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Fiber cones, analytic spreads of the canonical and anticanonical ideals and limit Frobenius complexity of Hibi rings

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Abstract. Let $\mathcal{R}_{\mathbb{K}}[H]$ be the Hibi ring over a field $\mathbb{K}$ on a finite distributive lattice H, P the set of join-irreducible elements of H and ω the canonical ideal of $\mathcal{R}_{\mathbb{K}}[\mathrm{H}]$. We show the powers $\omega^{(n)}$ of ω in the group of divisors $\mathrm{Div}(\mathcal{R}_{\mathbb{K}}[H])$ is identical with the ordinary powers of ω, describe the $\mathbb{K}$-vector space basis of $\omega^{(n)}$ for $n \in \mathbb{Z}$. Further, we show that the fiber cones $\bigoplus_{n\geq 0} \omega^n/\mathfrak{m}\omega^n$ and $\bigoplus_{n\geq 0}(\omega^{(-1)})^n/\mathfrak{m}(\omega^{(-1)})^n$ of ω and $\omega^{(-1)}$ are sum of the Ehrhart rings, defined by sequences of elements of P with a certain condition, which are polytopal complex version of Stanley–Reisner rings. Moreover, we show that the analytic spread of ω and $\omega^{(-1)}$ are maximum of the dimensions of these Ehrhart rings. Using these facts, we show that the question of Page about Frobenius complexity is affirmative: $\lim_{p\to\infty} \mathrm{cx}_F(\mathcal{R}_{\mathbb{K}}[H]) = \dim(\bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)}) - 1$, where p is the characteristic of the field $\mathbb{K}$.

1. Introduction.

Lyubeznik and Smith [**LS**] defined the ring of Frobenius operators: let R be a commutative ring with prime characteristic p and M an R-module. Let ${}^eM$ denote the R-module whose additive module structure is that of M and the action of R is defined by e-times iterated Frobenius map: $r \cdot m = r^{p^e} m$, where the right hand side is the original action of R on M. $\mathrm{Hom}_R(M, {}^eM)$ is an additive group, which is denoted by $\mathcal{F}^e(M)$. Since for any $\varphi \in \mathrm{Hom}_R(M, {}^eM)$ and $\phi \in \mathrm{Hom}(M, {}^{e'}M)$, $\phi \circ \varphi \in \mathrm{Hom}_R(M, {}^{e+e'}M)$, we see that $\bigoplus_{e\geq 0} \mathcal{F}^e(M)$ has a structure of noncommutative ring which is denoted by $\mathcal{F}(M)$ and called the ring of Frobenius operators on M in [**LS**].

In [**LS**], they studied the relation of finite generation of $\mathcal{F}(M)$ over $\mathcal{F}^0(M)$ and good behaviors of tight closure operation, e.g., commutativity of tight closure of an ideal and localization of a ring. Despite the fact that it is now known that tight closure does not commute with localization [**BM**], problem of finite generation of $\mathcal{F}(M)$, especially the case where R is a local ring and M is the injective hull E of the residue field of R is important: see [**KSSZ**]. Moreover, Enescu and Yao [**EY1**] defined the Frobenius complexity of a local ring by taking $\log_p$ of the complexity of $\mathcal{F}(E)$: the complexity of an $\mathbb{N}$-graded ring is a measure of infinite generation over its degree 0 part. They took $\log_p$ in the definition of Frobenius complexity because there is substantial evidence that, in important cases, there is a limit as $p \to \infty$ of Frobenius complexity. Above all they

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showed in [**EY2**] that if $m > n \geq 2$, then the determinantal ring obtained by modding out the 2-minors of an $m \times n$ matrix of indeterminates with base field prime characteristic p has limit Frobenius complexity $m-1$ as $p \to \infty$.

Page [**Pag**] generalized this result to non-Gorenstein anticanonical level Hibi rings: let $\mathcal{R}_{\mathbb{K}}[H]$ be a Hibi ring over a field $\mathbb{K}$ of characteristic p on a distributive lattice H and P the set of join-irreducible elements of H. Then the Frobenius complexity of $\mathcal{R}_{\mathbb{K}}[H]$ approaches to $\#P_{\mathrm{nonmin}}$ as $p \to \infty$, where $P_{\mathrm{nonmin}} := \{z \in P \mid z$ is not in any maximal chain of P of minimal length$\}$. See Definition 2.6 for the definition of anticanonical level property.

In the case where $\mathcal{R}_{\mathbb{K}}[H]$ is anticanonical level, $\#P_{\mathrm{nonmin}}$ is equal to the analytic spread of $\omega^{(-1)}$ minus 1, where ω is the canonical module of $\mathcal{R}_{\mathbb{K}}[H]$ and $\omega^{(-1)}$ is the inverse element of ω in $\mathrm{Div}(\mathcal{R}_{\mathbb{K}}[H])$. Thus, Page raised a question if the limit of Frobenius complexity of an arbitrary non-Gorenstein Hibi ring is equal to the analytic spread of $\omega^{(-1)}$ minus 1 as $p \to \infty$ [**Pag**, Question 5.1].

The main purpose of this paper is to answer this question affirmatively (see Theorem 8.5). In order to accomplish this task, we first analyze the fiber cone of $\omega^{(-1)}$. Since the treatment is the same for the case of ω, we study the fiber cones of ω and $\omega^{(-1)}$ simultaneously. We show that the fiber cone of ω (resp. $\omega^{(-1)}$) is a finite sum of Ehrhart rings each of which is defined by a certain “sequence with Condition N” (see Definition 3.2) and express the analytic spread of ω (resp. $\omega^{(-1)}$) by the dimensions of the Ehrhart rings defined by these sequences. This expression, which is described by a polytopal complex, is interesting in its own right. After this, we show that the Frobenius complexity of $\mathcal{R}_{\mathbb{K}}[H]$ approaches to the analytic spread of $\omega^{(-1)}$ minus 1 by using the expression above.

This paper is organized as follows. First in Section 2, we recall the definition and basic facts of Hibi rings, study the n-th power $\omega^{(n)}$ of the canonical ideal ω of $\mathcal{R}_{\mathbb{K}}[H]$ in $\mathrm{Div}(\mathcal{R}_{\mathbb{K}}[H])$, where $n \in \mathbb{Z}$ and $\mathcal{R}_{\mathbb{K}}[H]$ is the Hibi ring over $\mathbb{K}$ on a finite distributive lattice H. We describe Laurent monomials in $\omega^{(n)}$ for $n \in \mathbb{Z}$ and show that for $n > 0$, $\omega^{(n)} = \omega^n$ and $\omega^{(-n)} = (\omega^{(-1)})^n$. See Theorem 2.9. Though this result for the case of negative powers is obtained by Page [**Pag**, Corollary 3.1 and Proposition 3.2], our proof is more down to earth and treat the cases of positive and negative powers simultaneously.

Next in Section 3, we recall the notion of a sequence with Condition N [**Miy2**, Definition 3.1] (in this paper we call a sequence with Condition N an N-sequence also), define the notion of a $q^{(n)}$-reduced N-sequence, where $n \in \mathbb{Z}$. See Definitions 3.2 and 3.3. We show that the Laurent monomial $\prod_{x \in P} T_x^{\nu(x)}$, where ν is a map from the set P of join-irreducible elements of H to $\mathbb{Z}$, is a generator of $\omega^{(n)}$ if there is a $q^{(n)}$-reduced N-sequence with a certain condition related to ν. Conversely, we construct for each $q^{(n)}$-reduced N-sequence, maps $\nu^{\downarrow}$ and $\nu^{\uparrow}$ form P to $\mathbb{Z}$ such that the Laurent monomials $\prod_{x \in P} T_x^{\nu^{\downarrow}(x)}$ and $\prod_{x \in P} T_x^{\nu^{\uparrow}(x)}$ are generators of $\omega^{(n)}$. From this, we deduce that $\mathcal{R}_{\mathbb{K}}[H]$ is level (resp. anticanonical level) if and only if $q^{(1)}$-reduced (resp. $q^{(-1)}$-reduced) N-sequence is the empty sequence only. Further, we show the degrees of the generators of $\omega^{(n)}$ are consecutive integers, i.e., if there are generators of degrees d_1 and d_2 of $\omega^{(n)}$ with $d_1 < d_2$, then for any integer d with $d_1 \leq d \leq d_2$, there is a generator of $\omega^{(n)}$ with degree d.

After these preparations, we define in Section 4, for each $q^{(\epsilon)}$-reduced N-sequence an integral convex polytope whose Ehrhart ring is standard, i.e., generated by elements of degree 1 over the base field, where $\epsilon = \pm 1$. We express the dimension of this convex polytope by the word of poset and the $q^{(\epsilon)}$-reduced N-sequence which defines this convex polytope. As a special case, we show that if the $q^{(1)}$-reduced (resp. $q^{(-1)}$-reduced) sequence under consideration is an empty sequence, then the dimension of this convex polytope is $\#P_{\mathrm{nonmax}}$ (resp. $\#P_{\mathrm{nonmin}}$), where $P_{\mathrm{nonmax}} := \{z \in P \mid z$ is not in any chain of P of maximal length$\}$.

