $\mathscr{X}_{\mathbf{h}}$ is contained in ${}_{\mathbf{A}}U_q^{-}$ (see Introduction). Thus σ-stable PBW-basis $L(\mathbf{c},\mathbf{h})$ in $\mathscr{X}_{\mathbf{h}}^{\sigma}$ is contained in ${}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$. By Lemma 1.8 such an $L(\mathbf{c},\mathbf{h})$ can be written as

$$L(\mathbf{c},\mathbf{h}) = \widetilde{f}_{\eta_1}^{(\gamma_1)} R_{\eta_1}(\widetilde{f}_{\eta_2}^{(\gamma_2)}) \cdots (R_{\eta_1}\cdots R_{\eta_{\underline{\nu}-1}})(\widetilde{f}_{\eta_{\underline{\nu}}}^{(\gamma_{\underline{\nu}})}), \tag{1.11.1}$$

where $\underline{\mathbf{c}} = (\gamma_1,\dots,\gamma_{\underline{\nu}})$ and

$$\mathbf{c} = (c_1,\dots,c_{\nu}) = (\underbrace{\gamma_1,\dots,\gamma_1}_{|\eta_1|\text{-times}},\underbrace{\gamma_2,\dots,\gamma_2}_{|\eta_2|\text{-times}},\dots,\underbrace{\gamma_{\underline{\nu}},\dots,\gamma_{\underline{\nu}}}_{|\eta_{\underline{\nu}}|\text{-times}}). \tag{1.11.2}$$

For each $L(\mathbf{c},\mathbf{h}) \in \mathscr{X}_{\mathbf{h}}^{\sigma}$, put $E(\underline{\mathbf{c}},\underline{\mathbf{h}}) = \pi(L(\mathbf{c},\mathbf{h}))$ under the correspondence in (1.11.2). By Lemma 1.8 (i), any element $x \in {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$ can be written as an $\mathbf{A}'$-linear combination of σ-stable PBW-basis modulo J. Thus we have

(1.11.3) The set $\{E(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$ generates $\mathbf{V}_q$ as $\mathbf{A}'$-module.

1.12. It is known, for any Cartan datum X, that there exists a canonical symmetric bilinear form $(\ ,\ )$ on $\mathbf{U}_q^{-}$, which satisfies the property,

$$(L(\mathbf{c},\mathbf{h}),L(\mathbf{c}',\mathbf{h}))=\prod_{k=1}^{\nu}(f_{i_k}^{(c_k)},f_{i_k}^{(c_k')})=\prod_{k=1}^{\nu}\delta_{c_k,c_k'}\prod_{d=1}^{c_k}\frac{1}{1-q_{i_k}} \tag{1.12.1}$$

for $\mathbf{c}=(c_1,\dots,c_\nu),\mathbf{c}'=(c_1',\dots,c_\nu')$. In particular, $(L(\mathbf{c},\mathbf{h}),L(\mathbf{c}',\mathbf{h}))=0$ if $\mathbf{c}\neq\mathbf{c}'$, and the form $(\ ,\ )$ is non-degenerate. Assume that X is as in 1.2. Then σ preserves the form, namely, $(\sigma(x),\sigma(y))=(x,y)$ for any $x,y\in\mathbf{U}_q^-$.

Let $\mathbf{F}(q)$ be the field of rational functions over $\mathbf{F}$, and put ${}_{\mathbf{F}(q)}\mathbf{V}_q=\mathbf{V}_q\otimes_{\mathbf{A}'}\mathbf{F}(q)$. Then the form $(\ ,\ )$ on $\mathbf{U}_q^-$ induces a symmetric bilinear form on ${}_{\mathbf{F}(q)}\mathbf{V}_q$ (note that $(\sum_i\sigma^i(x),\sum_i\sigma^i(y))=0$ in $\mathbf{F}(q)$). We have $(E(\underline{\mathbf{c}},\underline{\mathbf{h}}),E(\underline{\mathbf{c}}',\underline{\mathbf{h}}'))=0$ if $\underline{\mathbf{c}}\neq\underline{\mathbf{c}}'$, and $(E(\underline{\mathbf{c}},\underline{\mathbf{h}}),E(\underline{\mathbf{c}},\underline{\mathbf{h}}))\neq 0$. Thus $\{E(\underline{\mathbf{c}},\underline{\mathbf{h}})\mid\underline{\mathbf{c}}\in\mathbf{N}^{\underline{\nu}}\}$ gives rise to an orthogonal basis of ${}_{\mathbf{F}(q)}\mathbf{V}_q$.

Put ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-={}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-\otimes_{\mathbf{A}'}\mathbf{F}(q)$. We can regard $\{L(\underline{\mathbf{c}},\underline{\mathbf{h}})\mid\underline{\mathbf{c}}\in\mathbf{N}^{\underline{\nu}}\}$ as an $\mathbf{F}(q)$-basis of ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$. The map $\Phi:{}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-\to\mathbf{V}_q$ induces an algebra homomorphism ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-\to{}_{\mathbf{F}(q)}\mathbf{V}_q$, which we denote also by Φ. We need a lemma.

LEMMA 1.13. *Assume that $\underline{X}$ has rank 2, and $\underline{\mathbf{h}}=(\eta_1,\dots,\eta_{\underline{\nu}})$. Then for $k=1,\dots,\underline{\nu}$, we have*

$$\Phi\big(\underline{T}_{\eta_1}\cdots\underline{T}_{\eta_{k-1}}(\underline{f}_{\eta_k})\big)=\pi\big(R_{\eta_1}\cdots R_{\eta_{k-1}}(\widetilde{f}_{\eta_k})\big). \tag{1.13.1}$$

Lemma 1.13 will be proved in Section 4. We continue the discussion assuming the lemma. By using Lemma 1.13, we can prove the following theorem.

THEOREM 1.14. *Let $\mathbf{h}$ and $\underline{\mathbf{h}}$ be as in* 1.6.

(i) *For any $\underline{\mathbf{c}}\in\mathbf{N}^{\underline{\nu}}$, we have $\Phi(L(\underline{\mathbf{c}},\underline{\mathbf{h}}))=E(\underline{\mathbf{c}},\underline{\mathbf{h}})$.*

(ii) *Φ gives an algebra isomorphism ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-\xrightarrow{\sim}{}_{\mathbf{F}(q)}\mathbf{V}_q$.*

PROOF. Since R_η's satisfy the braid relation, we can define $R_w=R_{\eta_1}\cdots R_{\eta_k}$ for a reduced expression $w=\underline{s}_{\eta_1}\cdots\underline{s}_{\eta_k}\in\underline{W}$. Let $\underline{\Delta}^+$ be the set of positive roots in $\bigoplus_{\eta\in\underline{I}}\mathbf{Q}\alpha_\eta$. We consider the following statement.

(1.14.1) Assume that, $w(\alpha_\eta)\in\underline{\Delta}^+$. Then $\pi\big(R_w(\widetilde{f}_\eta)\big)=\Phi\big(\underline{T}_w(\underline{f}_\eta)\big)$.

Note that (1.14.1) certainly holds in the case where $\underline{X}$ has rank 2, in view of Lemma 1.13. We prove (1.14.1) by induction on $l(w)$. (1.14.1) holds if $l(w)=0$. Thus we assume that $l(w)>0$, and choose $\eta'\in\underline{I}$ such that $l(ws_{\eta'})=l(w)-1$. From the assumption in (1.14.1), $\eta'\neq\eta$. It is known that there exist $w',w''\in\underline{W}$ such that $w=w'w''$, which satisfy the condition

(i) w'' is contained in the subgroup of $\underline{W}$ generated by s_η and $s_{\eta'}$,
(ii) $l(w)=l(w')+l(w'')$,
(iii) $l(w's_\eta)=l(w')+1$, $l(w's_{\eta'})=l(w')+1$.

By applying (1.14.1) to the case $\underline{X}$ has rank 2, we see that $\pi(R_{w''}(\widetilde{f}_\eta))=\Phi(\underline{T}_{w''}(\underline{f}_\eta))$. Since $w\neq w'$, we have $l(w')<l(w)$. Also note that $w'(\alpha_\eta),w'(\alpha_{\eta'})\in\underline{\Delta}^+$. Thus by induction, we have

$$\pi\big(R_{w'}(\widetilde{f}_\eta)\big) = \Phi\big(\underline{T}_{w'}(\underline{f}_\eta)\big), \quad \pi\big(R_{w'}(\widetilde{f}_{\eta'})\big) = \Phi\big(\underline{T}_{w'}(\underline{f}_{\eta'})\big).$$

Since $R_w(\widetilde{f}_\eta) = R_{w'}R_{w''}(\widetilde{f}_\eta)$ and $\underline{T}_w(\underline{f}_\eta) = \underline{T}_{w'}\underline{T}_{w''}(\underline{f}_\eta)$, (1.14.1) holds for w. Thus (1.14.1) is proved.

Now the claim (i) in the theorem follows from (1.14.1). Let Z be the $\mathbf{F}(q)$-subspace of ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$ spanned by $\{L(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$. Since $\{E(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$ is a basis of ${}_{\mathbf{F}(q)}\mathbf{V}_q$, Φ gives an isomorphism $Z \xrightarrow{\sim} {}_{\mathbf{F}(q)}\mathbf{V}_q$ by (i), and so Z is an algebra over $\mathbf{F}(q)$. Since $\underline{f}_\eta^{(a)} = ([a]_{q_\eta}^!)^{-1}\underline{f}_\eta^a$ is contained in Z, we see that $Z = {}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$. Thus (ii) holds. The theorem is proved. □

1.15. We follow the point of view explained in Introduction. In the simply laced case, the properties (i), (ii) and (iii) in (0.1.1) are known to hold. Hence there exists the canonical basis $\{b(\mathbf{c},\mathbf{h}) \mid \mathbf{c} \in \mathbf{N}^\nu\}$ in $\mathscr{L}_{\mathbf{Z}}(\infty)$, which is characterized by the following properties,

$$\begin{aligned} &\overline{b(\mathbf{c},\mathbf{h})} = b(\mathbf{c},\mathbf{h}), \\ &b(\mathbf{c},\mathbf{h}) \equiv L(\mathbf{c},\mathbf{h}) \mod q\mathscr{L}_{\mathbf{Z}}(\infty), \end{aligned} \tag{1.15.1}$$

where $x \mapsto \overline{x}$ is the bar involution in $\mathbf{U}_q^-$. Note that $\{b(\mathbf{c},\mathbf{h}) \mid \mathbf{c} \in \mathbf{N}^\nu\}$ is independent of the choice of $\mathbf{h}$, which we denote by $\mathbf{B}$.

We define a total order on $\mathbf{N}^\nu$ by making use of the lexicographic order, i.e., for $\mathbf{c} = (c_1,\dots,c_\nu), \mathbf{d} = (d_1,\dots,d_\nu) \in \mathbf{N}^\nu$, $\mathbf{c} < \mathbf{d}$ if and only if there exists k such that $c_i = d_i$ for $i < k$ and $c_k < d_k$. Then the second formula in (1.15.1) can be written more precisely as

$$b(\mathbf{c},\mathbf{h}) = L(\mathbf{c},\mathbf{h}) + \sum_{\mathbf{c}<\mathbf{d}} a_{\mathbf{d}} L(\mathbf{d},\mathbf{h}) \tag{1.15.2}$$

with $a_{\mathbf{d}} \in q\mathbf{Z}[q]$.

1.16. We choose $\mathbf{h}$ and $\underline{\mathbf{h}}$ as in 1.6. Since σ permutes the PBW-basis $L(\mathbf{c},\mathbf{h})$, σ permutes the canonical basis $\mathbf{B}$. We denote by $\mathbf{B}^\sigma$ the set of σ-stable canonical basis of $\mathbf{U}_q^-$. Take $\mathbf{b} = \mathbf{b}(\mathbf{c},\mathbf{h}) \in \mathbf{B}^\sigma$. Then $L(\mathbf{c},\mathbf{h})$ is σ-stable, and $\mathbf{c}$ is obtained from $\underline{\mathbf{c}}$ as in 1.11. Since $\mathbf{b} \in {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$, one can consider $\pi(\mathbf{b})$. Then we can write as

$$\pi(\mathbf{b}) = E(\underline{\mathbf{c}},\underline{\mathbf{h}}) + \sum_{\underline{\mathbf{c}}<\underline{\mathbf{d}}} a_{\underline{\mathbf{d}}} E(\underline{\mathbf{d}},\underline{\mathbf{h}}) \tag{1.16.1}$$

with $a_{\underline{\mathbf{d}}} \in q\mathbf{F}[q]$. The total order $\underline{\mathbf{c}} < \underline{\mathbf{d}}$ on $\mathbf{N}^{\underline{\nu}}$ is defined similarly. The bar involution can be defined on $\mathbf{V}_q$, and the map π is compatible with those bar involutions. Thus we have

$$\overline{\pi(\mathbf{b})} = \pi(\mathbf{b}). \tag{1.16.2}$$

Let $\widetilde{\mathscr{L}_{\mathbf{F}}}(\infty)$ be the $\mathbf{F}[q]$-submodule of $\mathbf{V}_q$ generated by $E(\underline{\mathbf{c}},\underline{\mathbf{h}})$. Then the set $\{\pi(\mathbf{b}) \mid \mathbf{b} \in \mathbf{B}^\sigma\}$ gives rise to an $\mathbf{F}[q]$-basis of $\widetilde{\mathscr{L}_{\mathbf{F}}}(\infty)$ satisfying the properties (1.16.1) and (1.16.2). Note that the set $\{\pi(\mathbf{b}) \mid \mathbf{b} \in \mathbf{B}^\sigma\}$ is characterized by those properties, and this set is

independent of the choice of $\underline{\mathbf{h}}$, which we call the canonical basis of $\mathbf{V}_q$.

Let $\underline{\mathscr{L}}_{\mathbf{F}}(\infty)$ be the $\mathbf{F}[q]$-submodule of ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$ generated by $\{L(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$. We have the following result.

PROPOSITION 1.17. *There exists a unique $\mathbf{F}[q]$-basis $\{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$ in $\underline{\mathscr{L}}_{\mathbf{F}}(\infty)$ satisfying the following properties,*

$$\overline{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}})} = \mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}), \tag{1.17.1}$$

$$\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) = L(\underline{\mathbf{c}},\underline{\mathbf{h}}) + \sum_{\underline{\mathbf{c}}<\underline{\mathbf{d}}} a_{\underline{\mathbf{d}}} L(\underline{\mathbf{d}},\underline{\mathbf{h}}), \qquad (a_{\underline{\mathbf{d}}} \in q\mathbf{F}[q]).$$

Moreover, the set $\{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$ is independent of the choice of $\underline{\mathbf{h}}$, and $\underline{\mathscr{L}}_{\mathbf{F}}(\infty)$ does not depend on the choice of $\underline{\mathbf{h}}$.

PROOF. It is clear that the map $\Phi : {}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^- \to {}_{\mathbf{F}(q)}\mathbf{V}_q$ is compatible with the bar involutions. Then the proposition immediately follows from Theorem 1.14. □

1.18. For any $\underline{X}$, we consider the following statements corresponding to (iii) and (iii′) in (0.1.1).

(1.18.1) PBW-basis $\underline{\mathscr{X}}_{\underline{\mathbf{h}}}$ gives an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$.

(1.18.2) Any element $L(\underline{\mathbf{c}},\underline{\mathbf{h}}) \in \underline{\mathscr{X}}_{\underline{\mathbf{h}}}$ is contained in ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$.