In Section 5, we show that the Ehrhart ring defined by the convex polytope above is isomorphic to a graded subalgebra of the fiber cone $\bigoplus_{n\geq 0} \omega^n/\mathfrak{m}\omega^n$ (resp. $\bigoplus_{n\geq 0}(\omega^{(-1)})^n/\mathfrak{m}(\omega^{(-1)})^n$) of ω (resp. $\omega^{(-1)}$) if $\epsilon = 1$ (resp. $\epsilon = -1$), where $\mathfrak{m}$ is the irrelevant maximal ideal of $\mathcal{R}_{\mathbb{K}}[H]$. Further, we show that $\bigoplus_{n\geq 0} \omega^n/\mathfrak{m}\omega^n$ (resp. $\bigoplus_{n\geq 0}(\omega^{(-1)})^n/\mathfrak{m}(\omega^{(-1)})^n$) is the sum of finite number of these types of subalgebras. Since the dimension of a graded ring is computed by the Hilbert function, we conclude that the analytic spread of ω (resp. $\omega^{(-1)}$) is the maximum of the dimensions of these Ehrhart rings. We also note that gluing of these Ehrhart rings in $\bigoplus_{n\geq 0} \omega^n/\mathfrak{m}\omega^n$ (resp. $\bigoplus_{n\geq 0}(\omega^{(-1)})^n/\mathfrak{m}(\omega^{(-1)})^n$) is a generalization of Stanley–Reisner rings to polytopal complexes.

In Section 6, we recall the definition of complexity of (not necessarily commutative) $\mathbb{N}$-graded ring and Frobenius complexity. We also define the notion of strong left R-skew algebra and show that if $A = \bigoplus_{n\geq 0} A_n$ is a strong left A_0-skew algebra and A_0 is a commutative local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}A$ is a graded two sided ideal of A and the complexity of A and $A/\mathfrak{m}A$ coincide.

In Section 7, we recall the operation T-construction defined by Katzman et al. [**KSSZ**] and define the T-complexity of a commutative $\mathbb{N}$-graded ring of characteristic p. By the result of Katzman et al. [**KSSZ**, Theorem 3.3] and the results of previous sections, we see that the Frobenius complexity of a Hibi ring can be computed by the T-complexities of Ehrhart rings appeared in Section 5. We state key lemmas to compute the limit T-complexities of Ehrhart rings.

Finally in Section 8, by using the results up to the previous section, we show that the Frobenius complexities of Hibi rings approaches to analytic spread of the anticanonical ideal minus 1.

2. Posets and Hibi rings.

In this paper, all rings and algebras are assumed to have identity element and, up to Section 5, assumed to be commutative unless stated otherwise. We also assume that a ring homomorphism maps the identity element to the identity element. We denote by $\mathbb{N}$ the set of nonnegative integers, by $\mathbb{Z}$ the set of integers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_{>0}$ the set of positive real numbers and by $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. We use letter p to express a prime number.

We denote the cardinality of a set X by $\#X$. For two sets X and Y, we denote by $X \setminus Y$ the set $\{x \in X \mid x \notin Y\}$. We use this notation not only the case where $X \supset Y$ but also the case where $X \not\supset Y$. We denote the set of maps from X to Y by Y^X. If X is a finite set, we identify $\mathbb{R}^X$ with the Euclidean space $\mathbb{R}^{\#X}$.

Next we recall some definitions concerning finite partially ordered sets (poset for short). Let Q be a finite poset. A chain in Q is a totally ordered subset of Q. For a chain X in Q, we define the length of X as $\#X-1$. The maximum length of chains in Q is called the rank of Q and denoted by $\mathrm{rank}Q$. If every maximal chain of Q has the same length, we say that Q is pure. If $I \subset Q$ and $x \in I$, $y \in Q$, $y \le x \Rightarrow y \in I$, then we say that I is a poset ideal of Q. If x, $y \in Q$, $x < y$ and there is no $z \in Q$ with $x < z < y$, we say that y covers x and denoted by $x \lessdot y$ or $y \gtrdot x$. For x, $y \in Q$ with $x \le y$, we set $[x,y]_Q := \{z \in Q \mid x \le z \le y\}$. We denote $[x,y]_Q$ by $[x,y]$ if there is no fear of confusion. Let ∞ be a new element which is not contained in Q. We define a new poset Q^+ as follows. The base set of Q^+ is $Q \cup \{\infty\}$ and $x < y$ in $Q^+ \iff x < y$ in Q or $x \in Q$ and $y = \infty$.

Definition 2.1. Let Q be an arbitrary poset and let x and y be elements of Q with $x \le y$. A saturated chain from x to y is a sequence of elements z_0, $z_1, \dots, z_t$ of Q such that

$$x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_t = y.$$

Note that the length of the chain z_0, $z_1, \dots, z_t$ is t.

Definition 2.2. Let Q, x and y be as above. We define

$$\mathrm{dist}(x,y) := \min\{t \mid \text{there is a saturated chain from } x \text{ to } y \text{ with length } t\}$$

and call $\mathrm{dist}(x,y)$ the distance of x and y. Further, for $n \in \mathbb{Z}$, we define

$$q^{(n)}\mathrm{dist}(x,y) := \max\left\{ nt \;\middle|\; \begin{array}{l}\text{there is a saturated chain from } x \text{ to } y \text{ with}\\ \text{length } t\end{array}\right\}$$

and call $q^{(n)}\mathrm{dist}(x,y)$ the n-th quasi-distance of x and y.

Note that $q^{(-1)}\mathrm{dist}(x,y) = -\mathrm{dist}(x,y)$ and $q^{(1)}\mathrm{dist}(x,y) = \mathrm{rank}([x,y])$. Note also that $q^{(n)}\mathrm{dist}(x,z)+q^{(n)}\mathrm{dist}(z,y) \le q^{(n)}\mathrm{dist}(x,y)$ for any x, z, y with $x \le z \le y$. Further, $q^{(n)}\mathrm{dist}(x,y) = n$ if $x \lessdot y$, $q^{(n)}\mathrm{dist}(x,x) = 0$ and $q^{(mn)}\mathrm{dist}(x,y) = mq^{(n)}\mathrm{dist}(x,y)$ for any positive integer m.

Definition 2.3. For a poset Q and $n \in \mathbb{Z}$, we set

$$\mathcal{T}^{(n)}(Q) := \{\nu\colon Q^+ \to \mathbb{Z} \mid \nu(\infty) = 0,\, \nu(x) - \nu(y) \ge n \text{ if } x \lessdot y \text{ in } Q^+\}.$$

Note that if x is a maximal element of Q and $\nu \in \mathcal{T}^{(n)}(Q)$, then $\nu(x) \ge n$ since $x \lessdot \infty$ in Q^+. Note also that if $\nu \in \mathcal{T}^{(n)}(Q)$, x, $y \in Q^+$ and $x \le y$, then $\nu(x) - \nu(y) \ge q^{(n)}\mathrm{dist}(x,y)$.

In the following, we identify a map $\nu\colon Q^+ \to \mathbb{R}$ with $\nu(\infty) = 0$ with the restriction $\nu\,|_Q\colon Q \to \mathbb{R}$.

Next we define operations of maps from a set to $\mathbb{Z}$.

Definition 2.4. Let X be a set. For ν, $\nu' \in \mathbb{Z}^X$ and a positive integer n, we define maps $\nu \pm \nu'$, $\max\{\nu, \nu'\}$, $\min\{\nu, \nu'\}$, $n\nu$ and $\lfloor \nu/n \rfloor \in \mathbb{Z}^X$ by $(\nu \pm \nu')(x) = \nu(x) \pm \nu'(x)$, $\max\{\nu, \nu'\}(x) = \max\{\nu(x), \nu'(x)\}$, $\min\{\nu, \nu'\}(x) = \min\{\nu(x), \nu'(x)\}$, $(n\nu)(x) = n\nu(x)$ and $\lfloor \nu/n \rfloor(x) = \lfloor \nu(x)/n \rfloor$ for $x \in X$, where $\lfloor \nu(x)/n \rfloor$ is the largest integer not exceeding $\nu(x)/n$.

We note the following fact which is easily proved.

LEMMA 2.5. *Let m, ℓ be integers and n an integer greater than 1. Suppose that ν_1, $\nu_1' \in \mathcal{T}^{(m)}(Q)$, $\nu_2 \in \mathcal{T}^{(\ell)}(Q)$, $\nu \in \mathcal{T}^{(n)}(Q)$ and $\nu' \in \mathcal{T}^{(-n)}(Q)$. Then it holds that $\nu_1 + \nu_2 \in \mathcal{T}^{(m+\ell)}(Q)$, $\max\{\nu_1, \nu_1'\}$, $\min\{\nu_1, \nu_1'\} \in \mathcal{T}^{(m)}(Q)$, $n\nu_1 \in \mathcal{T}^{(nm)}(Q)$, $\lfloor \nu/n \rfloor \in \mathcal{T}^{(1)}(Q)$, $\nu - \lfloor \nu/n \rfloor \in \mathcal{T}^{(n-1)}(Q)$. $\lfloor \nu'/n \rfloor \in \mathcal{T}^{(-1)}(Q)$ and $\nu' - \lfloor \nu'/n \rfloor \in \mathcal{T}^{(-n+1)}(Q)$.*

Here we note the following fact. Let R be a Noetherian normal domain and I a fractional ideal. I is said to be divisorial if $R :_{Q(R)} (R :_{Q(R)} I) = I$, i.e., I is reflexive as an R-module, where $Q(R)$ is the fraction field of R. It is known that the set of divisorial ideals form a group, denoted by $\mathrm{Div}(R)$, by the operation $I \cdot J := R :_{Q(R)} (R :_{Q(R)} IJ)$. We denote the n-th power of I in this group by $I^{(n)}$, where $n \in \mathbb{Z}$. Note that if $I \subsetneq R$, then $I^{(n)}$ is identical with the n-th symbolic power of I. Note also that the inverse element of I in $\mathrm{Div}(R)$ is $R :_{Q(R)} I$.

Suppose further that R is a standard graded ring over a field $\mathbb{K}$ (resp. affine semigroup ring generated by Laurent monomials in the Laurent polynomial ring $\mathbb{K}[X_1^{\pm 1}, \ldots, X_s^{\pm 1}]$ over a field $\mathbb{K}$ with weight so that R is a standard graded ring, where $X_1, \ldots, X_s$ are indeterminates). Let I be a divisorial ideal generated by homogeneous elements (resp. Laurent monomials) $m_1, \ldots, m_\ell$. Then $R :_{Q(R)} I = \bigcap_{i=1}^{\ell} Rm_i^{-1}$. Thus, $R :_{Q(R)} I$ is an R-submodule of $S^{-1}R$ generated by homogeneous elements, where $S = \{x \in R \mid x \neq 0,\ x$ is a homogeneous element$\}$ (resp. of $\mathbb{K}[X_1^{\pm 1}, \ldots, X_s^{\pm 1}]$ generated by Laurent monomials). Therefore, the set of divisorial ideals generated by homogeneous elements in $S^{-1}R$ (resp. Laurent monomials) form a subgroup of $\mathrm{Div}(R)$. It is known that the canonical module ω of R is reflexive and is isomorphic to an ideal. Therefore $\omega \in \mathrm{Div}(R)$. Thus, if ω is generated by homogeneous elements (resp. Laurent monomials), the inverse element $\omega^{(-1)}$ of ω in $\mathrm{Div}(R)$ is also an R-submodule of $S^{-1}R$ (resp. $\mathbb{K}[X_1^{\pm 1}, \ldots, X_s^{\pm 1}]$) generated by homogeneous elements in $S^{-1}R$ (resp. Laurent monomials).

Taking into account of this fact, we recall the definition of level (resp. anticanonical level) property.

DEFINITION 2.6 ([**Sta1**], [**Pag**]). Let R be a standard graded Cohen–Macaulay algebra over a field. If the degree of all the generators of the canonical module ω of R are the same, we say that R is level. Moreover, if R is normal (thus, is a domain) and the degree of all the generators of $\omega^{(-1)}$ are the same, we say that R is anticanonical level.

As is noted in [**Pag**, Example 3.4], level property does not imply anticanonical level property nor anticanonical level property does not imply level property.

Now we recall the definition of a Hibi ring. A lattice is a poset L such that for any elements α and $\beta \in L$, there are the minimum upper bound of $\{\alpha, \beta\}$, denoted by $\alpha \vee \beta$ and the maximum lower bound of $\{\alpha, \beta\}$, denoted by $\alpha \wedge \beta$. A lattice L is distributive if $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ for any α, β and $\gamma \in L$.

Let $\mathbb{K}$ be a field, H a finite distributive lattice with unique minimal element x_0, P the set of join-irreducible elements of H, i.e., $P = \{\alpha \in H \mid \alpha = \beta \vee \gamma \Rightarrow \alpha = \beta$ or $\alpha = \gamma\}$. Note that we treat x_0 as a join-irreducible element. It is known that H is isomorphic to the set of nonempty poset ideals of P ordered by inclusion.

Let $\{T_x\}_{x \in P}$ be a family of indeterminates indexed by P.

Definition 2.7 ([**Hib**]). We set

$$\mathcal{R}_{\mathbb{K}}[H] := \mathbb{K}\left[\prod_{x \in I} T_x \,\middle|\, I \text{ is a nonempty poset ideal of } P\right].$$

It is easily verified that if we set $\alpha = \bigvee_{x \in I} x$ for a nonempty poset ideal I, then $I = \{x \in P \mid x \leq \alpha \text{ in } H\}$. Further, for $\alpha \in H$, $\{x \in P \mid x \leq \alpha\}$ is a nonempty poset ideal of P. Thus, $\mathcal{R}_{\mathbb{K}}[H] = \mathbb{K}[\prod_{x \leq \alpha} T_x \mid \alpha \in H]$.

$\mathcal{R}_{\mathbb{K}}[H]$ is called the Hibi ring over $\mathbb{K}$ on H nowadays. Hibi [**Hib**, Section 2 b)] showed that $\mathcal{R}_{\mathbb{K}}[H]$ is a normal affine semigroup ring and thus is Cohen–Macaulay by the result of Hochster [**Hoc**]. Further, he showed [**Hib**, Section 3 d)] that $\mathcal{R}_{\mathbb{K}}[H]$ is Gorenstein if and only if P is pure. Moreover, by setting $\deg T_{x_0} = 1$ and $\deg T_x = 0$ for $x \in P \setminus \{x_0\}$, $\mathcal{R}_{\mathbb{K}}[H]$ is a standard graded $\mathbb{K}$-algebra. We denote the graded canonical module of $\mathcal{R}_{\mathbb{K}}[H]$ by ω.

For $\nu\colon P \to \mathbb{Z}$, we denote the Laurent monomial $\prod_{x \in P} T_x^{\nu(x)}$ by T^ν. Note that $\deg T^\nu = \nu(x_0)$. It is shown by Hibi [**Hib**] and is easily verified that

$$\mathcal{R}_{\mathbb{K}}[H] = \bigoplus_{\nu \in \mathcal{T}^{(0)}(P)} \mathbb{K} T^\nu$$

and therefore by the description of the canonical module of a normal affine semigroup ring by Stanley [**Sta2**, p. 82], we see that

$$\omega = \bigoplus_{\nu \in \mathcal{T}^{(1)}(P)} \mathbb{K} T^\nu.$$

We call this ideal the canonical ideal of $\mathcal{R}_{\mathbb{K}}[H]$ and $\omega^{(-1)}$ the anticanonical ideal of $\mathcal{R}_{\mathbb{K}}[H]$.

Next we state the following.

Lemma 2.8. *Let x and y be elements of P^+ with $x \lessdot y$ and $n \in \mathbb{Z}$. Then there exists $\nu \in \mathcal{T}^{(n)}(P)$ such that $\nu(x) - \nu(y) = n$.*

Proof. For $z \in P^+$, set

$$\nu(z) = \begin{cases} q^{(n)}\mathrm{dist}(z, \infty) & \text{if } z \not\leq y, \\ \max\{q^{(n)}\mathrm{dist}(z, \infty), q^{(n)}\mathrm{dist}(x, \infty) - n + q^{(n)}\mathrm{dist}(z, y)\} & \text{if } z \leq y. \end{cases}$$

Then it is easily verified that ν satisfies the required condition. □

Now we state the following.

THEOREM 2.9. *For a positive integer n,*

$$\omega^n = \omega^{(n)} = \bigoplus_{\nu \in \mathcal{T}^{(n)}(P)} \mathbb{K}T^\nu,$$
$$(\omega^{(-1)})^n = \omega^{(-n)} = \bigoplus_{\nu \in \mathcal{T}^{(-n)}(P)} \mathbb{K}T^\nu.$$

PROOF. Let ν be an arbitrary element of $\mathcal{T}^{(-n)}(P)$ and let $\nu_1, \dots, \nu_n$ be arbitrary elements in $\mathcal{T}^{(1)}(P)$. Then $\nu + \nu_1 + \cdots + \nu_n \in \mathcal{T}^{(0)}(P)$. Therefore,

$$\bigoplus_{\nu \in \mathcal{T}^{(-n)}(P)} \mathbb{K}T^\nu \subset (\mathcal{R}_{\mathbb{K}}[H] \colon \omega^n) = \omega^{(-n)}.$$

In order to prove the reverse inclusion, first note that $\omega^{(-n)}$ is a $\mathbb{Z}^{\#P}$-graded $\mathcal{R}_{\mathbb{K}}[H]$-submodule of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in P]$ and therefore a $\mathbb{K}$-vector subspace of $\mathbb{K}[T_x^{\pm 1} \mid x \in P]$ which has a basis consisting of Laurent monomials.