As was explained in Introduction, the proof of (1.18.1) is reduced to the case of rank 2, namely the case of type B_2 and G_2, and in that case, (1.18.2) was proved by Lusztig [**L1**] and Xi [**X1**]. In any case, the computation in the case of G_2 is not easy. (1.18.2) can be proved by computing the commutation relations of root vectors, which is relatively easy compared to (1.18.1).

In the discussion below, we only assume that (1.18.2) holds for ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$, and will prove that (1.18.1) holds for ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$.

1.19. We return to our original setting, and consider the map $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \to \mathbf{V}_q$. By (1.18.2), the PBW-basis $\underline{\mathscr{X}}_{\underline{\mathbf{h}}} = \{L(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$ is contained in ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$. Since $\{E(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$ is an $\mathbf{A}'$-basis of $\mathbf{V}_q$, we see that Φ is surjective, by Theorem 1.14 (i). Let ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ be the $\mathbf{A}'$-module generated by $\{L(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$. Again by Theorem 1.14, ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ is an $\mathbf{A}'$-submodule of ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$, which is independent of the choice of $\underline{\mathbf{h}}$. We show that

$${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^- = {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-. \tag{1.19.1}$$

By (1.18.2), we know that ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^- \subset {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$. On the other hand, for each $\eta \in \underline{I}$, one can find a sequence $\underline{\mathbf{h}} = (\eta_1, \dots, \eta_{\underline{N}})$ such that $\eta_1 = \eta$. This implies that ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ is invariant under the left multiplication by $\underline{f}_\eta^{(a)}$. Since this is true for any η, we see that ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ is contained in ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$. Thus (1.19.1) holds.

Summing up the above arguments, we have the following integral form of Theorem 1.14.

PROPOSITION 1.20. *Assume that* (1.18.2) *holds for* ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$. *Then* Φ *induces an isomorphism* ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \simeq \mathbf{V}_q$. *In particular, the PBW-basis* $\underline{\mathscr{X}}_{\underline{\mathbf{h}}}$ *gives an* $\mathbf{A}'$*-basis of* ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$.

As a corollary, we have

COROLLARY 1.21. *Assume that* (1.18.2) *holds for* ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$. *Then* (1.18.1) *also holds.*

PROOF. Let ${}_{\mathbf{A}}\widehat{\underline{\mathbf{U}}}_q^-$ be the inverse limit of ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-/\varepsilon^n({}_{\mathbf{A}}\underline{\mathbf{U}}_q^-)$. Then ${}_{\mathbf{A}}\widehat{\underline{\mathbf{U}}}_q^-$ has a natural structure of the module over $\mathbf{Z}_\varepsilon[q,q^{-1}] = \varprojlim \mathbf{A}/\varepsilon^n\mathbf{A}$, where $\mathbf{Z}_\varepsilon$ is the ring of ε-adic integers. We have a natural embedding ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^- \subset {}_{\mathbf{A}}\widehat{\underline{\mathbf{U}}}_q^-$. Now take $x \in {}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$. (1.18.1) shows that x can be written as a linear combination of PBW-basis with coefficients in $\mathbf{A}$ modulo $\varepsilon({}_{\mathbf{A}}\underline{\mathbf{U}}_q^-)$. We regard x as an element in ${}_{\mathbf{A}}\widehat{\underline{\mathbf{U}}}_q^-$. Then x can be written as a linear combination of PBW-basis with coefficients in $\mathbf{Z}_\varepsilon[q,q^{-1}]$. On the other hand, we know that x is a linear combination of PBW-basis with coefficients in $\mathbf{Q}(q)$. Thus those coefficients belong to $\mathbf{A} = \mathbf{Z}[q,q^{-1}]$, and we obtain (1.18.1). □

1.22. We assume that (1.18.2) holds for $\underline{\mathbf{U}}_q^-$. Then by Corollary 1.21, we have

(1.22.1) In $\underline{\mathbf{U}}_q^-$, $\overline{L(\underline{\mathbf{c}},\underline{\mathbf{h}})}$ is a linear combination of various $L(\underline{\mathbf{d}},\underline{\mathbf{h}})$ with coefficients in $\mathbf{A}$.

Then by [**L3**, Lemma 24.2.1], one can define a basis $\{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$ of $\underline{\mathbf{U}}_q^-$, satisfying the properties

$$\overline{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}})} = \mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}), \tag{1.22.2}$$

$$\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) = L(\underline{\mathbf{c}},\underline{\mathbf{h}}) + \sum_{\underline{\mathbf{c}}<\underline{\mathbf{d}}} a_{\underline{\mathbf{d}}} L(\underline{\mathbf{d}},\underline{\mathbf{h}}), \qquad (a_{\underline{\mathbf{d}}} \in q\mathbf{Z}[q]).$$

In this construction, we cannot give the independence of the basis $\{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$ from $\underline{\mathbf{h}}$. But by using the almost orthogonality of PBW-basis (1.12.1), one can prove a weaker property, namely, the independence from $\underline{\mathbf{h}}$, up to sign (see [**L3**, Theorem 14.2.3]); if we fix $\mathbf{h},\mathbf{h}'$, then for any $\underline{\mathbf{c}}$, there exists a unique $\underline{\mathbf{c}}'$ such that

$$\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) = \pm\mathbf{b}(\underline{\mathbf{c}}',\underline{\mathbf{h}}'). \tag{1.22.3}$$

We denote by $\underline{\mathbf{B}}$ the set of canonical basis $\{b(\underline{\mathbf{c}},\mathbf{h})\}$ in $\underline{\mathbf{U}}_q^-$. On the other hand, let $\underline{\mathbf{B}}'$ be the canonical basis in ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ given in Proposition 1.17. We temporally write them as $\{\mathbf{b}'(\underline{\mathbf{c}},\underline{\mathbf{h}})\}$. Then the image of $\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}})$ under the natural map ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^- \to {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ coincides with $\mathbf{b}'(\underline{\mathbf{c}},\underline{\mathbf{h}})$, and this gives a bijection $\underline{\mathbf{B}} \xrightarrow{\sim} \underline{\mathbf{B}}'$. In the case where $\varepsilon = 2$, this does not give a new information on the sign of $\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}})$. But in the case where $\varepsilon = 3$, we have the following result.

PROPOSITION 1.23. *Assume that* $\varepsilon = 3$, *and* $\underline{X}$ *is of type* G_2. *Then the canonical basis* $\{\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) \mid \underline{\mathbf{c}} \in \mathbf{N}^{\underline{\nu}}\}$ *is independent of the choice of* $\underline{\mathbf{h}}$, *namely, if we fix* $\underline{\mathbf{h}},\underline{\mathbf{h}}'$, *then for any* $\underline{\mathbf{c}}$, *there exists a unique* $\underline{\mathbf{c}}'$ *such that*

$$\mathbf{b}(\underline{\mathbf{c}},\underline{\mathbf{h}}) = \mathbf{b}(\underline{\mathbf{c}}',\underline{\mathbf{h}}').$$

PROOF. By (1.22.3), we have $\mathbf{b}'(\underline{\mathbf{c}},\underline{\mathbf{h}}) = a\mathbf{b}'(\underline{\mathbf{c}}',\underline{\mathbf{h}}')$ for some $a = \pm 1$. But by Proposition 1.17, $\mathbf{b}'(\underline{\mathbf{c}},\underline{\mathbf{h}})$ is determined uniquely as an element in $\underline{\mathscr{L}}_{\mathbf{F}}(\infty)$, which is independent of the choice of $\underline{\mathbf{h}}$. It follows that $a \equiv 1 \mod 3$. This implies that $a = 1$, and the proposition is proved. □

REMARK 1.24. By Proposition 1.20, we have a natural bijection $\underline{\mathbf{B}}' \xrightarrow{\sim} \mathbf{B}^\sigma$. By the discussion in 1.22, we have $\underline{\mathbf{B}} \simeq \underline{\mathbf{B}}'$. Hence

$$\mathbf{B}^\sigma \simeq \underline{\mathbf{B}}' \simeq \underline{\mathbf{B}}. \tag{1.24.1}$$

Thus we have a natural correspondence $\mathbf{B}^\sigma \leftrightarrow \underline{\mathbf{B}}$ between the set of σ-stable canonical basis of $\mathbf{U}_q^-$ and the set of canonical basis of $\underline{\mathbf{U}}_q^-$. This is nothing but the reformulation, by our context of elementary setting, of Lusztig's result [**L4**, 1.12 (b)] (see also [**L3**, Theorem 14.4.9]) obtained by geometric considerations.

2. PBW-bases for affine quantum groups.

2.1. In Beck and Nakajima [**BN**], the PBW-bases were constructed in the case of affine quantum groups. In this section, by making use of their PBW-bases, we shall extend the results in the previous section to the case of affine quantum groups.

Let $\mathfrak{g}$ be an untwisted affine Lie algebra associated to the simply laced Cartan datum X, with the vertex set I, and $\mathfrak{g}_0$ the simple Lie algebra over $\mathbf{C}$ with the vertex set I_0 associated to the simply laced Cartan datum X_0 such that

$$\begin{aligned} L\mathfrak{g}_0 &= \mathfrak{g}_0 \otimes_{\mathbf{C}} \mathbf{C}[t,t^{-1}], \\ \mathfrak{g} &= L\mathfrak{g}_0 \oplus \mathbf{C}c \oplus \mathbf{C}d, \end{aligned}$$

where c is the center of $\mathfrak{g}$ and d is the degree operator. Here $L\mathfrak{g}_0 \oplus \mathbf{C}c$ is the central extension of the Loop algebra $L\mathfrak{g}_0$.

Let $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Delta_0}(\mathfrak{g}_0)_\alpha$ be the root space decomposition of $\mathfrak{g}_0$ with respect to a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, where Δ_0 is the set of roots in $\mathfrak{g}_0$. Then $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbf{C}c \oplus \mathbf{C}d$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ is decomposed as

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\substack{\alpha\in\Delta_0 \\ m\in\mathbf{Z}}} (\mathfrak{g}_0)_\alpha \otimes t^m\Big) \oplus \Big(\bigoplus_{m\in\mathbf{Z}-\{0\}} \mathfrak{h}_0 \otimes t^m\Big). \tag{2.1.1}$$

We define $\delta \in \mathfrak{h}^*$ by $\langle d,\delta\rangle = 1, \langle \mathfrak{h}_0 \oplus \mathbf{C}c, \delta\rangle = 0$. We regard $\alpha \in \Delta_0 \subset \mathfrak{h}_0^*$ as an element in $\mathfrak{h}^*$ by $\alpha(c) = 0, \alpha(d) = 0$. Then $(\mathfrak{g}_0)_\alpha \otimes t^m, \mathfrak{h}_0 \otimes t^m$ correspond to the root space with root $\alpha + m\delta$, $m\delta$, respectively, and (2.1.1) gives a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Δ (resp. Δ^+) be the set of roots (resp. the set of positive roots) in $\mathfrak{g}$. Also Δ_0^+ be the set of positive roots in Δ_0. Then Δ^+ is given by

$$\Delta^+ = \Delta_>^{\mathrm{re},+} \sqcup \Delta_<^{\mathrm{re},+} \sqcup \mathbf{Z}_{>0}\delta, \tag{2.1.2}$$

where

$$\Delta_{>}^{\mathrm{re},+} = \{\alpha + m\delta \mid \alpha \in \Delta_0^+, m \in \mathbf{Z}_{\geq 0}\},$$
$$\Delta_{<}^{\mathrm{re},+} = \{\alpha + m\delta \mid \alpha \in -\Delta_0^+, m \in \mathbf{Z}_{>0}\}.$$

$\Delta_{>}^{\mathrm{re},+} \sqcup \Delta_{<}^{\mathrm{re},+}$ is the set of positive real roots, and $\mathbf{Z}_{>0}\delta$ is the set of positive imaginary roots. The simple roots Π are given by

$$\Pi = \{\alpha_i \mid i \in I_0\} \sqcup \{\alpha_0 = \delta - \theta\},$$

where θ is the highest root in Δ_0^+.

2.2. Let $\sigma : I \to I$ be the permutation as in 1.2. We assume that σ preserves I_0. Thus if X is irreducible, X has type $A_{2n+1}^{(1)}(n \geq 1), D_n^{(1)}(n \geq 4), E_6^{(1)}$ for $\varepsilon = 2$, and $D_4^{(1)}$ for $\varepsilon = 3$. Correspondingly, $\underline{X}$ has type $D_{n+2}^{(2)}, (n \geq 1), A_{2n-3}^{(2)}, (n \geq 4), E_6^{(2)}$ and $D_4^{(3)}$ under the notation of the table in [**Ka**, Section 4.8]. Let $\underline{I}_0$ be the set of σ-orbits in I_0, and $\underline{X}_0$ be the corresponding Cartan datum. Then $\underline{X}_0$ has type $B_{n+1}, C_{n-1}, F_4, G_2$, respectively.

σ induces a Lie algebra automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$, and let $\mathfrak{g}^\sigma$ be the subalgebra of $\mathfrak{g}$ consisting of σ-fixed elements. σ preserves $\mathfrak{g}_0$, and $\sigma(c) = c, \sigma(d) = d$. We define $\mathfrak{g}_0^\sigma$ similarly. Then $\mathfrak{g}_0^\sigma$ is a simple Lie algebra, and $\mathfrak{g}^\sigma = L\mathfrak{g}_0^\sigma \oplus \mathbf{C}c \oplus \mathbf{C}d$ is the affine Lie algebra associated to $\mathfrak{g}_0^\sigma$. Note that $\mathfrak{g}^\sigma$ is isomorphic to the affine Lie algebra $\underline{\mathfrak{g}}$ associated to $\underline{X}$, which is the twisted affine Lie algebra of type $X_k^{(r)}$ given above (here r coincides with ε). Moreover $\mathfrak{g}_0^\sigma$ is isomorphic to $\underline{\mathfrak{g}}_0$ associated to $\underline{X}_0$. We have $\mathfrak{h}^\sigma = \mathfrak{h}_0^\sigma \oplus \mathbf{C}c \oplus \mathbf{C}d$, and $\mathfrak{h}^\sigma \simeq \underline{\mathfrak{h}}$, $\mathfrak{h}_0^\sigma \simeq \underline{\mathfrak{h}}_0$ (Cartan subalgebras of $\underline{\mathfrak{g}}$ and $\underline{\mathfrak{g}}_0$).

Note that σ acts on Δ^+, leaving Δ_0^+ invariant. Moreover, $\sigma(\delta) = \delta$. Thus $\Delta_{>}^{\mathrm{re},+}$ and $\Delta_{<}^{\mathrm{re},+}$ are stable by σ.

Let $\underline{\Delta}^+$ (resp. $\underline{\Delta}^{\mathrm{re},+}$, $\underline{\Delta}^{\mathrm{im},+}$) be the set of positive roots (resp. positive real roots, positive imaginary roots) in the root system $\underline{\Delta}$ of $\underline{\mathfrak{g}}$. Since $\underline{\mathfrak{g}}$ is twisted of type $X_n^{(r)}$, by [**Ka**, Proposition 6.3], $\underline{\Delta}^{\mathrm{re},+}$ can be written as $\underline{\Delta}^{\mathrm{re},+} = \underline{\Delta}_{>}^{\mathrm{re},+} \sqcup \underline{\Delta}_{<}^{\mathrm{re},+}$ and $\underline{\Delta}^{\mathrm{im},+} = \mathbf{Z}_{>0}\delta$, where

$$\underline{\Delta}_{>}^{\mathrm{re},+} = \{\alpha + m\delta \mid \alpha \in (\underline{\Delta}_0^+)_s, m \in \mathbf{Z}_{\geq 0}\} \sqcup \{\alpha + mr\delta \mid \alpha \in (\underline{\Delta}_0^+)_l, m \in \mathbf{Z}_{\geq 0}\}, \quad (2.2.1)$$
$$\underline{\Delta}_{<}^{\mathrm{re},+} = \{\alpha + m\delta \mid \alpha \in -(\underline{\Delta}_0^+)_s, m \in \mathbf{Z}_{>0}\} \sqcup \{\alpha + mr\delta \mid \alpha \in -(\underline{\Delta}_0^+)_l, m \in \mathbf{Z}_{>0}\}.$$

Here $(\underline{\Delta}_0^+)_s$ (resp. $(\underline{\Delta}_0^+)_l$) is the set of positive short roots (resp. positive long roots) in the root system $\underline{\Delta}_0$ of $\underline{\mathfrak{g}}_0$.