Let T^ν, $\nu \colon P \to \mathbb{Z}$, be an arbitrary Laurent monomial in $\omega^{(-n)}$. We extend ν to a map from P^+ to $\mathbb{Z}$ by setting $\nu(\infty) = 0$. Let x and y be arbitrary elements of P^+ with $x \lessdot y$. Then by Lemma 2.8, we see that there is $\nu' \in \mathcal{T}^{(1)}(P)$ such that $\nu'(x) - \nu'(y) = 1$. Since $(T^{\nu'})^n \in \omega^n$, we see that

$$T^{\nu + n\nu'} = T^\nu (T^{\nu'})^n \in \mathcal{R}_{\mathbb{K}}[H].$$

Thus $(\nu + n\nu')(x) - (\nu + n\nu')(y) \geq 0$ and we see that $\nu(x) - \nu(y) \geq -n$.

Since x and y are arbitrary, we see that $\nu \in \mathcal{T}^{(-n)}(P)$. Thus we see that

$$\omega^{(-n)} \subset \bigoplus_{\nu \in \mathcal{T}^{(-n)}(P)} \mathbb{K}T^\nu$$

and therefore

$$\omega^{(-n)} = \bigoplus_{\nu \in \mathcal{T}^{(-n)}(P)} \mathbb{K}T^\nu.$$

From this fact, we can show that

$$\omega^{(n)} = (\mathcal{R}_{\mathbb{K}}[H] \colon \omega^{(-n)}) = \bigoplus_{\nu \in \mathcal{T}^{(n)}(P)} \mathbb{K}T^\nu$$

by a similar way.

Next assume that ν is an arbitrary element of $\mathcal{T}^{(n)}(P)$. By using Lemma 2.5 repeatedly, we see that there are $\nu_1, \dots, \nu_n \in \mathcal{T}^{(1)}(P)$ such that $\nu = \nu_1 + \cdots + \nu_n$. Therefore, $T^\nu \in \omega^n$. Since ν is an arbitrary element of $\mathcal{T}^{(n)}(P)$, we see that

$$\omega^{(n)} = \bigoplus_{\nu \in \mathcal{T}^{(n)}(P)} \mathbb{K}T^\nu \subset \omega^n.$$

Thus, we see that $\omega^{(n)} = \omega^n$, since the reverse inclusion holds in general. We see that $(\omega^{(-1)})^n = \omega^{(-n)}$ by the same way. $\square$

Remark 2.10. By Theorem 2.9, we see that symbolic Rees algebras $R = \bigoplus_{n\geq 0}\omega^{(n)}$ and $R' = \bigoplus_{n\geq 0}\omega^{(-n)}$ are ordinary Rees algebras and therefore Noetherian. Thus, by applying the result of Goto et al. [**GHNV**, Theorems (4.5) and (4.8)] to ω and $\omega^{(-1)}$, we see that R' is Gorenstein and the canonical module of R is isomorphic to $\omega^{(2)}R$.

In our case, we can describe the canonical modules of these rings explicitly. Let X be a new indeterminate and we embed the above rings in the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in P][X^{\pm 1}]$ by identifying R with

$$\bigoplus_{n\geq 0}\omega^{(n)}X^n = \bigoplus_{\substack{n\geq 0\\ \nu\in\mathcal{T}^{(n)}(P)}} \mathbb{K}T^\nu X^n$$

and R' with

$$\bigoplus_{n\geq 0}\omega^{(-n)}X^{-n} = \bigoplus_{\substack{n\geq 0\\ \nu\in\mathcal{T}^{(-n)}(P)}} \mathbb{K}T^\nu X^{-n}.$$

Then we see that these rings are normal by Hochster's criterion [**Hoc**] and therefore Cohen–Macaulay. Further, by Stanley's description of the canonical module of a normal affine semigroup ring, we see that the canonical module of R (resp. R') is $\bigoplus_{\substack{n>0\\ \nu\in\mathcal{T}^{(n+1)}(P)}} \mathbb{K}T^\nu X^n$ (resp. $\bigoplus_{\substack{n>0\\ \nu\in\mathcal{T}^{(-n+1)}(P)}} \mathbb{K}T^\nu X^{-n}$). Thus, we see by Theorem 2.9 that the canonical module of R' is generated by X^{-1}. Further, we see that the canonical module of R is generated by $\{T^\nu X \mid \nu \in \mathcal{T}^{(2)}(P)\}$.

3. Generators of $\omega^{(n)}$ and $q^{(n)}$-reduced N-sequence.

In this section, we state a characterization of a Laurent monomial T^ν to be a generator of $\omega^{(n)}$, where $n \in \mathbb{Z}$.

First, we introduce an order on $\mathcal{T}^{(n)}(P)$ and describe generators of $\omega^{(n)}$ with it, where $n \in \mathbb{Z}$. Since $\omega^{(n)}$ is a finitely generated $\mathbb{Z}^{\#P}$-graded $\mathcal{R}_{\mathbb{K}}[H]$-submodule of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in P]$, there is a unique minimal set of Laurent monomials which generate $\omega^{(n)}$ as an $\mathcal{R}_{\mathbb{K}}[H]$-module. We call an element of this set a generator of $\omega^{(n)}$. By Theorem 2.9, we see that for $\nu \in \mathcal{T}^{(n)}(P)$, T^ν is a generator of $\omega^{(n)}$ if and only if there are no $\nu_1 \in \mathcal{T}^{(n)}(P)$ and $\nu_2 \in \mathcal{T}^{(0)}(P)$ such that $\nu \neq \nu_1$ and $\nu = \nu_1 + \nu_2$. On account of this fact, we make the following.

Definition 3.1. Let $n \in \mathbb{Z}$ and ν, $\nu' \in \mathcal{T}^{(n)}(P)$. We define the relation $\leq$ on $\mathcal{T}^{(n)}(P)$ by

$$\nu \leq \nu' \iff \nu' - \nu \in \mathcal{T}^{(0)}(P).$$

It is easily verified that $\leq$ is an order relation on $\mathcal{T}^{(n)}(P)$. Further, by the above argument, we see that for $\nu \in \mathcal{T}^{(n)}(P)$, T^ν is a generator of $\omega^{(n)}$ if and only if ν is a minimal element of $\mathcal{T}^{(n)}(P)$.

In the rest of this section, we fix $n \in \mathbb{Z}$. First we state the following (cf. [**Miy2**, Definition 3.1]).

Definition 3.2. We say a (possibly empty) sequence $(y_0, x_1, y_1, x_2, \ldots, y_{t-1}, x_t)$ of elements $P \setminus \{x_0\}$ satisfies Condition N if

(1) $y_0 > x_1 < y_1 > x_2 < \cdots < y_{t-1} > x_t$ and

(2) $y_i \not\geq x_j$ if $i \leq j-2$.

We also say that the sequence $(y_0, x_1, \ldots, y_{t-1}, x_t)$ is an N-sequence.

When considering an N-sequence, we add x_0 at the beginning and $y_t = \infty$ at the end of the sequence and consider the sequence $(x_0, y_0, \ldots, x_t, y_t)$. In particular, when $t = 0$, we consider the sequence (x_0, ∞).

In order to simplify description, we set the following.

Notation. Let w_0, z_0, w_1, $z_1, \ldots, w_\ell$, z_ℓ be elements of P^+ with $w_0 < z_0 > w_1 < z_1 > \cdots > w_\ell < z_\ell$. We set

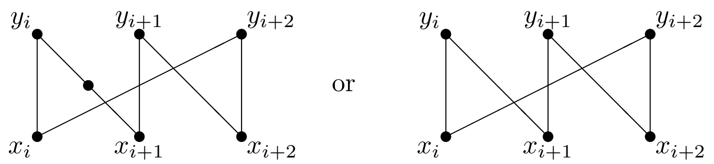
$$q^{(n)}(w_0, z_0, w_1, z_1, \ldots, w_\ell, z_\ell) := \sum_{i=0}^{\ell} q^{(n)}\mathrm{dist}(w_i, z_i) - \sum_{i=1}^{\ell} q^{(n)}\mathrm{dist}(w_i, z_{i-1}).$$

Next we define the following property of an N-sequence.

Definition 3.3. Let m be an integer and $(y_0, x_1, \ldots, y_{t-1}, x_t)$ an N-sequence. Set $y_t = \infty$. If for any $0 \leq i < j \leq t$ with $x_i \leq y_j$, $q^{(m)}\mathrm{dist}(x_i, y_j) < q^{(m)}(x_i, y_i, \ldots, x_j, y_j)$, we say that $(y_0, x_1, \ldots, y_{t-1}, x_t)$ is $q^{(m)}$-reduced. We treat the empty sequence as a $q^{(m)}$-reduced N-sequence.

Note that an N-sequence $(y_0, x_1, \ldots, y_{t-1}, x_t)$ is $q^{(1)}$-reduced (resp. $q^{(-1)}$-reduced) if and only if it is $q^{(m)}$-reduced (resp. $q^{(-m)}$-reduced) for any $m > 0$.

Example 3.4. If there is a part of the sequence of the following form



then it is not $q^{(1)}$-reduced. Later, we seek a sequence $(y_0, x_1, \ldots, y_{t-1}, x_t)$ with Condition N such that $q^{(\epsilon)}(x_0, y_0, \ldots, x_t, \infty)$ as large as possible, where $\epsilon = 1$ or -1. If there is a part of the first kind in the N-sequence, we can replace it with $(y_0, x_1, \ldots, x_i, y_{i+2}, x_{i+2}, \ldots, x_t)$ and obtain a sequence with larger $q^{(1)}(x_0, y_0, \ldots)$. Further, if there is a part of the second kind in the N-sequence, we apply the replacement above and remove redundancy. In fact, there is no redundancy of this kind is key to Lemma 3.9 and Section 4.

Now we begin to analyze the property of generating system of $\omega^{(n)}$. First we state the following (cf. [**Miy2**, Lemma 3.2]).

Lemma 3.5. *Let ν be an element of $\mathcal{T}^{(n)}(P)$. If there is a possibly empty sequence $(z_0, w_1, \ldots, z_{\ell-1}, w_\ell)$ of elements of $P \setminus \{x_0\}$ such that $z_0 > w_1 < \cdots < z_{\ell-1} > w_\ell$ and $\nu(w_i) - \nu(z_i) = q^{(n)}\mathrm{dist}(w_i, z_i)$ for any $0 \leq i \leq \ell$, where we set $w_0 = x_0$ and $z_\ell = \infty$, then ν is a minimal element of $\mathcal{T}^{(n)}(P)$.*

The proof is almost identical with that of [**Miy2**, Lemma 3.2]. Thus, we omit it.

Next we state a strong converse of this lemma.

Lemma 3.6. *Let ν be a minimal element of $\mathcal{T}^{(n)}(P)$. Then there is a possibly empty $q^{(n)}$-reduced N-sequence $(y_0, x_1, \ldots, y_{t-1}, x_t)$ such that*

$$\nu(x_i) - \nu(y_i) = q^{(n)}\mathrm{dist}(x_i, y_i) \quad \textit{for} \;\; 0 \leq i \leq t, \tag{3.1}$$

where we set $y_t = \infty$.

Since the proof is almost identical with [**Miy2**, Lemma 3.3], we omit it. Note that the N-sequence we constructed in the proof of [**Miy2**, Lemma 3.3] is $q^{(1)}$-reduced. By Lemmas 3.5 and 3.6, we see that $\nu \in \mathcal{T}^{(n)}(P)$ is a minimal element of $\mathcal{T}^{(n)}(P)$ if and only if there exists an N-sequence which satisfies equations (3.1).

Noting that there are only finitely many N-sequences, we make the following.

Definition 3.7. We set $q_0^{(n)} := q^{(n)}\mathrm{dist}(x_0, \infty)$ and

$$q_{\max}^{(n)} := \max\left\{ q^{(n)}(x_0, y_0, \ldots, x_t, \infty) \;\middle|\; \begin{array}{l}(y_0, x_1, \ldots, y_{t-1}, x_t) \text{ is a } q^{(n)}\text{-reduced} \\ \text{N-sequence}\end{array} \right\}.$$

By the fact that $\nu(x_0) = \nu(x_0) - \nu(\infty) \geq q^{(n)}\mathrm{dist}(x_0, \infty)$, for $\nu \in \mathcal{T}^{(n)}(P)$, we see that $q_0^{(n)} \leq \nu(x_0)$ for any element ν of $\mathcal{T}^{(n)}(P)$. Further, by the same way as the proof of [**Miy2**, Corollary 3.5], we see that $\nu(x_0) \leq q_{\max}^{(n)}$ for any minimal element ν of $\mathcal{T}^{(n)}(P)$ by Lemma 3.6. Thus, $q_0^{(n)} \leq d \leq q_{\max}^{(n)}$ is a necessary condition that there is a generator of $\omega^{(n)}$ with degree d, since $\deg T^\nu = \nu(x_0)$.

We show that this is also a sufficient condition in the rest of this section.

Definition 3.8. Let $(y_0, x_1, \ldots, y_{t-1}, x_t)$ be a $q^{(n)}$-reduced N-sequence. Set $y_t = \infty$. We define

$$\mu_{(y_0, x_1, \ldots, y_{t-1}, x_t)}(x_i) := q^{(n)}(x_i, y_i, \ldots, x_t, \infty)$$

and

$$\mu_{(y_0, x_1, \ldots, y_{t-1}, x_t)}(y_i) := q^{(n)}(x_i, y_i, \ldots, x_t, \infty) - q^{(n)}\mathrm{dist}(x_i, y_i)$$

for $0 \le i \le t$. We also define

$$\nu^{\downarrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}(z) := \max\{\mu_{(y_0,x_1,\ldots,y_{t-1},x_t)}(y_j) + q^{(n)}\mathrm{dist}(z,y_j) \mid y_j \ge z\}$$

and

$$\nu^{\uparrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}(z) := \min\{\mu_{(y_0,x_1,\ldots,y_{t-1},x_t)}(x_i) - q^{(n)}\mathrm{dist}(x_i,z) \mid x_i \le z\}$$

for $z \in P^+$.

Note that the definition of $\nu^{\uparrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ is different from that of [**Miy2**, Definition 3.6]. Here we define $\nu^{\downarrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ and $\nu^{\uparrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ for $q^{(n)}$-reduced N-sequences only. Next we state basic properties of $\nu^{\downarrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ and $\nu^{\uparrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$.

Lemma 3.9. *Let $(y_0,x_1,\ldots,y_{t-1},x_t)$ be a $q^{(n)}$-reduced N-sequence. We denote $\mu_{(y_0,x_1,\ldots,y_{t-1},x_t)}$, $\nu^{\downarrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ and $\nu^{\uparrow}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ by μ, $\nu^{\downarrow}$ and $\nu^{\uparrow}$ respectively. Then $\nu^{\downarrow}$, $\nu^{\uparrow}$ are minimal elements of $\mathcal{T}^{(n)}(P)$, $\nu^{\downarrow}(x_i) = \nu^{\uparrow}(x_i) = \mu(x_i)$ and $\nu^{\downarrow}(y_i) = \nu^{\uparrow}(y_i) = \mu(y_i)$ for $0 \le i \le t$.*

Proof. Suppose z, $z' \in P^+$ and $z \lessdot z'$. Then it is easily verified that

$$\nu^{\downarrow}(z) - \nu^{\downarrow}(z') \ge n \quad \text{and} \quad \nu^{\uparrow}(z) - \nu^{\uparrow}(z') \ge n. \tag{3.2}$$

Next we show that $\nu^{\downarrow}(x_i) = \mu(x_i)$ for $0 \le i \le t$. Take j with $y_j \ge x_i$ and $\nu^{\downarrow}(x_i) = q^{(n)}\mathrm{dist}(x_i,y_j) + \mu(y_j)$. Then $j \ge i-1$ since $(y_0,x_1,\ldots,y_{t-1},x_t)$ satisfies Condition N. If $j > i$, then since $(y_0,x_1,\ldots,y_{t-1},x_t)$ is $q^{(n)}$-reduced, we see that

$$\begin{aligned}\nu^{\downarrow}(x_i) &= q^{(n)}\mathrm{dist}(x_i,y_j) + \mu(y_j)\\ &< q^{(n)}(x_i,y_i,\ldots,x_j,y_j) + \mu(y_j)\\ &= \mu(x_i)\\ &= q^{(n)}\mathrm{dist}(x_i,y_i) + \mu(y_i).\end{aligned} \tag{3.3}$$

This contradicts the definition of $\nu^{\downarrow}$. Thus, $j = i$ or $i-1$. Since $q^{(n)}\mathrm{dist}(x_i,y_i)+\mu(y_i) = \mu(x_i)$ and $q^{(n)}\mathrm{dist}(x_i,y_{i-1}) + \mu(y_{i-1}) = \mu(x_i)$ if $i > 0$, we see that $\nu^{\downarrow}(x_i) = \mu(x_i)$ by the definition of $\nu^{\downarrow}$. We also see that $\nu^{\uparrow}(y_i) = \mu(y_i)$ for $0 \le i \le t$ by the same way.