2.3. Let $\mathfrak{h}^{*0} = \{\lambda \in \mathfrak{h}^* \mid \langle c, \lambda \rangle = 0\} = \{\lambda \in \mathfrak{h}^* \mid (\lambda, \delta) = 0\}$ be the subspace of $\mathfrak{h}^*$. Then $\mathfrak{h}^{*0} = \bigoplus_{i \in I_0} \mathbf{C}\alpha_i \oplus \mathbf{C}\delta$. We define a map

$$\mathrm{cl} : \mathfrak{h}^{*0} \to \mathfrak{h}_0^*$$

by $\mathrm{cl}(\alpha_i) = \alpha_i$ $(i \in I_0)$ and $\mathrm{cl}(\delta) = 0$, where $\mathfrak{h}_0^* = (\mathfrak{h}_0)^*$. Then cl induces an isomorphism $\mathfrak{h}^{*0}/\mathbf{C}\delta \xrightarrow{\sim} \mathfrak{h}_0^*$. σ acts on $\mathfrak{h}^{*0}$ and on $\mathfrak{h}_0^*$, and cl is compatible with those σ-actions. Hence cl induces a map $(\mathfrak{h}^{*0})^\sigma \to (\mathfrak{h}_0^*)^\sigma$. The restriction map $\mathfrak{h}^* \to (\mathfrak{h}^\sigma)^*$ induces an isomorphism $(\mathfrak{h}^*)^\sigma \xrightarrow{\sim} (\mathfrak{h}^\sigma)^*$, which implies that $(\mathfrak{h}^{*0})^\sigma \simeq \underline{\mathfrak{h}}^{*0}$ since $\mathfrak{h}^\sigma \simeq \underline{\mathfrak{h}}$. Similarly we have $(\mathfrak{h}_0^*)^\sigma \simeq$

$(\mathfrak{h}_0^\sigma)^* \simeq \underline{\mathfrak{h}}_0^*$. Under those identifications, the induced map $(\mathfrak{h}^{*0})^\sigma \to (\mathfrak{h}_0^*)^\sigma$ coincides with the map $\mathrm{cl} : \underline{\mathfrak{h}}^{*0} \to \underline{\mathfrak{h}}_0^*$ defined for $\underline{\mathfrak{g}}$ similarly to $\mathfrak{g}$.

2.4. Let Q_{cl} be the image of $\bigoplus_{i\in I_0} \mathbf{Z}\alpha_i$ in $\mathfrak{h}^{*0}/\mathbf{C}\delta$. Then Q_{cl} can be identified with the root lattice of $\mathfrak{g}_0$ via cl. We define $t : \mathfrak{h}^{*0} \to GL(\mathfrak{h}^*)$ by

$$t(\xi)(\lambda) = \lambda + (\lambda,\delta)\xi - \left\{(\lambda,\xi) + \frac{(\xi,\xi)}{2}(\lambda,\delta)\right\}\delta, \quad (\xi \in \mathfrak{h}^{*0}, \lambda \in \mathfrak{h}^*), \tag{2.4.1}$$

which induces a map $t : \mathfrak{h}^{*0}/\mathbf{C}\delta \to GL(\mathfrak{h}^*)$, and consider the restriction of t on Q_{cl}. Note that in the case where $\lambda \in \mathfrak{h}^{*0}$, (2.4.1) can be written in a simple form

$$t(\xi)(\lambda) = \lambda - (\lambda,\xi)\delta. \tag{2.4.2}$$

Let W be the Weyl group of $\mathfrak{g}$ and W_0 the Weyl group of $\mathfrak{g}_0$. Then we have an exact sequence

$$1 \longrightarrow Q_{\mathrm{cl}} \xrightarrow{t} W \longrightarrow W_0 \longrightarrow 1. \tag{2.4.3}$$

Put

$$P_{\mathrm{cl}} = \{\lambda \in \mathfrak{h}^{*0} \mid (\lambda,\alpha_i) \in \mathbf{Z} \text{ for any } i \in I_0\}/\mathbf{C}\delta.$$

Then P_{cl} is identified with the weight lattice of $\mathfrak{g}_0$ via cl. We define an extended affine Weyl group $\widetilde{W}$ by $\widetilde{W} = P_{\mathrm{cl}} \rtimes W_0$ (note that $\mathfrak{g}$ is simply laced).

Let $\underline{W}$ be the Weyl group of $\underline{\mathfrak{g}}$ and $\underline{W}_0$ the Weyl group of $\underline{\mathfrak{g}}_0$. Let $(\ ,\ )_1$ be the non-degenerate symmetric bilinear form on $\underline{\mathfrak{h}}_0^*$, normalized that $(\alpha_i,\alpha_i)_1 = 2$ for a short root α_i $(i \in \underline{I}_0)$ (see 1.2). The form $(\ ,\ )_1$ is extended uniquely to a non-degenerate symmetric bilinear form $(\ ,\ )_1$ on $\underline{\mathfrak{h}}^*$ by the condition that $(\lambda,\delta) = \langle c,\lambda\rangle$ for any $\lambda \in \underline{\mathfrak{h}}^*$. For $\alpha \in \underline{\Delta}_0$, put $\alpha^\vee = 2\alpha/(\alpha,\alpha)_1$. Put $\underline{Q}_{\mathrm{cl}} = \bigoplus_{\eta\in\underline{I}_0} \mathbf{Z}\alpha_\eta$ and $\underline{Q}_{\mathrm{cl}}^\vee = \bigoplus_{\eta\in\underline{I}_0} \mathbf{Z}\alpha_\eta^\vee$. Since $\underline{\mathfrak{g}}$ is the dual of the untwisted algebra, we have $\underline{Q}_{\mathrm{cl}} \subset \underline{Q}_{\mathrm{cl}}^\vee$. As in (2.4.1), we can define a map $t : \underline{\mathfrak{h}}^{*0}/\mathbf{C}\delta \to GL(\underline{\mathfrak{h}}^*)$, and we have an exact sequence

$$1 \longrightarrow \underline{Q}_{\mathrm{cl}}^\vee \xrightarrow{t} \underline{W} \longrightarrow \underline{W}_0 \longrightarrow 1. \tag{2.4.4}$$

For each $i \in I_0$, let ω_i be the fundamental weight of $(\underline{\Delta}_0, \underline{\mathfrak{h}}_0^*)$, defined by $(\omega_i,\alpha_j) = \delta_{ij}$ $(i,j \in I_0)$. Then under the isomorphism $\mathrm{cl} : \mathfrak{h}^{*0}/\mathbf{C}\delta \simeq \underline{\mathfrak{h}}_0^*$, $P_{\mathrm{cl}} \simeq \bigoplus_{i\in I_0} \mathbf{Z}\omega_i$. The action of σ on $\mathfrak{h}^{*0}$ induces an action of σ on P_{cl}, which is given by $\omega_i \mapsto \omega_{\sigma(i)}$ $(i \in I_0)$. Thus we have an action of σ on $\widetilde{W}$, which preserves W_0. On the other hand, we define the fundamental coweight $\underline{\omega}_\eta^\vee$ of $(\underline{\Delta}_0, \underline{\mathfrak{h}}_0^*)$ by $(\underline{\omega}_\eta^\vee, \alpha_{\eta'})_1 = \delta_{\eta\eta'}$ $(\eta,\eta' \in \underline{I}_0)$, and put $\widetilde{\underline{\omega}}_\eta = |\eta|\underline{\omega}_\eta^\vee$. We define $\widetilde{\underline{P}}_{\mathrm{cl}} = \bigoplus_{\eta\in\underline{I}_0} \mathbf{Z}\widetilde{\underline{\omega}}_\eta$, which we regard as a lattice of $\underline{\mathfrak{h}}^{*0}/\mathbf{C}\delta$ dual to $\underline{Q}_{\mathrm{cl}}^\vee$. Define the extended affine Weyl group by $\widetilde{\underline{W}} = \widetilde{\underline{P}}_{\mathrm{cl}} \rtimes \underline{W}_0$. Since the map

$$(P_{\mathrm{cl}})^\sigma \xrightarrow{\sim} \widetilde{\underline{P}}_{\mathrm{cl}}, \quad \sum_{i\in\eta} \omega_i \mapsto |\eta|\underline{\omega}_\eta^\vee = \widetilde{\underline{\omega}}_\eta \tag{2.4.5}$$

is compatible with the action of $W_0^\sigma \simeq \underline{W}_0$, we have an isomoprhism

$$\widetilde{W}^\sigma = (P_{\mathrm{cl}})^\sigma \rtimes W_0^\sigma \simeq \underline{\widetilde{P}}_{\mathrm{cl}} \rtimes \underline{W}_0 = \widetilde{\underline{W}}. \tag{2.4.6}$$

Let $\mathscr{T} = \{w \in \widetilde{W} \mid w(\Delta^+) \subset \Delta^+\}$, which is a subgroup of the automorphism group of the ambient diagram. Then we have $\widetilde{W} = \mathscr{T} \ltimes W$. Similarly we define $\underline{\mathscr{T}} = \{w \in \underline{W} \mid w(\underline{\Delta}^+) \subset \underline{\Delta}^+\}$ so that $\widetilde{\underline{W}} = \underline{\mathscr{T}} \ltimes \underline{W}$. The action of σ on $\widetilde{W}$ preserves $\mathscr{T}$, and we have $\mathscr{T}^\sigma = \underline{\mathscr{T}}$.

2.5. Following [**BN**, 3.1], put

$$\xi = \sum_{i \in I_0} \omega_i \in P_{\mathrm{cl}}, \tag{2.5.1}$$

and consider $t(\xi) \in \widetilde{W}$, which we simply denote by ξ. Here $\xi \in \widetilde{W}^\sigma \simeq \widetilde{\underline{W}} = \underline{\mathscr{T}} \ltimes \underline{W}$, and one can express ξ as

$$\xi = s_{\eta_1} \cdots s_{\eta_{\underline{\nu}}} \tau \tag{2.5.2}$$

with $\tau \in \underline{\mathscr{T}} = \mathscr{T}^\sigma$, where $w = s_{\eta_1} \cdots s_{\eta_{\underline{\nu}}}$ is a reduced expression of $w \in \underline{W}$ (w is the $\underline{W}$-component of ξ). Accordingly, we obtain a reduced expression of $w = s_{i_1} \cdots s_{i_\nu} \in W$ such that

$$w = \Big(\prod_{k_1 \in \eta_1} s_{k_1}\Big) \cdots \Big(\prod_{k_{\underline{\nu}} \in \eta_{\underline{\nu}}} s_{k_{\underline{\nu}}}\Big) = s_{i_1} \cdots s_{i_\nu}. \tag{2.5.3}$$

As in [**BN**, 3.1], we define a doubly infinite sequence attached to $\mathfrak{g}$

$$\mathbf{h} = (\dots, i_{-1}, i_0, i_1, \dots) \tag{2.5.4}$$

by setting $i_{k+\nu} = \tau(i_k)$ for $k \in \mathbf{Z}$. Then for any integer $m < p$, the product $s_{i_m} s_{i_{m+1}} \cdots s_{i_p} \in W$ is a reduced expression. Similarly, we define a doubly infinite sequence

$$\underline{\mathbf{h}} = (\dots, \eta_{-1}, \eta_0, \eta_1, \dots) \tag{2.5.5}$$

by the condition that $\eta_{k+\underline{\nu}} = \tau(\eta_k)$ for $k \in \mathbf{Z}$, which satisfies the property that $s_{\eta_m} s_{\eta_{m+1}} \cdots s_{\eta_p} \in \underline{W}$ is a reduced expression for $m < p$. Note that $\xi \in (P_{\mathrm{cl}})^\sigma$, and under the isomorphism $(P_{\mathrm{cl}})^\sigma \simeq \underline{\widetilde{P}}_{\mathrm{cl}}$ in (2.4.5), ξ coincides with the element $\sum_{\eta \in \underline{I}_0} \widetilde{\underline{\omega}}_\eta$. Thus the sequence (2.5.5) is exactly the sequence defined in [**BN**, 3.1] attached to $\underline{\mathfrak{g}}$.

By (2.4.2), for $\beta = \alpha + m\delta \in \Delta_>^{\mathrm{re},+}$ and $n \in \mathbf{Z}$, $(n\xi)^{-1}(\beta) = \beta + n(\xi, \beta)\delta = \alpha + (m + n(\xi, \beta))\delta$. Since $(\xi, \beta) > 0$ by (2.5.1), $(n\xi)^{-1}(\beta) \in \Delta^-$ if $n < 0$ is small enough. Similar argument holds also for $\beta \in \Delta_<^{\mathrm{re},+}$ by replacing $n < 0$ by $n > 0$. It follows that

$$\bigcup_{n \in \mathbf{Z}_{<0}} (\Delta_>^{\mathrm{re},+} \cap w^n(\Delta^-)) = \Delta_>^{\mathrm{re},+}, \qquad \bigcup_{n \in \mathbf{Z}_{>0}} (\Delta_<^{\mathrm{re},+} \cap w^n(\Delta^-)) = \Delta_<^{\mathrm{re},+}. \tag{2.5.6}$$

Similar formulas hold also for the root system $\underline{\Delta}^+$ of $\underline{\mathfrak{g}}$. As a corollary of (2.5.6), we have

(2.5.7) Let $\mathbf{h}$ be as in (2.5.4). Then any $i \in I$ appears in the infinite sequence $(\dots, i_{-1}, i_0, i_1, \dots)$. Similarly, let $\underline{\mathbf{h}}$ be as in (2.5.5). Then any $\eta \in \underline{I}$ appears in the infinite sequence $(\dots, \eta_{-1}, \eta_0, \eta_1, \dots)$.