Next we show that $\nu^{\uparrow}(x_i) = \mu(x_i)$ for $0 \le i \le t$. Take j with $x_j \le x_i$ and $\nu^{\uparrow}(x_i) = \mu(x_j) - q^{(n)}\mathrm{dist}(x_j,x_i)$. Since $x_j \le x_i < y_{i-1}$ and $(y_0,x_1,\ldots,y_{t-1},x_t)$ satisfies Condition N, we see that $j \le i$. If $j < i$, then, since $(y_0,x_1,\ldots,y_{t-1},x_t)$ is $q^{(n)}$-reduced, we see that

$$\begin{aligned}\nu^{\uparrow}(x_i) &= \mu(x_j) - q^{(n)}\mathrm{dist}(x_j,x_i)\\ &= \mu(x_j) - q^{(n)}\mathrm{dist}(x_j,x_i) - q^{(n)}\mathrm{dist}(x_i,y_i) + q^{(n)}\mathrm{dist}(x_i,y_i)\\ &\ge \mu(x_j) - q^{(n)}\mathrm{dist}(x_j,y_i) + q^{(n)}\mathrm{dist}(x_i,y_i)\\ &> \mu(x_j) - q^{(n)}(x_j,y_j,\ldots,x_i,y_i) + q^{(n)}\mathrm{dist}(x_i,y_i)\end{aligned}$$

$$
\begin{aligned}
&= \mu(y_i) + q^{(n)}\mathrm{dist}(x_i, y_i) \\
&= \mu(x_i) \\
&= \mu(x_i) - q^{(n)}\mathrm{dist}(x_i, x_i). \qquad (3.4)
\end{aligned}
$$

This contradicts the definition of $\nu^{\uparrow}$. Thus we see that $j = i$ and

$$\nu^{\uparrow}(x_i) = \mu(x_i) - q^{(n)}\mathrm{dist}(x_i, x_i) = \mu(x_i).$$

We also see that $\nu^{\downarrow}(y_i) = \mu(y_i)$ for $0 \le i \le t$ by the same way. In particular, $\nu^{\downarrow}(\infty) = \nu^{\uparrow}(\infty) = \mu(\infty) = 0$. Therefore, we see that $\nu^{\downarrow}$, $\nu^{\uparrow} \in \mathcal{T}^{(n)}(P)$ by inequalities (3.2). By Lemma 3.5, we see that $\nu^{\downarrow}$ and $\nu^{\uparrow}$ are minimal elements of $\mathcal{T}^{(n)}(P)$. $\square$

Next we note the following.

Lemma 3.10. *Let ν be a minimal element of $\mathcal{T}^{(n)}(P)$ and k a positive integer. Set*

$$\nu'(z) = \max\{\nu(z) - k, q^{(n)}\mathrm{dist}(z, \infty)\}$$

for $z \in P^{+}$. Then ν' is also a minimal element of $\mathcal{T}^{(n)}(P)$.

Proof. It is easily verified that $\nu' \in \mathcal{T}^{(n)}(P)$. The rest is proved along the same line with [**Miy2**, Lemma 3.11]. $\square$

Now we state the following.

Theorem 3.11. *There exists a generator of $\omega^{(n)}$ with degree d if and only if $q_0^{(n)} \le d \le q_{\max}^{(n)}$.*

Proof. "Only if" part is already proved after Definition 3.7.

Let d be an integer with $q_0^{(n)} \le d \le q_{\max}^{(n)}$. Set $k = q_{\max}^{(n)} - d$ and take a $q^{(n)}$-reduced N-sequence $(y_0, x_1, \dots, y_{t-1}, x_t)$ such that $q^{(n)}(x_0, y_0, \dots, x_t, \infty) = q_{\max}^{(n)}$. Then $\nu^{\downarrow}_{(y_0, x_1, \dots, y_{t-1}, x_t)}$ is a minimal element of $\mathcal{T}^{(n)}(P)$ with

$$\nu^{\downarrow}_{(y_0, x_1, \dots, y_{t-1}, x_t)}(x_0) = \mu_{(y_0, x_1, \dots, y_{t-1}, x_t)}(x_0) = q^{(n)}(x_0, y_0, \dots, x_t, \infty) = q_{\max}^{(n)}$$

by Lemma 3.9. Thus,

$$\nu' \colon P^{+} \to \mathbb{Z},\ z \mapsto \max\{\nu^{\downarrow}_{(y_0, x_1, \dots, y_{t-1}, x_t)}(z) - k, q^{(n)}\mathrm{dist}(z, \infty)\}$$

is a minimal element of $\mathcal{T}^{(n)}(P)$ by Lemma 3.10. Since $\nu'(x_0) = q_{\max}^{(n)} - k = d$, we see that $T^{\nu'}$ is a generator of $\omega^{(n)}$ with degree d. $\square$

For any nonempty $q^{(n)}$-reduced N-sequence $(y_0, x_1, \dots, y_{t-1}, x_t)$, $q_0^{(n)} = q^{(n)}\mathrm{dist}(x_0, \infty) < q^{(n)}(x_0, y_0, \dots, x_t, \infty) \le q_{\max}^{(n)}$. Therefore, we obtain the following result from Theorem 3.11.

THEOREM 3.12. *$\mathcal{R}_{\mathbb{K}}[H]$ is level (resp. anticanonical level) if and only if $q^{(1)}$-reduced (resp. $q^{(-1)}$-reduced) N-sequence is the empty sequence only.*

Note that for level case, Theorem 3.12 is another expression of [**Miy2**, Theorem 3.9] using the notion of a $q^{(1)}$-reduced N-sequence.

As a corollary, we see that the anticanonical counterpart of [**Miy1**, Theorem 3.3] (see also [**Miy2**, Corollary 3.10]) also holds.

COROLLARY 3.13. *If $\{z \in P \mid z \geq w\}$ is pure for any $w \in P \setminus \{x_0\}$, then $\mathcal{R}_{\mathbb{K}}[H]$ is level and anticanonical level.*

PROOF. Suppose that there exists a $q^{(-1)}$-reduced N-sequence $(y_0, x_1, \dots, y_{t-1}, x_t)$ with $t > 0$. Set $y_t := \infty$. Then

$$\begin{aligned}
&q^{(-1)}(x_0, y_0, \dots, x_t, y_t) \\
&= \sum_{i=0}^{t} q^{(-1)}\mathrm{dist}(x_i, y_i) - \sum_{i=1}^{t} q^{(-1)}\mathrm{dist}(x_i, y_{i-1}) \\
&= q^{(-1)}\mathrm{dist}(x_0, y_0) + \sum_{i=1}^{t} \left(q^{(-1)}\mathrm{dist}(x_i, \infty) - q^{(-1)}\mathrm{dist}(y_i, \infty)\right) \\
&\quad - \sum_{i=1}^{t} \left(q^{(-1)}\mathrm{dist}(x_i, \infty) - q^{(-1)}\mathrm{dist}(y_{i-1}, \infty)\right) \\
&= q^{(-1)}\mathrm{dist}(x_0, y_0) + q^{(-1)}\mathrm{dist}(y_0, \infty) - q^{(-1)}\mathrm{dist}(\infty, \infty) \\
&\leq q^{(-1)}\mathrm{dist}(x_0, \infty),
\end{aligned}$$

contradicting the fact that $(y_0, x_1, \dots, y_{t-1}, x_t)$ is $q^{(-1)}$-reduced. Thus, there is no $q^{(-1)}$-reduced N-sequence except the empty sequence and we see by Theorem 3.12 that $\mathcal{R}_{\mathbb{K}}[H]$ is anticanonical level.

The level property is proved by the same way. $\square$

4. Convex polytope associated to a $q^{(\epsilon)}$-reduced N-sequence.

Let ϵ be ± 1. In this section, we construct a convex polytope associated to a $q^{(\epsilon)}$-reduced N-sequence. Fix a $q^{(\epsilon)}$-reduced N-sequence $(y_0, x_1, \dots, y_{t-1}, x_t)$. We set $y_t := \infty$. We define a convex polytope from $(y_0, x_1, \dots, y_{t-1}, x_t)$ and study the Ehrhart ring defined by this polytope.

Here we establish the notation of the Ehrhart ring. Let W be a finite set and C a rational convex polytope in $\mathbb{R}^W$, i.e., a convex polytope whose vertices are in $\mathbb{Q}^W$. Also let $\mathbb{K}$ be a field, $\{X_w\}_{w \in W}$ a family of indeterminates indexed by W and Y an indeterminate. Then the Ehrhart ring $\mathbb{K}[C]$ of C over $\mathbb{K}$ in $\mathbb{K}[X_w^{\pm 1} \mid w \in W][Y]$ is the subring of $\mathbb{K}[X_w^{\pm 1} \mid w \in W][Y]$ generated by $\{\prod_{w \in W} X_w^{\nu(w)} T^n \mid n \in \mathbb{N},\ \nu \in nC \cap \mathbb{Z}^W\}$ over $\mathbb{K}$. It is known that $\dim \mathbb{K}[C] = \dim C + 1$, see e.g., [**Mat**, (14.C) Theorem 23].

Let n be a positive integer. Note that $(y_0, x_1, \dots, y_{t-1}, x_t)$ is a $q^{(n\epsilon)}$-reduced N-sequence. We set

$$C^{(n\epsilon)} := \left\{ \nu\colon P^+ \to \mathbb{R} \;\middle|\; \begin{array}{l} \nu(\infty) = 0,\ \nu(z) - \nu(z') \geq n\epsilon \text{ for any } z, \\ z' \in P^+ \text{ with } z \lessdot z' \text{ and } \nu(x_i) - \nu(y_i) = \\ q^{(n\epsilon)}\mathrm{dist}(x_i, y_i) \text{ for } 0 \leq i \leq t \end{array} \right\}.$$

For any $\nu \in C^{(n\epsilon)}$ and $z \in P$, $\nu(z) \geq q^{(n\epsilon)}\mathrm{dist}(z,\infty)$ and $\nu(z) = \nu(x_0) - (\nu(x_0) - \nu(z)) \leq q^{(n\epsilon)}_{\max} - q^{(n\epsilon)}\mathrm{dist}(x_0, z)$ by the argument after Definition 3.7. Thus, $C^{(n\epsilon)}$ is bounded, i.e., $C^{(n\epsilon)}$ is a convex polytope. Since $q^{(n\epsilon)}\mathrm{dist}(x_i, y_i) = nq^{(\epsilon)}\mathrm{dist}(x_i, y_i)$ for $0 \leq i \leq t$, we see that $C^{(n\epsilon)} = nC^{(\epsilon)}$. Further, if $n \geq 2$ and $\nu \in C^{(n\epsilon)} \cap \mathbb{Z}^P$, then it is easily verified that $\lfloor \nu/n \rfloor \in C^{(\epsilon)} \cap \mathbb{Z}^P$ and $\nu - \lfloor \nu/n \rfloor \in C^{((n-1)\epsilon)} \cap \mathbb{Z}^P$. Thus, we see by induction that any element $\nu \in C^{(n\epsilon)} \cap \mathbb{Z}^P$ can be written as a sum of n elements of $C^{(\epsilon)} \cap \mathbb{Z}^P$. Since $C^{(n\epsilon)} = nC^{(\epsilon)}$, we see that the Ehrhart ring defined by $C^{(\epsilon)}$ is a standard graded ring, i.e., generated by the degree 1 part over the base field. In particular, $C^{(\epsilon)}$ is integral, i.e., all the vertices of $C^{(\epsilon)}$ are contained in $\mathbb{Z}^P$.

Next we consider the dimension of $C^{(\epsilon)}$. Set

$$G_i = \{ z \in [x_i, y_i]_{P^+} \mid q^{(\epsilon)}\mathrm{dist}(x_i, z) + q^{(\epsilon)}\mathrm{dist}(z, y_i) = q^{(\epsilon)}\mathrm{dist}(x_i, y_i) \}$$

for $0 \leq i \leq t$ and $G = G_0 \cup \cdots \cup G_t$. Note that x_i, $y_i \in G_i$ for $0 \leq i \leq t$. Further for any $\nu \in C^{(\epsilon)}$ and $z \in G_i$,

$$\nu(z) = \nu(y_i) + q^{(\epsilon)}\mathrm{dist}(z, y_i), \tag{4.1}$$

since

$$\begin{aligned} \nu(z) - \nu(y_i) &\geq q^{(\epsilon)}\mathrm{dist}(z, y_i), \\ \nu(x_i) - \nu(z) &\geq q^{(\epsilon)}\mathrm{dist}(x_i, z), \\ q^{(\epsilon)}\mathrm{dist}(x_i, z) + q^{(\epsilon)}\mathrm{dist}(z, y_i) &= q^{(\epsilon)}\mathrm{dist}(x_i, y_i) \text{ and} \\ \nu(x_i) - \nu(y_i) &= q^{(\epsilon)}\mathrm{dist}(x_i, y_i). \end{aligned}$$

Therefore,

$$\dim C^{(\epsilon)} \leq \#(P \setminus G) + t. \tag{4.2}$$

We show the reverse inequality by showing that there are affinely independent elements of $C^{(\epsilon)}$ consisting of $\#(P \setminus G) + t + 1$ elements. First we make the following.

Definition 4.1. Let s be an integer with $0 \leq s \leq t$. We set $\mu = \mu_{(y_0, x_1, \ldots, y_{t-1}, x_t)}$,

$$\mu_s(x_i) := \begin{cases} \mu(x_i) & \text{if } i \geq s \\ \mu(x_i) - 1 & \text{if } i < s \end{cases}$$

and

$$\mu_s(y_i) := \begin{cases} \mu(y_i) & \text{if } i \geq s \\ \mu(y_i) - 1 & \text{if } i < s. \end{cases}$$

Further, we define maps $\nu_s^{\downarrow}$ and $\nu_s^{\uparrow}$ from P^+ to $\mathbb{Z}$ by

$$\nu_s^{\downarrow}(z) := \max\{q^{(\epsilon)}\mathrm{dist}(z, y_i) + \mu_s(y_i) \mid y_i \geq z\}$$

and

$$\nu_s^{\uparrow}(z) := \min\{\mu_s(x_i) - q^{(\epsilon)}\mathrm{dist}(x_i, z) \mid x_i \leq z\}.$$

Note that it is easily verified that $\nu_s^{\downarrow}(z) - \nu_s^{\downarrow}(z') \geq \epsilon$ and $\nu_s^{\uparrow}(z) - \nu_s^{\uparrow}(z') \geq \epsilon$ for any z, $z' \in P^+$ with $z \lessdot z'$. Further, since the inequalities (3.3) and (3.4) are strict, we can show that

$$\nu_s^{\downarrow}(x_i) = \nu_s^{\uparrow}(x_i) = \mu_s(x_i) \tag{4.3}$$

and

$$\nu_s^{\downarrow}(y_i) = \nu_s^{\uparrow}(y_i) = \mu_s(y_i) \tag{4.4}$$

for any i with $0 \leq i \leq t$ by the same way as the proof of Lemma 3.9. In particular, $\nu_s^{\downarrow}(\infty) = \nu_s^{\uparrow}(\infty) = 0$. Therefore, we see that $\nu_s^{\downarrow}$, $\nu_s^{\uparrow} \in \mathcal{T}^{(\epsilon)}(P)$ for any s with $0 \leq s \leq t$.

Since $\nu_s^{\downarrow}(x_i) = \nu_s^{\uparrow}(x_i) = \mu_s(x_i)$ and $\nu_s^{\downarrow}(y_i) = \nu_s^{\uparrow}(y_i) = \mu_s(y_i)$, we see that $\nu_s^{\downarrow}(x_i) - \nu_s^{\downarrow}(y_i) = q^{(\epsilon)}\mathrm{dist}(x_i, y_i)$ and $\nu_s^{\uparrow}(x_i) - \nu_s^{\uparrow}(y_i) = q^{(\epsilon)}\mathrm{dist}(x_i, y_i)$ for any $0 \leq i \leq t$. Therefore, $\nu_s^{\downarrow}$, $\nu_s^{\uparrow} \in C^{(\epsilon)}$ for any $0 \leq s \leq t$. Note that $\mu_0 = \mu$.

Next we state the following.