2.6. Let $\mathbf{U}_q^-$ (resp. $\underline{\mathbf{U}}_q^-$) be the negative part of the quantum enveloping algebra $\mathbf{U}_q$ (resp. $\underline{\mathbf{U}}_q$) associated to X (resp. $\underline{X}$). We follow the notation in 1.5. We fix $\mathbf{h}$ as in (2.5.4), and define $\beta_k \in \Delta^+$ for $k \in \mathbf{Z}$ by

$$\beta_k = \begin{cases} s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \le 0, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0. \end{cases} \tag{2.6.1}$$

Then, as in [**BN**, 3.1], we have

$$\Delta_>^{\mathrm{re},+} = \{\beta_k \mid k \in \mathbf{Z}_{\le 0}\}, \qquad \Delta_<^{\mathrm{re},+} = \{\beta_k \mid k \in \mathbf{Z}_{>0}\}. \tag{2.6.2}$$

We define root vectors $f_{\beta_k}^{(c)} \in \mathbf{U}_q^-$ by

$$f_{\beta_k}^{(c)} = \begin{cases} T_{i_0} T_{i_{-1}} \cdots T_{i_{k+1}}(f_{i_k}^{(c)}), & \text{if } k \le 0, \\ T_{i_1}^{-1} T_{i_2}^{-1} \cdots T_{i_{k-1}}^{-1}(f_{i_k}^{(c)}), & \text{if } k > 0. \end{cases} \tag{2.6.3}$$

We fix $p \in \mathbf{Z}$, and let $\mathbf{c}_{+_p} = (c_p, c_{p-1}, \dots) \in \mathbf{N}^{\mathbf{Z}_{\le p}}, \mathbf{c}_{-_p} = (c_{p+1}, c_{p+2}, \dots) \in \mathbf{N}^{\mathbf{Z}_{>p}}$ be functions which are almost everywhere 0. We define $L(\mathbf{c}_{+_p}), L(\mathbf{c}_{-_p}) \in \mathbf{U}_q^-$ by

$$\begin{aligned} L(\mathbf{c}_{+_p}) &= f_{i_p}^{(c_p)} T_{i_p}(f_{i_{p-1}}^{(c_{p-1})}) T_{i_p} T_{i_{p-1}}(f_{i_{p-2}}^{(c_{p-2})}) \cdots \\ L(\mathbf{c}_{-_p}) &= \cdots T_{i_{p+1}}^{-1} T_{i_{p+2}}^{-1}(f_{i_{p+3}}^{(c_{p+3})}) T_{i_{p+1}}^{-1}(f_{i_{p+2}}^{(c_{p+2})}) f_{i_{p+1}}^{(c_{p+1})}. \end{aligned} \tag{2.6.4}$$

In the case where $p = 0$, we simply write $\mathbf{c}_{+_p}, \mathbf{c}_{-_p}$ as $\mathbf{c}_+, \mathbf{c}_-$. Thus $(\mathbf{c}_{+_p}, \mathbf{c}_{-_p})$ is obtained from $(\mathbf{c}_+, \mathbf{c}_-)$ by the shift by p. Note that $L(\mathbf{c}_+)$ (resp. $L(\mathbf{c}_-)$) coincides with $f_{\beta_0}^{(c_0)} f_{\beta_{-1}}^{(c_{-1})} f_{\beta_{-2}}^{(c_{-2})} \cdots$ (resp. $\cdots f_{\beta_3}^{(c_3)} f_{\beta_2}^{(c_2)} f_{\beta_1}^{(c_1)}$). A similar discussion works for $\underline{\mathbf{U}}_q^-$. We fix $\underline{\mathbf{h}}$ as in (2.5.5). $\beta_k \in \underline{\Delta}^+$ for $k \in \mathbf{Z}$ is defined similarly to (2.6.1), and the root vectors $\underline{f}_{\beta_k} \in \underline{\mathbf{U}}_q^-$ are defined as in (2.6.3). For $\underline{\mathbf{c}}_{+_p} = (\gamma_p, \gamma_{p-1}, \dots) \in \mathbf{N}^{\mathbf{Z}_{\le p}}$, $\underline{\mathbf{c}}_{-_p} = (\gamma_{p+1}, \gamma_{p+2}, \dots) \in \mathbf{N}^{\mathbf{Z}_{>p}}$, define $L(\underline{\mathbf{c}}_{+_p}), L(\underline{\mathbf{c}}_{-_p}) \in \underline{\mathbf{U}}_q^-$ similarly to (2.6.4).

It is known by [**BN**, Remark 3.6], for $i \in I_0, \eta \in \underline{I}_0$,

$$f_{k\delta+\alpha_i} = T_{-\omega_i}^k f_i, \ (k \ge 0), \qquad f_{k\delta-\alpha_i} = T_{-\omega_i}^{-k} T_i f_i, \ (k > 0), \tag{2.6.5}$$

$$\underline{f}_{k|\eta|\delta+\alpha_\eta} = T_{-\widetilde{\omega}_\eta}^k \underline{f}_\eta, \ (k \ge 0), \qquad \underline{f}_{k|\eta|\delta-\alpha_\eta} = T_{-\widetilde{\omega}_\eta}^{-k} T_\eta \underline{f}_\eta, \ (k \ge 0). \tag{2.6.6}$$

2.7. For $i \in I_0, \eta \in \underline{I}_0, k > 0$, put

$$\widetilde{\psi}_{i,k} = f_{k\delta-\alpha_i} f_i - q^2 f_i f_{k\delta-\alpha_i}, \tag{2.7.1}$$

$$\underline{\widetilde{\psi}}_{\eta,k|\eta|} = \underline{f}_{k|\eta|\delta-\alpha_\eta} \underline{f}_\eta - q_\eta^2 \underline{f}_\eta \underline{f}_{k|\eta|\delta-\alpha_\eta}. \tag{2.7.2}$$

It is known that $\widetilde{\psi}_{i,k}$ $(i \in I_0, k \in \mathbf{Z}_{>0})$ are mutually commuting, and similarly, $\underline{\psi}_{\eta,k|\eta|}$

($\eta \in \underline{I}_0, k \in \mathbf{Z}_{>0}$) are mutually commuting. For each $i \in I_0, k \in \mathbf{Z}_{>0}$, we define $\widetilde{P}_{i,k} \in \mathbf{U}_q^-$ by the following recursive identity;

$$\widetilde{P}_{i,k} = \frac{1}{[k]_q} \sum_{s=1}^{k} q^{s-k} \widetilde{\psi}_{i,s} \widetilde{P}_{i,k-s}. \tag{2.7.3}$$

Similarly, for $\eta \in \underline{I}_0, k \in \mathbf{Z}_{>0}$, we define $\underline{\widetilde{P}}_{\eta,k|\eta|} \in \underline{\mathbf{U}}_q^-$ by

$$\underline{\widetilde{P}}_{\eta,k|\eta|} = \frac{1}{[k]_{q_\eta}} \sum_{s=1}^{k} q_{|\eta|}^{s-k} \underline{\widetilde{\psi}}_{\eta,s|\eta|} \underline{\widetilde{P}}_{\eta,(k-s)|\eta|}. \tag{2.7.4}$$

For a fixed $i \in I_0$, regarding $\widetilde{P}_{i,k}$ ($k \in \mathbf{Z}_{>0}$) as elementary symmetric functions, we define Schur polynomials by making use of the determinant formula; for each partition $\rho^{(i)}$, put

$$S_{\rho^{(i)}} = \det\bigl(\widetilde{P}_{i,\rho'_k - k + m}\bigr)_{1 \le k,m \le t} \tag{2.7.5}$$

where $(\rho'_1, \dots, \rho'_t)$ is the dual partition of $\rho^{(i)}$. For an $|I_0|$-tuple of partitions $\mathbf{c}_0 = (\rho^{(i)})_{i \in I_0}$, we define $S_{\mathbf{c}_0}$ by

$$S_{\mathbf{c}_0} = \prod_{i \in I_0} S_{\rho^{(i)}}. \tag{2.7.6}$$

Similarly, for a fixed $\eta \in \underline{I}_0$, choose a partition $\underline{\rho}^{(\eta)}$, and define a Schur polynomial by

$$\underline{S}_{\underline{\rho}^{(\eta)}} = \det\bigl(\underline{\widetilde{P}}_{\eta,(\rho'_k - k + m)|\eta|}\bigr)_{1 \le k,m \le t} \tag{2.7.7}$$

where $(\rho'_1, \dots, \rho'_t)$ is the dual partition of $\underline{\rho}^{(\eta)}$. For an $\underline{I}_0$-tuple of partitions $\underline{\mathbf{c}}_0 = (\underline{\rho}^{(\eta)})_{\eta \in \underline{I}_0}$, we define

$$\underline{S}_{\underline{\mathbf{c}}_0} = \prod_{\eta \in \underline{I}_0} \underline{S}_{\underline{\rho}^{(\eta)}}. \tag{2.7.8}$$

We denote by $\mathscr{C}$ the set of triples $\mathbf{c} = (\mathbf{c}_+, \mathbf{c}_0, \mathbf{c}_-)$, where $\mathbf{c}_+ \in \mathbf{N}^{\mathbf{Z}_{\le 0}}, \mathbf{c}_- \in \mathbf{N}^{\mathbf{Z}_{>0}}$, and $\mathbf{c}_0$ is an I_0-tuple of partitions. For each $\mathbf{c} \in \mathscr{C}, p \in \mathbf{Z}$, we define $L(\mathbf{c}, p) \in \mathbf{U}_q^-$ by

$$L(\mathbf{c}, p) = \begin{cases} L(\mathbf{c}_{+_p}) \times \bigl(T_{i_{p+1}}^{-1} T_{i_{p+2}}^{-1} \cdots T_{i_0}^{-1}(S_{\mathbf{c}_0})\bigr) \times L(\mathbf{c}_{-_p}), & \text{if } p \le 0, \\ L(\mathbf{c}_{+_p}) \times \bigl(T_{i_p} \cdots T_{i_2} T_{i_1}(S_{\mathbf{c}_0})\bigr) \times L(\mathbf{c}_{-p}), & \text{if } p > 0. \end{cases} \tag{2.7.9}$$

Similarly, we denote by $\underline{\mathscr{C}}$ the set of triples $\underline{\mathbf{c}} = (\underline{\mathbf{c}}_+, \underline{\mathbf{c}}_0, \underline{\mathbf{c}}_-)$, where $\underline{\mathbf{c}}_+ \in \mathbf{N}^{\mathbf{Z}_{\le 0}}, \underline{\mathbf{c}}_- \in \mathbf{N}^{\mathbf{Z}_{>0}}$, and $\underline{\mathbf{c}}_0$ is the set of $\underline{I}_0$-tuples of partitions. We define $L(\underline{\mathbf{c}}, p) \in \underline{\mathbf{U}}_q^-$ in a similar way as in (2.7.9). The following results are proved in [**BN**]. Note that Lemma 3.39 in [**BN**] can be applied to the case where X is simply laced.

PROPOSITION 2.8 ([**BN**, Proposition 3.16]). *$L(\mathbf{c}, p) \in {}_{\mathbf{A}}\mathbf{U}_q^-$, and $L(\underline{\mathbf{c}}, p) \in {}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$.*

PROPOSITION 2.9 ([**BN**, Theorem 3.13 (i), Lemma 3.39]). *We fix* $\mathbf{h}$ *and* p *as before.*

(i) *For various* $\mathbf{c} \in \mathscr{C}$, $L(\mathbf{c}, p)$ *are almost orthonormal, namely,*

$$(L(\mathbf{c}, p), L(\mathbf{c}', p)) \in \delta_{\mathbf{c},\mathbf{c}'} + q\mathbf{Z}[[q]] \cap \mathbf{Q}(q).$$

In particular, for a fixed $\mathbf{h}, p$, $\{L(\mathbf{c}, p) \mid \mathbf{c} \in \mathscr{C}\}$ *gives a* $\mathbf{Q}(q)$*-basis of* $\mathbf{U}_q^-$.

Similarly, $L(\underline{\mathbf{c}}, p)$ *are almost orthonormal, and* $\{L(\underline{\mathbf{c}}, p) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ *gives a* $\mathbf{Q}(q)$*-basis of* $\underline{\mathbf{U}}_q^-$.

(ii) $\{L(\mathbf{c}, p) \mid \mathbf{c} \in \mathscr{C}\}$ *gives an* $\mathbf{A}$*-basis of* ${}_{\mathbf{A}}\mathbf{U}_q^-$.

2.10. We first fix $\underline{\mathbf{h}}$ as in (2.5.5), then construct $\mathbf{h}$ as in (2.5.4) from $\underline{\mathbf{h}}$ by making use of the relation (2.5.3). We also fix $\underline{p} > 0$, and consider the sequence $w_{\underline{p}} = s_{\eta_{\underline{p}}} s_{\eta_{\underline{p}-1}} s_{\eta_{\underline{p}-2}} \cdots$ in $\underline{W} \simeq W^\sigma$. Then $w_{\underline{p}}$ determines an integer $p > 0$ such that $w_{\underline{p}}$ corresponds to $w_p = s_{i_p} s_{i_{p-1}} s_{i_{p-2}} \cdots$ in W. For each s_{η_k} appearing in $w_{\underline{p}}$, let I_k be an interval in $\mathbf{Z}$ such that $s_{\eta_k} = \prod_{j \in I_k} s_{i_j}$ corresponds to a subexpression of w_p as above. Put $F_{\eta_k}(\mathbf{c}_{\pm_p}) = \prod_{j \in I_k} f_{i_j}^{(c_j)}$. We also define $R_\eta = \prod_{j \in \eta} T_j$ for $\eta \in \underline{I}_0$. Then σ commutes with R_η. Note that $L(\mathbf{c}_{+_p}), L(\mathbf{c}_{-_p})$ can be expressed as

$$\begin{aligned} L(\mathbf{c}_{+_p}) &= F_{\eta_{\underline{p}}}(\mathbf{c}_{+_p}) R_{\eta_{\underline{p}}} \big(F_{\eta_{\underline{p}-1}}(\mathbf{c}_{+_p})\big) R_{\eta_{\underline{p}}} R_{\eta_{\underline{p}-1}} \big(F_{\eta_{\underline{p}-2}}(\mathbf{c}_{+_p})\big) \cdots, \\ L(\mathbf{c}_{-_p}) &= \cdots R_{\eta_{\underline{p}+1}}^{-1} R_{\eta_{\underline{p}+2}}^{-1} \big(F_{\eta_{\underline{p}+3}}(\mathbf{c}_{-_p})\big) R_{\eta_{\underline{p}+1}}^{-1} \big(F_{\eta_{\underline{p}+2}}(\mathbf{c}_{-_p})\big) F_{\eta_{\underline{p}+1}}(\mathbf{c}_{-_p}). \end{aligned} \tag{2.10.1}$$

We have a lemma.

LEMMA 2.11. *Take* $\mathbf{h}, p$ *as in* 2.10.

(i) σ *permutes the PBW-basis* $\{L(\mathbf{c}, p)\}$ *of* $\mathbf{U}_q^-$, *namely,* $\sigma(L(\mathbf{c}, p)) = L(\mathbf{c}', p)$ *for some* $\mathbf{c}' \in \mathscr{C}$.

(ii) *Let* $\mathbf{c} = (\mathbf{c}_+, \mathbf{c}_0, \mathbf{c}_-) \in \mathscr{C}$. *Then* $L(\mathbf{c}, p)$ *is* σ*-stable if and only if* c_j *is constant for each* $j \in I_k$ *corresponding to* s_{η_k} *in* $w_{\underline{p}}$, *and* $\rho^{(i)}$ *is constant on* $i \in \eta$ *for each* $\eta \in \underline{I}_0$. *In particular, the set of* σ*-stable PBW-basis in* $\mathbf{U}_q^-$ *with respect to* $\mathbf{h}, p$ *is in bijection with the set of PBW-basis* $\{L(\underline{\mathbf{c}}, \underline{p})\}$ *in* $\underline{\mathbf{U}}_q^-$ *if* $\mathbf{h}, p$ *are obtained from* $\underline{\mathbf{h}}, \underline{p}$.

PROOF. By (2.10.1), we have $\sigma(L(\mathbf{c}_{+_p})) = L(\mathbf{c}'_{+_p})$, $\sigma(L(\mathbf{c}_{-_p})) = L(\mathbf{c}'_{-_p})$ for some $\mathbf{c}'_{+_p} \in \mathbf{N}^{\mathbf{Z}_{\le p}}, \mathbf{c}'_{-_p} \in \mathbf{N}^{\mathbf{Z}_{>p}}$. On the other hand, since $\sigma(f_{k\delta \pm \alpha_i}) = f_{k\delta \pm \alpha_{\sigma(i)}}$ for $i \in I_0, k > 0$ by (2.6.5), we have $\sigma(\widetilde{\psi}_{i,k}) = \widetilde{\psi}_{\sigma(i),k}$, and so $\sigma(\widetilde{P}_{i,k}) = \widetilde{P}_{\sigma(i),k}$. This implies that $\sigma(S_{\rho^{(i)}}) = S_{\rho^{(\sigma(i))}}$ for each $i \in I_0$. We see that $\sigma(S_{\mathbf{c}_0}) = S_{\mathbf{c}'_0}$ for some I_0-tuple of partitions $\mathbf{c}'_0$. Thus we obtain (i). (ii) follows from (i). □

2.12. We apply the discussion in 1.5 to the affine case, and we can define a homomorphism $\pi : {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to \mathbf{V}_q$. For any $\eta \in \underline{I}$, and $a \in \mathbf{N}$, we define $\widetilde{f}_\eta^{(a)} = \prod_{i \in I} f_i^{(a)}$, and put $g_\eta^{(a)} = \pi(\widetilde{f}_\eta^{(a)})$ as in 1.9. Then Proposition 1.10 still holds for the affine case, and we can define an algebra homomorphism $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \to \mathbf{V}_q$ of $\mathbf{A}'$-algebras. Assume

that $\mathbf{h}, p$ are obtained from $\underline{\mathbf{h}}, \underline{p}$ as in 2.10. We denote by $\mathscr{X}_{\mathbf{h},p}$ the set of PBW-basis $\{L(\mathbf{c},p) \mid \mathbf{c} \in \mathscr{C}\}$ of $\mathbf{U}_q^-$, and $\mathscr{X}_{\mathbf{h},p}^{\sigma}$ the subset of $\mathscr{X}_{\mathbf{h},p}$ consisting of σ-stable PBW-basis. Similarly, we denote by $\underline{\mathscr{X}}_{\underline{\mathbf{h}},\underline{p}}$ the set of PBW-basis $\{L(\underline{\mathbf{c}},\underline{p}) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ of $\underline{\mathbf{U}}_q^-$. By Lemma 2.11 (ii), we have a natural bijection $\mathscr{X}_{\mathbf{h},p}^{\sigma} \simeq \underline{\mathscr{X}}_{\underline{\mathbf{h}},\underline{p}}$, by $L(\mathbf{c},p) \leftrightarrow L(\underline{\mathbf{c}},\underline{p})$. We put $E(\underline{\mathbf{c}},\underline{p}) = \pi(L(\mathbf{c},p))$ under this correspondence. Then by Lemma 2.11 (i), and by Proposition 2.9 (see the discussion in 1.12), we see that $\{E(\underline{\mathbf{c}},\underline{p})\}$ gives rise to an $\mathbf{A}'$-basis of $\mathbf{V}_q$.

Assume that $L(\mathbf{c},p) \in \mathscr{X}_{\mathbf{h},p}^{\sigma}$ corresponds to $L(\underline{\mathbf{c}},\underline{p}) \in \underline{\mathscr{X}}_{\underline{\mathbf{h}},\underline{p}}$ with $\mathbf{c} = (\mathbf{c}_+, \mathbf{c}_0, \mathbf{c}_-)$, $\underline{\mathbf{c}} = (\underline{\mathbf{c}}_+, \underline{\mathbf{c}}_0, \underline{\mathbf{c}}_-)$. We consider $L(\mathbf{c}_{+_p}), L(\mathbf{c}_{-_p}) \in \mathbf{U}_q^{-,\sigma}$ and $L(\underline{\mathbf{c}}_{+_{\underline{p}}}), L(\underline{\mathbf{c}}_{-_{\underline{p}}}) \in \underline{\mathbf{U}}_q^-$. The following result can be proved in a similar way as in Theorem 1.14 (i).

PROPOSITION 2.13. *$\Phi(L(\underline{\mathbf{c}}_{+_{\underline{p}}})) = \pi(L(\mathbf{c}_{+_p}))$ and $\Phi(L(\underline{\mathbf{c}}_{-_{\underline{p}}})) = \pi(L(\mathbf{c}_{-_p}))$.*

2.14. Let $\mathbf{c}_0 = (\rho^{(i)})_{i\in I_0}$ be an I_0-tuple of partitions appearing in $\mathbf{c}$, and $\underline{\mathbf{c}}_0 = (\underline{\rho}^{(\eta)})_{\eta\in \underline{I}_0}$ be an $\underline{I}_0$-tuple of partitions appearing in $\underline{\mathbf{c}}$ as in 2.7. We have $\rho^{(i)} = \underline{\rho}^{(\eta)}$ if $i \in \eta$ for each $\eta \in \underline{I}_0$. Then $S_{\mathbf{c}_0} \in \mathbf{U}_q^{-,\sigma}$, and we consider $\pi(S_{\mathbf{c}_0}) \in \mathbf{V}_q$. On the other hand, we can consider $\underline{S}_{\underline{\mathbf{c}}_0} \in \underline{\mathbf{U}}_q^-$. We show a lemma.

LEMMA 2.15. $\Phi(\underline{S}_{\underline{\mathbf{c}}_0}) = \pi(S_{\mathbf{c}_0})$.

PROOF. Take $i \in I_0$ such that $i \in \eta$. We consider $\prod_{i\in\eta} f_{k\delta+\alpha_i} \in \mathbf{U}_q^{-,\sigma}$ and $\underline{f}_{k|\eta|\delta+\alpha_\eta} \in \underline{\mathbf{U}}_q^-$, and similar elements obtained by replacing α_i by $-\alpha_i$, α_η by $-\alpha_\eta$. By applying Proposition 2.13 for the case where $p = 0$, we have

$$\Phi\big(\underline{f}_{k|\eta|\delta+\alpha_\eta}\big) = \pi\Big(\prod_{i\in\eta} f_{k\delta+\alpha_i}\Big), \qquad \Phi\big(\underline{f}_{k|\eta|\delta-\alpha_\eta}\big) = \pi\Big(\prod_{i\in\eta} f_{k\delta-\alpha_i}\Big). \tag{2.15.1}$$

Next we show, for $\eta \in \underline{I}_0, k > 0$, that

$$\Phi\big(\widetilde{\underline{\psi}}_{\eta,k|\eta|}\big) = \pi\Big(\prod_{i\in\eta} \widetilde{\psi}_{i,k}\Big). \tag{2.15.2}$$

It is known by [**B**], [**BCP**] that $T_{\omega_i}(f_{k\delta\pm\alpha_j}) = f_{k\delta\pm\alpha_j}$ for $i \neq j, k \geq 0$. Hence if $(\alpha_i,\alpha_j) = 0$, we have

$$f_j f_{k\delta-\alpha_i} = f_j T_{-\omega_i}^{-k} T_i(f_i) = T_{\omega_i}^{-k} T_i(f_j f_i) = T_{\omega_i}^{-k} T_i(f_i f_j) = f_{k\delta-\alpha_i} f_j \tag{2.15.3}$$

by (2.6.5). Again by using (2.6.5) we have

$$f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} = f_{k\delta-\alpha_j} f_{k\delta-\alpha_i}. \tag{2.15.4}$$

In the case where $|\eta| = 1$, (2.15.2) immediately follows from (2.15.1). We assume that $|\eta| = 2$, and put $\eta = \{i,j\}$. Then by using commutation relations (2.15.3), (2.15.4), we have

$$\begin{aligned}\widetilde{\psi}_{i,k}\widetilde{\psi}_{j,k} &= (f_{k\delta-\alpha_i} f_i - q^2 f_i f_{k\delta-\alpha_i})(f_{k\delta-\alpha_j} f_j - q^2 f_j f_{k\delta-\alpha_j}) \\ &= f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} f_i f_j + q^4 f_i f_j f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} - q^2 Z,\end{aligned}$$

where

$$\begin{aligned} Z &= f_{k\delta-\alpha_i} f_i f_j f_{k\delta-\alpha_j} + f_i f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} f_j \\ &= f_j f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} f_i + f_i f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} f_j \\ &= f_j f_{k\delta-\alpha_j} f_{k\delta-\alpha_i} f_i + \sigma(f_j f_{k\delta-\alpha_j} f_{k\delta-\alpha_i} f_i). \end{aligned}$$

Since $Z \in J$, we have

$$\pi\big(\widetilde{\psi}_{i,k}\widetilde{\psi}_{j,k}\big) = \pi(f_{k\delta-\alpha_i} f_{k\delta-\alpha_j} f_i f_j - q_\eta^2 f_i f_j f_{k\delta-\alpha_i} f_{k\delta-\alpha_j}).$$

Now (2.15.2) follows from (2.15.1). The proof for the case $|\eta| = 3$ is similar. Thus (2.15.2) is proved.

Since $\widetilde{\psi}_{i,k}$ and $\widetilde{\psi}_{j,\ell}$ commute for any pair, $\widetilde{\psi}_{i,k}$ commutes with $\widetilde{P}_{j,\ell}$ for any pair i, j, k, ℓ. Then by a similar argument as in the proof of (2.15.2), for each $\eta \in \underline{I}_0$ we have

$$\Phi\big(\widetilde{\underline{P}}_{\eta,k}\big) = \pi\Big(\prod_{i\in\eta} \widetilde{P}_{i,k}\Big). \tag{2.15.5}$$

(Note that $([k]_q)^{|\eta|} = [k]_{q_\eta}$ in $\mathbf{A}'$.)

Since $\widetilde{P}_{i,k}$ are commuting for any pair i, k, (2.15.5) implies, by a similar argument as above, that

$$\Phi(\underline{S}_{\rho^{(\eta)}}) = \pi\Big(\prod_{i\in\eta} S_{\rho^{(i)}}\Big) \tag{2.15.6}$$

for any $\eta \in \underline{I}_0$. Lemma 2.15 follows from this. □

The following result is an analogue of Theorem 1.14 and Proposition 1.20.

THEOREM 2.16. (i) *For any $\underline{\mathbf{c}} \in \underline{\mathscr{C}}$, we have $\Phi(L(\underline{\mathbf{c}}, \underline{0})) = E(\underline{\mathbf{c}}, \underline{0})$.*

(ii) *PBW-basis $\{L(\underline{\mathbf{c}}, \underline{0}) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ gives an $\mathbf{A}'$-basis of ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$.*

(iii) *Φ gives an isomorphism ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \xrightarrow{\sim} \mathbf{V}_q$.*

PROOF. (i) follows from Proposition 2.13 and Lemma 2.15. By Proposition 2.8, (the image of) $L(\underline{\mathbf{c}}, \underline{0})$ is contained in ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$. Hence the map $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \to \mathbf{V}_q$ is surjective. As in the proof of Theorem 1.14, Φ can be extended to the map ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^- \to {}_{\mathbf{F}(q)}\mathbf{V}_q$, which gives an isomorphism of $\mathbf{F}(q)$-algebras. Let ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ be the $\mathbf{A}'$-submodule of ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$ spanned by $L(\underline{\mathbf{c}}, 0)$. Then Φ gives an isomorphism ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^- \simeq \mathbf{V}_q$ of $\mathbf{A}'$-modules. In particular, ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ is an algebra over $\mathbf{A}'$. We note that

$${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^- = {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-. \tag{2.16.1}$$

In fact, ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^- \subset {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ by Proposition 2.8. Since $\{E(\underline{\mathbf{c}}, \underline{p}) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ is an $\mathbf{A}'$-basis of $\mathbf{V}_q$, $\{\Phi^{-1}(E(\underline{\mathbf{c}}, \underline{p})) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ gives an $\mathbf{A}'$-basis of ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ for any $\underline{p}$. Hence by (2.10.1),

${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ is invariant under the left multiplication by $\underline{f}^{(k)}_{\underline{\eta}_{\underline{p}}}$. By (2.5.7), for any $\eta \in \underline{I}$, there exists $\underline{p}$ such that $\eta = \eta_{\underline{p}}$. Thus ${}_{\mathbf{A}'}\widetilde{\underline{\mathbf{U}}}_q^-$ is invariant under the left multiplication by any $\underline{f}^{(k)}_{\underline{\eta}}$, and (2.16.1) follows. Now (ii) and (iii) follow from (2.16.1). The theorem is proved. □

COROLLARY 2.17. *For any $\underline{p} \in \mathbf{Z}$, the PBW-basis $\{L(\underline{\mathbf{c}}, \underline{p}) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ gives an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$.*

PROOF. By a similar argument as in the proof of Corollary 1.21, we see that $\{L(\underline{\mathbf{c}}, \underline{0}) \mid \underline{\mathbf{c}} \in \underline{\mathscr{C}}\}$ gives an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$ thanks to Theorem 2.16. Then by [**BN**, Lemma 3.39], $\{L(\underline{\mathbf{c}}, \underline{p})\}$ gives an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$. The corollary is proved. □

REMARK 2.18. In the case where $\mathfrak{g}$ is a simply laced affine algebra, the fact that $\{L(\mathbf{c}, p) \mid \mathbf{c} \in \mathscr{C}\}$ gives an $\mathbf{A}$-basis of ${}_{\mathbf{A}}\mathbf{U}_q^-$ (Proposition 2.9 (ii)) was known by [**BCP**] for $p = 0$, and was proved by [**BN**] for arbitrary p. Corollary 2.17 is a generalization of this fact to the case of twisted affine algebras. Once this is done, one can define the (signed) canonical basis $b(\mathbf{c}, p)$ parametrized by $L(\mathbf{c}, p)$ as in (1.22.2). The basis $\{b(\mathbf{c}, p) \mid \mathbf{c} \in \mathscr{C}\}$ is independent of the choice of $\mathbf{h}$ and p, up to ± 1. In [**BN**], in the simply laced case, this ambiguity of the sign was removed by using the theory of extremal weight modules due to [**K2**]. It is likely that our result makes it possible to extend their results to the case of twisted affine Lie algebras.

3. The proof of Proposition 1.10.

3.1. In this and next section we write $[a]_{q^i}$ as $[a]_i$ for any $i \in \mathbf{Z}$. Thus $[a]_q = [a]_1$ and $[a]_{q_\eta} = [a]_{|\eta|}$ since $(\alpha_\eta, \alpha_\eta)_1/2 = |\eta|$. ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ is the $\mathbf{A}'$-algebra with generators $\underline{f}^{(a)}_{\underline{\eta}}$ $(\eta \in \underline{I}, a \in \mathbf{N})$ with fundamental relations

$$\sum_{k=0}^{1-a_{\eta\eta'}} (-1)^k \underline{f}^{(k)}_{\underline{\eta}} \underline{f}_{\underline{\eta}'} \underline{f}^{(1-a_{\eta\eta'}-k)}_{\underline{\eta}} = 0, \qquad (\eta \neq \eta'), \tag{3.1.1}$$

$$[a]^!_{|\eta|} \underline{f}^{(a)}_{\underline{\eta}} = \underline{f}^a_{\underline{\eta}}, \qquad (a \in \mathbf{N}), \tag{3.1.2}$$

where $A = (a_{\eta\eta'})$ is the Cartan matrix of $\underline{X}$. In order to prove Proposition 1.10, it is enough to show that $g^{(a)}_\eta$ satisfies a similar relations as above, namely,

$$\sum_{k=0}^{1-a_{\eta\eta'}} (-1)^k g^{(k)}_\eta g_{\eta'} g^{(1-a_{\eta\eta'}-k)}_\eta = 0, \qquad (\eta \neq \eta'), \tag{3.1.3}$$

$$[a]^!_{|\eta|} g^{(a)}_\eta = g^a_\eta, \qquad (a \in \mathbf{N}). \tag{3.1.4}$$

First we show (3.1.4). We have

$$\widetilde{f}^{(a)}_\eta = \prod_{i \in \eta} f^{(a)}_i = ([a]^!_1)^{-|\eta|} \prod_{i \in \eta} f^a_i = ([a]^!_1)^{-|\eta|} \widetilde{f}^a_\eta.$$

Since $|\eta| = 1$ or ε, we have $([a]_1^!)^{|\eta|} = [a]_{|\eta|}^!$ in $\mathbf{A}' = \mathbf{F}[q, q^{-1}]$ with $\mathbf{F} = \mathbf{Z}/\varepsilon\mathbf{Z}$. Thus (3.1.4) follows.

For the proof of (3.1.3), we may assume that $\underline{X}$ is of rank 2. Here we change the notation from 1.3, and consider $\underline{I} = \{\underline{1}, \underline{2}\}$ with Cartan matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & a \\ -1 & 2 \end{pmatrix}$$

where $\underline{X}$ is of type $A_1 \times A_1$ in the first case, and $a = -1, -2, -3$ according to the cases $\underline{X}$ is of type A_2, B_2, G_2.

Assume that $\underline{X}$ is of type $A_1 \times A_1$. In this case, we have $(\alpha_i, \alpha_j) = 0$ for any $i, j \in I$ such that $i \neq j$. It is easily seen that $g_{\underline{1}} g_{\underline{2}} = g_{\underline{2}} g_{\underline{1}}$, which coincides with the relation (3.1.3). Thus (3.1.3) holds.

3.2. Assume that $\underline{X}$ is of type A_2. We have two possibilities for I, $\underline{i} = \{i\}$ or $\underline{i} = \{i, i'\}$ for $i = 1, 2$. In the former case, (3.1.3) clearly holds. So we may assume that $I = \{1, 2, 1', 2'\}$ with $\underline{1} = \{1, 1'\}, \underline{2} = \{2, 2'\}$, where $(\alpha_i, \alpha_j) = -1$ for $\{i, j\} = \{1, 2\}$ or $\{1', 2'\}$, and is equal to zero for other cases. We have $g_{\underline{1}} = \pi(f_1 f_{1'})$ and $g_{\underline{2}} = \pi(f_2 f_{2'})$. The relation (3.1.3) is given by

$$g_{\underline{1}} g_{\underline{2}}^{(2)} - g_{\underline{2}} g_{\underline{1}} g_{\underline{2}} + g_{\underline{2}}^{(2)} g_{\underline{1}} = 0. \tag{3.2.1}$$

By (3.1.4), this is equivalent to

$$g_{\underline{1}} g_{\underline{2}}^2 - (q^2 + q^{-2}) g_{\underline{2}} g_{\underline{1}} g_{\underline{2}} + g_{\underline{2}}^2 g_{\underline{1}} = 0. \tag{3.2.2}$$

We show (3.2.2). It follows from the Serre relations for A_2, we have

$$\begin{aligned} f_1 f_2^2 - (q + q^{-1}) f_2 f_1 f_2 + f_2^2 f_1 = 0, \\ f_2 f_1^2 - (q + q^{-1}) f_1 f_2 f_1 + f_1^2 f_2 = 0, \end{aligned} \tag{3.2.3}$$

and formulas obtained form (3.2.3) by replacing f_1, f_2 by $f_{1'}, f_{2'}$. By multiplying these two formulas, and by using the commutation relations $f_i f_j = f_j f_i$ unless $\{i, j\} = \{1, 2\}$ nor $\{1', 2'\}$, we have

$$(f_1 f_{1'})(f_2 f_{2'})^2 + (q + q^{-1})^2 (f_2 f_{2'})(f_1 f_{1'})(f_2 f_{2'}) + (f_2 f_{2'})^2 (f_1 f_{1'}) + Z = 0,$$

where

$$Z = -(q + q^{-1})\big(f_2 f_{2'}^2 f_1 f_{1'} f_2 + f_{2'} f_2^2 f_{1'} f_1 f_{2'}\big).$$

Since $\varepsilon = 2$, and $Z \in J$, we obtain (3.2.2). Thus (3.1.3) is verified for $\underline{X}$ of type A_2.

3.3. Assume that $\underline{X}$ is of type B_2 and X is of type A_3. We have $I = \{1, 2, 2'\}$, $\underline{I} = \{\underline{1}, \underline{2}\}$ with $\underline{1} = \{1\}$ and $\underline{2} = \{2, 2'\}$, where $(\alpha_i, \alpha_j) = -1$ if $\{i, j\} = \{1, 2\}$ or $\{1, 2'\}$ and is equal to zero for all other $i \neq j$. By (3.1.4), (3.1.3) is equivalent to the formulas

$$g_{\underline{1}} g_{\underline{2}}^2 - (q^2 + q^{-2}) g_{\underline{2}} g_{\underline{1}} g_{\underline{2}} + g_{\underline{2}}^2 g_{\underline{1}} = 0, \tag{3.3.1}$$

$$g_{\underline{2}}g_{\underline{1}}^3 - [3]_1 g_{\underline{1}}g_{\underline{2}}g_{\underline{1}}^2 + [3]_1 g_{\underline{1}}^2 g_{\underline{2}}g_{\underline{1}} - g_{\underline{1}}^3 g_{\underline{2}} = 0. \tag{3.3.2}$$

We show (3.3.1). Here $\mathbf{U}_q^-$ satisfies the formulas (3.2.3) and the formulas obtained from (3.2.3) by replacing f_1, f_2 by $f_1, f_{2'}$. By multiplying $f_{2'}^2$ from the right on (3.2.3) for $f_1 f_2^2$, we have

$$f_1 f_2^2 f_{2'}^2 - (q+q^{-1}) f_2 f_1 f_{2'}^2 f_2 + f_2^2 f_1 f_{2'}^2 = 0. \tag{3.3.3}$$

Here by applying (3.2.3) for $f_1 f_{2'}^2$, we have

$$\begin{aligned} f_2(f_1 f_{2'}^2) f_2 &= (q+q^{-1})(f_2 f_{2'}) f_1 (f_2 f_{2'}) - f_2 f_{2'}^2 f_1 f_2, \\ f_2^2(f_1 f_{2'}^2) &= (q+q^{-1}) f_2^2 f_{2'} f_1 f_{2'} - (f_2 f_{2'})^2 f_1. \end{aligned}$$

Substituting these formulas into (3.3.3), we have

$$f_1(f_2 f_{2'})^2 - (q+q^{-1})^2 (f_2 f_{2'}) f_1 (f_2 f_{2'}) - (f_2 f_{2'})^2 f_1 + Z = 0,$$

where

$$Z = (q+q^{-1})\big(f_{2'}^2 f_2 f_1 f_2 + f_2^2 f_{2'} f_1 f_{2'}\big).$$

Since $\delta = 2$, $Z \in J$, we obtain (3.3.1).

Next we show (3.3.2). First note the following equality. By using (3.2.3) for $f_1 f_2 f_1$ and for $f_{2'} f_1^2$, we have

$$\begin{aligned} (q+q^{-1}) f_1 f_{2'} (f_1 f_2 f_1) &= f_1 f_{2'} (f_2 f_1^2 + f_1^2 f_2) \\ &= f_1 (f_2 f_{2'}) f_1^2 + f_1 (f_{2'} f_1^2) f_2 \\ &= f_1 (f_2 f_{2'}) f_1^2 + (q+q^{-1}) f_1^2 f_{2'} f_1 f_2 - f_1^3 f_{2'} f_2. \end{aligned} \tag{3.3.4}$$

Here by applying (3.2.3) for $f_2 f_1^2$ and for $f_{2'} f_1^2$ twice, we have

$$\begin{aligned} f_{2'} f_1 (f_2 f_1^2) &= f_{2'} f_1 \big((q+q^{-1}) f_1 f_2 f_1 - f_1^2 f_2\big) \\ &= f_{2'} f_1^2 \big((q+q^{-1}) f_2 f_1 - f_1 f_2\big) \\ &= \big((q+q^{-1}) f_1 f_{2'} f_1 - f_1^2 f_{2'}\big)\big((q+q^{-1}) f_2 f_1 - f_1 f_2\big) \\ &= (q+q^{-1})^2 f_1 f_{2'} f_1 f_2 f_1 - (q+q^{-1}) f_1 (f_{2'} f_1^2) f_2 \\ &\quad - (q+q^{-1}) f_1^2 f_{2'} f_2 f_1 + f_1^2 f_{2'} f_1 f_2 \\ &= (q+q^{-1})^2 f_1 f_{2'} f_1 f_2 f_1 - \big((q+q^{-1})^2 - 1\big) f_1^2 f_{2'} f_1 f_2 \\ &\quad - (q+q^{-1}) f_1^2 f_2 f_{2'} f_1 + (q+q^{-1}) f_1^3 f_2 f_{2'}. \end{aligned}$$

Substituting (3.3.4) into the last equality, we obtain

$$f_{2'} f_1 f_2 f_1^2 = (q+q^{-1}) f_1 (f_2 f_{2'}) f_1^2 - (q+q^{-1}) f_1^2 (f_2 f_{2'}) f_1 + f_1^2 f_{2'} f_1 f_2. \tag{3.3.5}$$

On the other hand, by applying (3.2.3) for $f_{2'} f_1^2$, we have

$$(f_{2'}f_1^2)f_2f_1 = (q+q^{-1})f_1f_{2'}f_1f_2f_1 - f_1^2f_{2'}f_2f_1$$
$$= f_1(f_2f_{2'})f_1^2 + (q+q^{-1})f_1^2f_{2'}f_1f_2 - f_1^3f_2f_{2'} - f_1^2f_{2'}f_2f_1. \tag{3.3.6}$$

The second identity is obtained by substituting (3.3.4) into the first identity. Now by applying (3.2.3) for $f_2f_1^2$ we have

$$f_{2'}f_2f_1^3 = f_{2'}(f_2f_1^2)f_1 = (q+q^{-1})f_{2'}f_1f_2f_1^2 - f_{2'}f_1^2f_2f_1.$$

By substituting (3.3.5) and (3.3.6) into the last formula, we have

$$(f_2f_{2'})f_1^3 = (q^2+1+q^{-2})f_1(f_2f_{2'})f_1^2 - (q^2+1+q^{-2})f_1^2(f_2f_{2'})f_1 + f_1^3(f_2f_{2'}). \tag{3.3.7}$$

Since $[3]_1 = q^2+1+q^{-2}$, by applying π, we obtain (3.3.2). Note that the formula (3.3.7) is obtained without appealing modulo 2. Thus (3.1.3) is verified for $\underline{X}$ of type B_2.

3.4. Assume that $\underline{X}$ is of type G_2 and X is of type D_4. We have $I = \{1,2,2',2''\}$, $\underline{I} = \{\underline{1},\underline{2}\}$ with $\underline{1} = \{1\}$ and $\underline{2} = \{2,2',2''\}$, where $(\alpha_i,\alpha_j) = -1$ if $\{i,j\} = \{1,2\},\{1,2'\}$ or $\{1,2''\}$, and is equal to zero for all other $i \neq j$. By (3.1.4), (3.1.3) is equivalent to the formulas

$$g_{\underline{1}}g_{\underline{2}}^2 - (q^3+q^{-3})g_{\underline{2}}g_{\underline{1}}g_{\underline{2}} + g_{\underline{2}}^2g_{\underline{1}} = 0, \tag{3.4.1}$$

$$g_{\underline{2}}g_{\underline{1}}^4 - [4]_1g_{\underline{1}}g_{\underline{2}}g_{\underline{1}}^3 + \begin{bmatrix}4\\2\end{bmatrix}_1 g_{\underline{1}}^2g_{\underline{2}}g_{\underline{1}}^2 - [4]_1g_{\underline{1}}^3g_{\underline{2}}g_{\underline{1}} + g_{\underline{1}}^4g_{\underline{2}} = 0, \tag{3.4.2}$$

where $[4]_1 = q^3+q+q^{-1}+q^{-3}$ and $\begin{bmatrix}4\\2\end{bmatrix}_1 = q^4+q^2+2+q^{-2}+q^{-4}$. We show (3.4.1). Here $\mathbf{U}_q^-$ satisfies the formulas (3.2.3) and the formulas obtained from (3.2.3) by replacing f_1,f_2 by $f_1,f_{2'}$ or $f_1,f_{2''}$. By multiplying $f_{2'}^2f_{2''}^2$ from the right on (3.2.3) for $f_1f_2^2$, we have

$$f_1f_2^2f_{2'}^2f_{2''}^2 - (q+q^{-1})f_2f_1f_{2'}^2f_{2''}^2f_2 + f_2^2f_1f_{2'}^2f_{2''}^2 = 0. \tag{3.4.3}$$

Concerning the middle term, by applying (3.2.3) for $f_1f_{2'}^2$, then for $f_1f_{2''}^2$, we have

$$f_2(f_1f_{2'}^2)f_{2''}^2f_2 = (q+q^{-1})f_2f_{2'}(f_1f_{2''}^2)f_{2'}f_2 - f_2f_{2'}^2f_1f_{2''}^2f_2$$
$$= (q+q^{-1})^2f_2f_{2'}f_{2''}f_1f_2f_{2'}f_{2''} - (q+q^{-1})f_2f_{2'}f_{2''}^2f_1f_2f_{2'}$$
$$- f_2f_{2'}^2f_1f_2f_{2''}^2. \tag{3.4.4}$$

Concerning the third term, by applying (3.2.3) for $f_1f_{2''}^2$, then for $f_1f_{2'}^2$, and finally for $f_2^2f_1$, we have

$$f_2^2(f_1f_{2''}^2)f_{2'}^2 = (q+q^{-1})f_2^2f_{2''}f_1f_{2''}f_{2'}^2 - f_2^2f_{2''}^2(f_1f_{2'}^2)$$
$$= (q+q^{-1})f_2^2f_{2''}f_1f_{2''}f_{2'}^2 - (q+q^{-1})f_{2'}f_{2''}^2(f_2^2f_1)f_{2'} + f_2^2f_{2''}^2f_{2'}^2f_1$$
$$= (q+q^{-1})f_2^2f_{2''}f_1f_{2''}f_{2'}^2 - (q+q^{-1})^2f_{2'}f_{2''}^2f_2f_1f_2f_{2'}$$
$$+ (q+q^{-1})f_{2'}f_{2''}^2f_1f_2^2f_{2'} + f_2^2f_{2''}^2f_{2'}^2f_1.$$

It follows that

$$f_1(f_2f_{2'}f_{2''})^2-(q+q^{-1})^3(f_2f_{2'}f_{2''})f_1(f_2f_{2'}f_{2''})+(f_2f_{2'}f_{2''})^2f_1+Z=0,$$

where

$$Z=(q+q^{-1})\big(f_2f_{2'}^2f_1f_{2''}^2f_2+f_{2''}f_2^2f_1f_{2'}^2f_{2''}+f_{2'}f_{2''}^2f_1f_2^2f_{2'}\big).$$

Since $\varepsilon=3$, $Z\in J$, we obtain (3.4.1).

3.5. It remains to prove (3.4.2). We shall prove the following formula in ${}_{\mathbf{A}'}\mathbf{U}_q^-$.

$$\begin{aligned}(f_2f_{2'}f_{2''})f_1^4-[4]_1f_1(f_2f_{2'}f_{2''})f_1^3+\begin{bmatrix}4\\2\end{bmatrix}_1f_1^2(f_2f_{2'}f_{2''})f_1^2\\ -[4]_1f_1^3(f_2f_{2'}f_{2''})f_1+f_1^4(f_2f_{2'}f_{2''})\equiv 0 \mod J.\end{aligned}\tag{3.5.1}$$

Clearly (3.5.1) will imply (3.4.2). The proof of (3.5.1) by the direct computation as in the case of B_2 seems to be difficult. Instead, we will prove (3.5.1) by making use of PBW-basis of $\mathbf{U}_q^-$.

Let $\mathbf{h}=(i_1,\dots,i_\nu)$ be a sequence associated to the longest element w_0 of W. Here W is of type D_4, and $\nu=12$. We choose $\mathbf{h}$ as

$$\mathbf{h}=(2,2',2'',1,2,2',2'',1,2,2',2'',1).\tag{3.5.2}$$

We define $\beta_k=s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k})$ for $k=1,\dots,\nu=12$. Then the set Δ^+ of positive roots is given as

$$\begin{aligned}\Delta^+&=\{\beta_1,\dots,\beta_{12}\}\\ &=\{2,2',2'',122'2'',12'2'',122'',122',1122'2'',12,12',12'',1\},\end{aligned}\tag{3.5.3}$$

where we use the notation for positive roots such as $12\leftrightarrow\alpha_1+\alpha_2$, $12'2''\leftrightarrow\alpha_1+\alpha_{2'}+\alpha_{2''}$, etc. For $k=1,\dots,\nu$, the root vector $f_{\beta_k}^{(c)}$ is defined by $f_{\beta_k}^{(c)}=T_{i_1}\cdots T_{i_{k-1}}(f_{i_k}^{(c)})$. Then PBW-basis of $\mathbf{U}_q^-$ is given as $\{L(\mathbf{c},\mathbf{h})\mid\mathbf{c}\in\mathbf{N}^{12}\}$, where for $\mathbf{c}=(c_1,\dots,c_{12})$,

$$L(\mathbf{c},\mathbf{h})=f_2^{(c_1)}f_{2'}^{(c_2)}f_{2''}^{(c_3)}f_{122'2''}^{(c_4)}f_{12'2''}^{(c_5)}f_{122''}^{(c_6)}f_{122'}^{(c_7)}f_{1122'2''}^{(c_8)}f_{12}^{(c_9)}f_{12'}^{(c_{10})}f_{12''}^{(c_{11})}f_1^{(c_{12})}.$$

We use the following commutation relations,

$$\begin{aligned}f_{12}&=f_1f_2-qf_2f_1,\quad\text{(similarly for }\ f_{12'},f_{12''}),\\ f_{122'}&=f_{12}f_{2'}-qf_{2'}f_{12}=f_{12'}f_2-qf_2f_{12'},\quad\text{(similarly for }\ f_{12'2''},f_{122''}),\\ f_{122'2''}&=f_{122'}f_{2''}-qf_{2''}f_{122'}=f_{12'2''}f_2-qf_2f_{12'2''}=f_{122''}f_{2'}-qf_{2'}f_{122''},\\ f_{1122'2''}&=f_{12''}f_{122'}-qf_{122'}f_{12''}=f_{12}f_{12'2''}-qf_{12'2''}f_{12}=f_{12'}f_{122''}-qf_{122''}f_{12'}.\end{aligned}\tag{3.5.4}$$

The following formulas are obtained by applying the commutation formula of Levendorskii and Soibelman [**LS**].

$$f_{122'2''}f_2=q^{-1}f_2f_{122'2''},\quad\text{(similarly for }\ f_{122'2''}f_{2'},f_{122'2''}f_{2''}),$$

$$
\begin{aligned}
f_{12'2''}f_{122'2''} &= q^{-1}f_{122'2''}f_{12'2''}, \quad \text{(similarly for } f_{122'}f_{122'2''}, f_{122''}f_{122'2''}\text{)},\\
f_{122''}f_{12'2''} &= f_{12'2''}f_{122''}, \quad \text{(similarly for } f_{122'}f_{122''}, f_{122'}f_{12'2''}\text{)},\\
f_{1122'2''}f_{122'} &= q^{-1}f_{122'}f_{1122'2''}, \quad \text{(similarly for } f_{1122'2''}f_{122''}, f_{1122'2''}f_{12'2''}\text{)},\\
f_{12}f_{1122'2''} &= q^{-1}f_{1122'2''}f_{12}, \quad \text{(similarly for } f_{12'}f_{1122'2''}, f_{12''}f_{1122'2''}\text{)},\\
f_{12'}f_{12} &= f_{12}f_{12'}, \quad \text{(similarly for } f_{12''}f_{12'}, f_{12''}f_{12}\text{)},\\
f_1f_{12} &= q^{-1}f_{12}f_1, \quad \text{(similarly for } f_1f_{12'}, f_1f_{12''}\text{)}.
\end{aligned}
$$

By using those relations, we obtain

$$
f_1f_{12'2''} = f_{12'2''}f_1 - (q-q^{-1})f_{12'}f_{12''}, \quad \text{(similarly for } f_1f_{122''}, f_1f_{122'}\text{)}.
$$

Also we can compute

$$
\begin{aligned}
f_1(f_2f_{2'}f_{2''}) = {} & f_{122'2''} + q(f_{2''}f_{122'} + f_{2'}f_{122''} + f_2f_{12'2''})\\
& + q^2(f_{2'}f_{2''}f_{12} + f_2f_{2''}f_{12'} + f_2f_{2'}f_{12''}) + q^3f_2f_{2'}f_{2''}f_1.
\end{aligned}
$$

It follows that

$$
f_1(f_2f_{2'}f_{2''}) \equiv f_{122'2''} + q^3f_2f_{2'}f_{2''}f_1 \mod J. \tag{3.5.5}
$$

By multiplying f_1 from the left on both sides of (3.5.5), we have

$$
\begin{aligned}
f_1^2(f_2f_{2'}f_{2''}) &\equiv f_1f_{122'2''} + q^3f_1(f_2f_{2'}f_{2''}f_1)\\
&\equiv f_1f_{122'2''} + q^3(f_{122'2''} + q^3f_2f_{2'}f_{2''}f_1)f_1\\
&= f_1f_{122'2''} + q^3f_{122'2''}f_1 + q^6f_2f_{2'}f_{2''}f_1^2,
\end{aligned}
$$

where we again used (3.5.5) in the second identity. On the other hand, we can compute

$$
\begin{aligned}
f_1f_{122'2''} &= qf_{122'2''}f_1 - q(q-q^{-1})\{f_{12'2''}f_{12} + f_{122''}f_{12'} + f_{122'}f_{12''}\}\\
&\quad + (q^{-1}-2q)f_{1122'2''}\\
&\equiv qf_{122'2''}f_1 + (q+q^{-1})f_{1122'2''} \mod J.
\end{aligned}
\tag{3.5.6}
$$

Hence we have

$$
f_1^2(f_2f_{2'}f_{2''}) \equiv (q+q^3)f_{122'2''}f_1 + (q+q^{-1})f_{1122'2''} + q^6f_2f_{2'}f_{2''}f_1^2. \tag{3.5.7}
$$

Next by multiplying f_1 from the left on both sides of (3.5.7), we have

$$
f_1^3(f_2f_{2'}f_{2''}) \equiv (q+q^3)f_1f_{122'2''}f_1 + (q+q^{-1})f_1f_{1122'2''} + q^6f_1f_2f_{2'}f_{2''}f_1^2. \tag{3.5.8}
$$

Here we can compute

$$
f_1f_{1122'2''} = q^{-1}f_{1122'2''}f_1 + (q-q^{-1})^2f_{12}f_{12'}f_{12''}. \tag{3.5.9}
$$

Thus by applying (3.5.6), (3.5.9) and (3.5.5) to (3.5.8), we have

$$f_1^3(f_2f_{2'}f_{2''}) \equiv (q^6+q^4+q^2)f_{122'2''}f_1^2 + (q^4+2q^2+2+q^{-2})f_{1122'2''}f_1 + (q^3+2q+2q^{-1}+q^{-3})f_{12}f_{12'}f_{12''} + q^9 f_2f_{2'}f_{2''}f_1^3. \tag{3.5.10}$$

Here we note that

$$f_1f_{12}f_{12'}f_{12''} = q^{-3}f_{12}f_{12'}f_{12''}f_1. \tag{3.5.11}$$

Then by multiplying f_1 from the left on both sides of (3.5.10), and by applying (3.5.6), (3.5.9), (3.5.11) and (3.5.5), we have

$$f_1^4(f_2f_{2'}f_{2''}) \equiv (q^9+q^7+q^5+q^3)f_{122'2''}f_1^3 + (q^7+2q^5+2q^{-1}+q^{-3})f_{1122'2''}f_1^2 + (q^6+2q^2+2q^{-2}+q^{-6})f_{12}f_{12'}f_{12''}f_1 + q^{12}f_2f_{2'}f_{2''}f_1^4. \tag{3.5.12}$$

Now (3.5.1) can be verified easily by (3.5.5), (3.5.7), (3.5.10) and (3.5.12). Thus (3.4.2) is verified, and (3.1.3) holds for the case $\underline{X}$ is of type G_2. This completes the proof of Proposition 1.10. □

REMARK 3.6. In the case where $\underline{X}$ is of type B_2, the equality (3.3.7) holds in $\mathbf{U}_q^-$. This is also true for the case of type G_2. In fact, a more precise computation shows that (3.5.1) holds in $\mathbf{U}_q^-$, without appealing modulo J nor modulo 3.

4. The proof of Lemma 1.13.

4.1. We consider the Cartan matrix as in 3.1. Since $\underline{X}$ has rank 2, $\underline{w}_0$ has two reduced expressions $\underline{\mathbf{h}} = (\eta_1, \dots, \eta_{\underline{\nu}})$ and $\underline{\mathbf{h}}' = (\eta_1', \dots, \eta_{\underline{\nu}}')$. Let $*$ be the anti-algebra automorphism of $\mathbf{U}_q^-$ and of $\underline{\mathbf{U}}_q^-$. It is known that

$$\Big(\underline{T}_{\eta_1}\cdots\underline{T}_{\eta_{k-1}}(\underline{f}_{\eta_k})\Big)^* = \underline{T}_{\eta_1'}\cdots\underline{T}_{\eta'_{\underline{\nu}-k}}\Big(\underline{f}_{\eta'_{\underline{\nu}-k+1}}\Big),$$

and the following formula is obtained from the corresponding formula for $\mathbf{U}_q^-$,

$$\Big(R_{\eta_1}\cdots R_{\eta_{k-1}}(\widetilde{f}_{\eta_k})\Big)^* = R_{\eta_1'}\cdots R_{\eta'_{\underline{\nu}-k}}\Big(\widetilde{f}_{\eta'_{\underline{\nu}-k+1}}\Big).$$

Thus we may verify (1.13.1) for a fixed $\underline{\mathbf{h}}$.

In the case where $\underline{X}$ has type $A_1 \times A_1$, there is nothing to prove.

4.2. Assume that $\underline{X}$ has type A_2. We write $I = \{1, 1', 2, 2'\}$ with $\underline{I} = \{\underline{1}, \underline{2}\}$, where $\underline{1} = \{1, 1'\}$, $\underline{2} = \{2, 2'\}$. Put $\underline{\mathbf{h}} = (\underline{2}, \underline{1}, \underline{2})$. Then $\underline{\Delta}^+ = \{\underline{2}, \underline{12}, \underline{1}\}$. We have

$$\underline{T}_{\underline{2}}(\underline{f}_{\underline{1}}) = \underline{f}_{\underline{1}}\underline{f}_{\underline{2}} - q^2\underline{f}_{\underline{2}}\underline{f}_{\underline{1}}, \quad \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}(\underline{f}_{\underline{2}}) = \underline{f}_{\underline{1}}.$$

We have

$$\begin{aligned} R_{\underline{2}}(\widetilde{f}_{\underline{1}}) &= T_2T_{2'}(f_1f_{1'}) = T_2(f_1)T_{2'}(f_{1'}) \\ &= (f_1f_2 - qf_2f_1)(f_{1'}f_{2'} - qf_{2'}f_{1'}) \\ &= f_1f_{1'}f_2f_{2'} + q^2f_2f_{2'}f_1f_{1'} - qZ, \end{aligned} \tag{4.2.1}$$

with

$$\begin{aligned} Z &= f_1f_2f_{2'}f_{1'} + f_2f_1f_{1'}f_{2'} \\ &= (f_{12} + qf_2f_1)f_{2'}f_{1'} + f_2f_1(f_{1'2'} + qf_{2'}f_{1'}) \\ &= f_{12}f_{2'}f_{1'} + f_{1'2'}f_2f_1 + 2qf_2f_1f_{2'}f_{1'}, \end{aligned}$$

where $f_{12} = T_2(f_1) = f_1f_2 - qf_2f_1$ and $f_{1'2'} = T_{2'}(f_{1'}) = f_{1'}f_{2'} - qf_{2'}f_{1'}$. Since $\sigma(f_{12}) = f_{1'2'}$, we see that $Z \in J$. Thus $\pi(R_{\underline{2}}(\widetilde{f}_{\underline{1}})) = g_{\underline{1}}g_{\underline{2}} - q^2g_{\underline{2}}g_{\underline{1}}$ and (1.13.1) holds for $\underline{T}_{\underline{2}}(\underline{f}_{\underline{1}})$. Moreover,

$$\begin{aligned} R_{\underline{2}}R_{\underline{1}}(\widetilde{f}_{\underline{2}}) &= T_2T_{2'}T_1T_{1'}(f_2f_{2'}) \\ &= T_2T_1(f_2)T_{2'}T_{1'}(f_{2'}) \\ &= f_1f_{1'}. \end{aligned} \tag{4.2.2}$$

Hence $\pi(R_{\underline{2}}R_{\underline{1}}(\widetilde{f}_{\underline{2}})) = g_1$, and (1.13.1) holds for $\underline{T}_{\underline{2}}\underline{T}_{\underline{1}}(\underline{f}_{\underline{2}})$. The lemma holds for $\underline{X}$ of type A_2.

4.3. Next assume that $\underline{X}$ has type B_2, and X has type A_3. We write $I = \{2, 1, 2'\}$ and $\underline{I} = \{\underline{1}, \underline{2}\}$, where $\underline{1} = \{1\}, \underline{2} = \{2, 2'\}$. Put $\mathbf{h} = (2, 2', 1, 2, 2', 1)$ and $\Delta^+ = \{2, 2', 122', 12', 12, 1\}$. Then $\underline{\mathbf{h}} = (\underline{2}, \underline{1}, \underline{2}, \underline{1})$ and $\underline{\Delta}^+ = \{\underline{2}, \underline{12}, \underline{112}, \underline{1}\}$. We define root vectors and PBW-bases of $\mathbf{U}_q^-$ and $\underline{\mathbf{U}}_q^-$ similarly to the case of G_2 in 3.5. Then we have

$$\begin{aligned} \underline{f}_{\underline{12}} &= \underline{T}_{\underline{2}}(\underline{f}_{\underline{1}}) = \underline{f}_{\underline{1}}\underline{f}_{\underline{2}} - q^2\underline{f}_{\underline{2}}\underline{f}_{\underline{1}}, \\ \underline{f}_{\underline{112}} &= \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}(\underline{f}_{\underline{2}}) = (q + q^{-1})^{-1}(\underline{f}_{\underline{1}}\underline{f}_{\underline{12}} - \underline{f}_{\underline{12}}\underline{f}_{\underline{1}}), \\ \underline{f}_{\underline{1}} &= \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}\underline{T}_{\underline{2}}(\underline{f}_{\underline{1}}). \end{aligned} \tag{4.3.1}$$

We compute

$$\begin{aligned} R_{\underline{2}}(\widetilde{f}_{\underline{1}}) &= T_2T_{2'}(f_1) = T_2(f_1f_{2'} - qf_{2'}f_1) \\ &= (f_1f_2 - qf_2f_1)f_{2'} - qf_{2'}(f_1f_2 - qf_2f_1) \\ &= f_1f_2f_{2'} + q^2f_2f_{2'}f_1 - q(f_2f_1f_{2'} + f_{2'}f_1f_2). \end{aligned} \tag{4.3.2}$$

Hence $\pi(R_{\underline{2}}(\widetilde{f}_{\underline{1}})) = g_{\underline{1}}g_{\underline{2}} - q^2g_{\underline{2}}g_{\underline{1}}$ and (1.13.1) holds for $\underline{f}_{\underline{12}}$. Also

$$\begin{aligned} R_{\underline{2}}R_{\underline{1}}R_{\underline{2}}(\widetilde{f}_{\underline{1}}) &= T_2(T_{2'}T_1T_{2'})T_2(f_1) \\ &= T_2(T_1T_{2'}T_1)T_2(f_1) \\ &= T_2T_1T_{2'}(f_2) \\ &= T_2T_1(f_2) = f_1. \end{aligned} \tag{4.3.3}$$

Hence $\pi(R_{\underline{2}}R_{\underline{1}}R_{\underline{2}}(\widetilde{f}_{\underline{1}})) = g_{\underline{1}}$, and (1.13.1) holds for $\underline{f}_{\underline{1}}$.

Finally consider

$$\begin{aligned} R_{\underline{2}}R_{\underline{1}}(\widetilde{f}_{\underline{2}}) &= T_2T_{2'}T_1(f_2f_{2'}) \\ &= T_{2'}(T_2T_1(f_2))T_2(T_{2'}T_1(f_{2'})) \\ &= T_{2'}(f_1)T_2(f_1) \\ &= f_{12'}f_{12}. \end{aligned} \tag{4.3.4}$$

Put

$$\begin{aligned} Z_{\underline{112}} &= f_1(f_1f_2f_{2'} - q^2f_2f_{2'}f_1) - (f_1f_2f_{2'} - q^2f_2f_{2'}f_1)f_1 \\ &= f_1^2f_2f_{2'} - (q^2+1)f_1f_2f_{2'}f_1 + q^2f_2f_{2'}f_1^2. \end{aligned} \tag{4.3.5}$$

Clearly $Z_{\underline{112}} \in \mathbf{U}_q^{-,\sigma}$, and $\pi(Z_{\underline{112}}) = (q+q^{-1})\Phi(\underline{f}_{\underline{112}})$ by (4.3.1). We express $Z_{\underline{112}}$ in terms of PBW-basis of $\mathbf{U}_q^-$. By using (3.2.3) for $f_1^2f_2$ and $f_1^2f_{2'}$, we have

$$f_1^2f_2f_{2'} = (q+q^{-1})f_1f_2f_1f_{2'} - (q+q^{-1})f_2f_1f_{2'}f_1 + f_2f_{2'}f_1^2. \tag{4.3.6}$$

$$\begin{aligned} f_1f_2f_1f_{2'} &= (qf_2f_1 + f_{12})(qf_{2'}f_1 + f_{12'}) \\ &= q^2f_2f_1f_{2'}f_1 + qf_{12}f_{2'}f_1 + qf_2f_1f_{12'} + f_{12}f_{12'} \\ &= q^3f_2f_{2'}f_1^2 + qf_{122'}f_1 + f_2f_{12'}f_1 + f_{12}f_{12'} + q^2(f_2f_{12'}f_1 + f_{2'}f_{12}f_1), \\ f_2f_1f_{2'}f_1 &= qf_2f_{2'}f_1^2 + f_2f_{12'}f_1. \end{aligned}$$

Here we have used the formula $f_1f_{12'} = q^{-1}f_{12'}f_1$. Moreover, by using $f_{12}f_{2'} = qf_{2'}f_{12} + f_{122'}$, we have

$$f_1f_2f_{2'}f_1 = q^2f_2f_{2'}f_1^2 + q(f_2f_{12'}f_1 + f_{2'}f_{12}f_1) + f_{122'}f_1. \tag{4.3.7}$$

Substituting these formulas into (4.3.5), we see that

$$Z_{\underline{112}} = (q+q^{-1})f_{12}f_{12'}. \tag{4.3.8}$$

Combining this with (4.3.4), (4.3.5), we obtain $\Phi(\underline{f}_{\underline{112}}) = \pi(R_{\underline{2}}R_{\underline{1}})(f_1)$. Thus the lemma holds for $\underline{X}$ of type B_2.

4.4. Finally assume that $\underline{X}$ has type G_2. We follow the notation in 3.5. Put $\underline{\mathbf{h}} = (\underline{2},\underline{1},\underline{2},\underline{1},\underline{2},\underline{1})$. Then $\underline{\Delta}^+ = \{\underline{2},\underline{12},\underline{11122},\underline{112},\underline{1112},\underline{1}\}$. We have

$$\begin{aligned} \underline{f}_{\underline{12}} &= \underline{T}_{\underline{2}}(\underline{f}_{\underline{1}}) = \underline{f}_{\underline{1}}\underline{f}_{\underline{2}} - q^3\underline{f}_{\underline{2}}\underline{f}_{\underline{1}}, \\ \underline{f}_{\underline{11122}} &= \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}(\underline{f}_{\underline{2}}) = [3]_1^{-1}(\underline{f}_{\underline{112}}\underline{f}_{\underline{12}} - q^{-1}\underline{f}_{\underline{12}}\underline{f}_{\underline{112}}), \\ \underline{f}_{\underline{112}} &= \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}\underline{T}_{\underline{2}}(\underline{f}_{\underline{1}}) = [2]_1^{-1}(\underline{f}_{\underline{1}}\underline{f}_{\underline{12}} - q\underline{f}_{\underline{12}}\underline{f}_{\underline{1}}), \\ \underline{f}_{\underline{1112}} &= \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}\underline{T}_{\underline{2}}\underline{T}_{\underline{1}}(\underline{f}_{\underline{2}}) = [3]_1^{-1}(\underline{f}_{\underline{1}}\underline{f}_{\underline{112}} - q^{-1}\underline{f}_{\underline{112}}\underline{f}_{\underline{1}}) \\ \underline{f}_{\underline{1}} &= \underline{T}_{\underline{2}}\underline{T}_{\underline{1}}\underline{T}_{\underline{2}}\underline{T}_{\underline{1}}\underline{T}_{\underline{2}}(\underline{f}_{\underline{1}}). \end{aligned} \tag{4.4.1}$$

First consider the case $\underline{f}_{\underline{12}}$. By using (4.3.2), we have

$$\begin{aligned}R_{\underline{2}}(\widetilde{f_{\underline{1}}}) &= T_2T_{2'}T_{2''}(f_1)\\ &= T_{2''}\big(f_1f_2f_{2'}+q^2f_2f_{2'}f_1-q(f_2f_1f_{2'}+f_{2'}f_1f_2)\big)\\ &= f_1(f_2f_{2'}f_{2''})-q^3(f_2f_{2'}f_{2''})f_1\\ &\quad -q(f_2f_1f_{2'}f_{2''}+f_{2'}f_1f_{2''}f_2+f_{2''}f_1f_2f_{2'})\\ &\quad +q^2(f_2f_{2'}f_1f_{2''}+f_{2'}f_{2''}f_1f_2+f_{2''}f_2f_1f_{2'}).\end{aligned}$$

Hence $\pi(R_{\underline{2}}(\widetilde{f_1}))=g_{\underline{1}}g_{\underline{2}}-q^3g_{\underline{2}}g_{\underline{1}}$ and (1.13.1) holds for $\underline{f}_{\underline{12}}$.

Next consider the case $\underline{f}_{\underline{112}}$. Put

$$\begin{aligned}Z_{\underline{12}} &= f_1(f_2f_{2'}f_{2''})-q^3(f_2f_{2'}f_{2''})f_1,\\ Z_{\underline{112}} &= f_1Z_{\underline{12}}-qZ_{\underline{12}}f_1.\end{aligned}$$

Then we have

$$Z_{\underline{112}}=f_1^2f_2f_{2'}f_{2''}-(q^3+q)f_1f_2f_{2'}f_{2''}f_1+q^4f_2f_{2'}f_{2''}f_1^2. \tag{4.4.2}$$

Clearly $Z_{\underline{112}}\in\mathbf{U}_q^{-,\sigma}$, and we have

$$\pi(Z_{\underline{112}})=(q+q^{-1})\Phi\big(\underline{f}_{\underline{112}}\big) \tag{4.4.3}$$

by (4.4.1). We express each term of $Z_{\underline{112}}$ in terms of PBW-basis. By (3.5.5), we have

$$f_1(f_2f_{2'}f_{2''})f_1\equiv f_{122'2''}f_1+q^3f_2f_{2'}f_{2''}f_1^2 \qquad \bmod J. \tag{4.4.4}$$

By (3.5.7), we have

$$f_1^2(f_2f_{2'}f_{2''})\equiv(q+q^3)f_{122'2''}f_1+(q+q^{-1})f_{1122'2''}+q^6f_2f_{2'}f_{2''}f_1^2 \qquad \bmod J. \tag{4.4.5}$$

Substituting these formulas into (4.4.2), we have $Z_{\underline{112}}\equiv(q+q^{-1})f_{1122'2''} \mod J$, which implies that

$$\pi(Z_{\underline{112}})=(q+q^{-1})\pi(f_{1122'2''}). \tag{4.4.6}$$

Note that by (3.5.2) and (3.5.3), we have

$$R_{\underline{2}}R_{\underline{1}}R_{\underline{2}}(f_1)=T_2T_{2'}T_{2''}T_1T_2T_{2'}T_{2''}(f_1)=f_{1122'2''}.$$

By comparing (4.4.3) and (4.4.6), we obtain

$$\pi(R_{\underline{2}}R_{\underline{1}}R_{\underline{2}}(f_1))=\Phi\big(\underline{f}_{\underline{112}}\big). \tag{4.4.7}$$

Thus (1.13.1) holds for $\underline{f}_{\underline{112}}$.

Next consider the case of $\underline{f}_{\underline{1112}}$. Put

$$Z_{\underline{1112}}=f_1Z_{\underline{112}}-q^{-1}Z_{\underline{112}}f_1.$$

It follows from the computation of $Z_{\underline{112}}$ in (4.4.2), we have

$$\begin{aligned} Z_{\underline{1112}} = {} & f_1^3 f_2 f_{2'} f_{2''} - (q^3 + q + q^{-1}) f_1^2 f_2 f_{2'} f_{2''} f_1 \\ & + (q^4 + q^2 + 1) f_1 f_2 f_{2'} f_{2''} f_1^2 - q^3 f_2 f_{2'} f_{2''} f_1^3. \end{aligned} \tag{4.4.8}$$

Clearly $Z_{\underline{1112}} \in \mathbf{U}_q^{-,\sigma}$, and we have

$$\pi(Z_{\underline{1112}}) = [2]_1[3]_1 \Phi\big(\underline{f}_{\underline{1112}}\big). \tag{4.4.9}$$

By (3.5.10), we have

$$\begin{aligned} f_1^3(f_2 f_{2'} f_{2''}) \equiv {} & (q^6 + q^4 + q^2) f_{122'2''} f_1^2 + (q^4 + 2q^2 + 2 + q^{-2}) f_{1122'2''} f_1 \\ & + (q^3 + 2q^2 + 2q^{-1} + q^{-3}) f_{12} f_{12'} f_{12''} + q^9 f_2 f_{2'} f_{2''} f_1^3. \end{aligned}$$

By this formula together with (3.5.7) and (3.5.5), we have $Z_{\underline{1112}} = [2]_1[3]_1 f_{12} f_{12'} f_{12''}$ mod J, which implies that.

$$\pi(Z_{\underline{1112}}) = [2]_1[3]_1 \pi(f_{12} f_{12'} f_{12''}). \tag{4.4.10}$$

Note that by (3.5.2) and (3.5.3), we have

$$\begin{aligned} R_{\underline{2}} R_{\underline{1}} R_{\underline{2}} R_{\underline{1}}(\widetilde{f}_{\underline{2}}) &= T_2 T_{2'} T_{2''} T_1 T_2 T_{2'} T_{2''} T_1 (f_2 f_{2'} f_{2''}) \\ &= f_{12} f_{12'} f_{12''}. \end{aligned}$$

By comparing (4.4.9) and (4.4.10), we obtain

$$\pi(R_{\underline{2}} R_{\underline{1}} R_{\underline{2}} R_{\underline{1}}(\widetilde{f}_{\underline{2}})) = \Phi(f_{\underline{1112}}).$$

Thus (1.13.2) holds for $\underline{f}_{\underline{1112}}$.

Finally consider the case of $\underline{f}_{\underline{11122}}$. Put

$$Z_{\underline{11122}} = f_{1122'2''} f_{122'2''} - q^{-1} f_{122'2''} f_{1122'2''}.$$

By (3.5.2) and (3.5.3), we have

$$R_{\underline{2}}(f_1) = T_2 T_{2'} T_{2''}(f_1) = f_{122'2''}.$$

Hence, by the previous computation, we know that $\pi(f_{122'2''}) = \Phi(\underline{f}_{\underline{12}})$. On the other hand, by (4.4.7), we have $\pi(f_{1122'2''}) = \Phi(\underline{f}_{\underline{112}})$. It follows, by (4.4.1), that

$$\pi(Z_{\underline{11122}}) = [3]_1 \Phi\big(\underline{f}_{\underline{11122}}\big). \tag{4.4.11}$$

We note, by (3.5.2) and (3.5.3), that

$$R_{\underline{2}} R_{\underline{1}}(\underline{f}_{\underline{2}}) = T_2 T_{2'} T_{2''} T_1 (f_2 f_{2'} f_{2''}) = f_{12'2''} f_{122''} f_{122'}.$$

Thus in order to prove (1.13.1) for $\underline{f}_{\underline{11122}}$, it is enough to see that

$$Z_{\underline{11122}} \equiv [3]_1 f_{12'2''} f_{122''} f_{122'} \mod J. \tag{4.4.12}$$

We shall express $Z_{\underline{11122}}$ in terms of the PBW-basis of $\mathbf{U}_q^-$. In the computation below, in addition to the formulas in 3.5, we need to use the following commutation relations, which are deduced from the formula of Levendorskii and Soibelman [**LS**] applied for the subalgebra of type A_3.

$$\begin{aligned} f_{12} f_2 &= q^{-1} f_2 f_{12}, \\ f_{122''} f_2 &= q^{-1} f_2 f_{122''}, \\ f_{122'} f_2 &= q^{-1} f_2 f_{122'}, \\ f_{12} f_{122'} &= q^{-1} f_{122'} f_{12}, \\ f_{12} f_{122''} &= q^{-1} f_{122''} f_{12}, \end{aligned} \tag{4.4.13}$$

and the formulas (two for each) by applying the operation σ on both sides. By using these relations, we have

$$\begin{aligned} f_{12} f_{12'2''} &= f_{122'2''} f_{12} + (q^{-1} - q) f_{122''} f_{122'}, \\ f_{1122'2''} f_2 &= f_2 f_{1122'2''} + (q^{-1} - q) f_{122''} f_{122'}, \end{aligned} \tag{4.4.14}$$

and the formulas (two for each) by applying the operation σ on both sides.

Now we can compute (note that the second formula in (4.4.14) is not used in this computation)

$$f_{1122'2''} f_{122'2''} = (q^2 - 2 + q^{-2}) f_{12'2''} f_{122''} f_{122'} + q^{-1} f_{122'2''} f_{1122'2''}.$$

Hence

$$\begin{aligned} Z_{\underline{11122}} &= f_{1122'2''} f_{122'2''} - q^{-1} f_{122'2''} f_{1122'2''} \\ &= (q^2 - 2 + q^{-2}) f_{12'2''} f_{122''} f_{122'} \\ &\equiv [3]_1 f_{12'2''} f_{122''} f_{122'} \mod J. \end{aligned}$$

Thus (4.4.12) holds, and (1.13.1) is proved for $f_{\underline{11122}}$. The lemma holds for $\underline{X}$ of type G_2.

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