LEMMA 4.2. *Let s be an integer with $0 \leq s \leq t$. Then $\nu_s^{\downarrow}(z) \leq \nu_s^{\uparrow}(z)$ for any $z \in P^+$.*

PROOF. Take i and j such that $x_i \leq z$, $\nu_s^{\uparrow}(z) = \mu_s(x_i) - q^{(\epsilon)}\mathrm{dist}(x_i, z)$ and $y_j \geq z$, $\nu_s^{\downarrow}(z) = \mu_s(y_j) + q^{(\epsilon)}\mathrm{dist}(z, y_j)$. Then $x_i \leq y_j$. Therefore $j \geq i-1$ since $(y_0, x_1, \ldots, y_{t-1}, x_t)$ satisfies Condition N. Further,

$$\begin{aligned}\nu_s^{\uparrow}(z) - \nu_s^{\downarrow}(z) &= \mu_s(x_i) - q^{(\epsilon)}\mathrm{dist}(x_i, z) - q^{(\epsilon)}\mathrm{dist}(z, y_j) - \mu_s(y_j)\\ &\geq \mu_s(x_i) - \mu_s(y_j) - q^{(\epsilon)}\mathrm{dist}(x_i, y_j).\end{aligned}$$

If $j = i-1$ or $j = i$, then

$$\mu_s(x_i) - \mu_s(y_j) - q^{(\epsilon)}\mathrm{dist}(x_i, y_j) \geq \mu(x_i) - \mu(y_j) - q^{(\epsilon)}\mathrm{dist}(x_i, y_j) = 0.$$

If $j > i$, then, since $(y_0, x_1, \ldots, y_{t-1}, x_t)$ is $q^{(\epsilon)}$-reduced, we see that

$$\begin{aligned}&\mu_s(x_i) - \mu_s(y_j) - q^{(\epsilon)}\mathrm{dist}(x_i, y_j)\\ &\quad\geq \mu(x_i) - \mu(y_j) - 1 - q^{(\epsilon)}\mathrm{dist}(x_i, y_j)\\ &\quad= q^{(\epsilon)}(x_i, y_i, \ldots, x_j, y_j) - 1 - q^{(\epsilon)}\mathrm{dist}(x_i, y_j)\\ &\quad\geq 0.\end{aligned}$$

Thus, $\nu_s^{\uparrow}(z) - \nu_s^{\downarrow}(z) \geq 0$. □

We set $\nu_{00} := \nu_t^{\downarrow}$ and $F_0 := \{z \in P \mid \nu_{00}(z) < \nu_t^{\uparrow}(z)\}$. Further, we set $F_0 = \{z_{01},$

$z_{02}, \dots, z_{0k(0)}\}$ so that $z_{01}, z_{02}, \dots, z_{0k(0)}$ is a linear extension of F_0, i.e., if $z_{0i} < z_{0j}$ then $i < j$.

For j with $1 \le j \le k(0)$, set $F_{0j} := \{z_{01}, \dots, z_{0j}\}$ and for $z \in P^+$

$$\nu_{0j}(z) = \begin{cases} \nu_{00}(z) & \text{if } z \notin F_{0j}, \\ \nu_{00}(z) + 1 & \text{if } z \in F_{0j}. \end{cases}$$

Then for any $1 \le j \le k(0)$, the following fact holds.

Lemma 4.3. *Let j be an integer with $1 \le j \le k(0)$. Then ν_{0j} is an element of $C^{(\epsilon)}$ such that $\nu_{0j}(x_i) = \mu_t(x_i)$ and $\nu_{0j}(y_i) = \mu_t(y_i)$ for any $0 \le i \le t$.*

Proof. Suppose z, $z' \in P^+$ and $z \lessdot z'$. If $z' \notin F_{0j}$ or $z \in F_{0j}$, then $\nu_{0j}(z) - \nu_{0j}(z') \ge \nu_{00}(z) - \nu_{00}(z') \ge \epsilon$. Assume that $z \notin F_{0j}$ and $z' \in F_{0j}$. Since $z_{01}, \dots, z_{0k(0)}$ is a linear extension of F_0, we see that $z \notin F_0$. Therefore, $\nu_t^{\uparrow}(z) = \nu_{00}(z)$. On the other hand, since $z' \in F_0$, we see that $\nu_t^{\uparrow}(z') \ge \nu_{00}(z') + 1$. Thus,

$$\nu_{0j}(z) - \nu_{0j}(z') = \nu_{00}(z) - (\nu_{00}(z') + 1) \ge \nu_t^{\uparrow}(z) - \nu_t^{\uparrow}(z') \ge \epsilon.$$

Since $\nu_t^{\uparrow}(x_i) = \nu_t^{\downarrow}(x_i) = \mu_t(x_i)$ and $\nu_t^{\uparrow}(y_i) = \nu_t^{\downarrow}(y_i) = \mu_t(y_i)$, we see that x_i, $y_i \notin F_0$ for any $0 \le i \le t$. Thus, $\nu_{0j}(x_i) = \mu_t(x_i)$ and $\nu_{0j}(y_i) = \mu_t(y_i)$ for any $0 \le i \le t$. In particular, $\nu_{0j}(\infty) = \nu_{0j}(y_t) = 0$ and $\nu_{0j}(x_i) - \nu_{0j}(y_i) = \mu_t(x_i) - \mu_t(y_i) = q^{(\epsilon)}\mathrm{dist}(x_i, y_i)$. Thus, we see that $\nu_{0j} \in C^{(\epsilon)}$. $\square$

By the above lemma, we see that $\nu_{0k(0)} \in \mathcal{T}^{(\epsilon)}(P)$. We set $\nu_{10} := \max\{\nu_{t-1}^{\downarrow}, \nu_{0k(0)}\}$. Then we see the following.

Lemma 4.4. *The map ν_{10} is an element of $C^{(\epsilon)}$ such that $\nu_{10}(x_i) = \mu_{t-1}(x_i)$, $\nu_{10}(y_i) = \mu_{t-1}(y_i)$ for $0 \le i \le t$ and $\nu_{10}(z) \le \nu_{t-1}^{\uparrow}(z)$ for any $z \in P^+$.*

Proof. The first part of the assertion follows from Lemmas 2.5 and 4.3, equalities (4.3), (4.4) and the definitions of μ_{t-1} and μ_t.

Let z be an arbitrary element of P^+. If $\nu_{10}(z) = \nu_{t-1}^{\downarrow}(z)$, then by Lemma 4.2, we see that $\nu_{10}(z) \le \nu_{t-1}^{\uparrow}(z)$. Suppose that $\nu_{10}(z) = \nu_{0k(0)}(z)$. If $z \notin F_0$, then

$$\nu_{0k(0)}(z) = \nu_t^{\downarrow}(z) = \nu_t^{\uparrow}(z) \le \nu_{t-1}^{\uparrow}(z)$$

by the definition of $\nu_t^{\uparrow}$ and $\nu_{t-1}^{\uparrow}$. If $z \in F_0$, then $\nu_{00}(z) < \nu_t^{\uparrow}(z)$ by the definition of F_0. Therefore,

$$\nu_{0k(0)}(z) = \nu_{00}(z) + 1 \le \nu_t^{\uparrow}(z) \le \nu_{t-1}^{\uparrow}(z). \qquad \square$$

We set $F_1 := \{z \in P \mid \nu_{10}(z) < \nu_{t-1}^{\uparrow}(z)\} \setminus F_0$ and set $F_1 = \{z_{11}, \dots, z_{1k(1)}\}$ so that $z_{11}, \dots, z_{1k(1)}$ is a linear extension of F_1. We also set $F_{1j} := \{z_{11}, \dots, z_{1j}\}$ and for $z \in P^+$,

$$\nu_{1j}(z) = \begin{cases} \nu_{10}(z) & \text{if } \ z \notin F_{1j}, \\ \nu_{10}(z)+1 & \text{if } \ z \in F_{1j} \end{cases}$$

for $1 \le j \le k(1)$. Then by the same argument as the proof of Lemma 4.3, we see that for any $1 \le j \le k(1)$, ν_{1j} is an element of $C^{(\epsilon)}$ with $\nu_{1j}(x_i) = \mu_{t-1}(x_i)$ and $\nu_{1j}(y_i) = \mu_{t-1}(y_i)$ for any $0 \le i \le t$.

Set $\nu_{20} := \max\{\nu_{1k(1)}, \nu_{t-2}^{\downarrow}\}$. Then by the same argument as the proof of Lemma 4.4, we see that ν_{20} is an element of $C^{(\epsilon)}$ such that $\nu_{20}(x_i) = \mu_{t-2}(x_i)$ and $\nu_{20}(y_i) = \mu_{t-2}(y_i)$ for $0 \le i \le t$ and $\nu_{20}(z) \le \nu_{t-2}^{\uparrow}(z)$ for any $z \in P^+$. Thus, we can repeat this argument by setting $F_2 := \{z \in P \mid \nu_{20}(z) < \nu_{t-2}^{\uparrow}(z)\} \setminus (F_0 \cup F_1)$ and taking a linear extension $F_2 = \{z_{21}, \dots, z_{2k(2)}\}$, setting $F_{2j} := \{z_{21}, \dots, z_{2j}\}$ and for $z \in P^+$,

$$\nu_{2j}(z) = \begin{cases} \nu_{20}(z) & \text{if } \ z \notin F_{2j}, \\ \nu_{20}(z)+1 & \text{if } \ z \in F_{2j} \end{cases}$$

for $1 \le j \le k(2)$ and so on.

Finally, we define $k(0) + k(1) + \cdots + k(t) + t + 1$ elements

$$\nu_{00}, \nu_{01}, \dots, \nu_{0k(0)}, \nu_{10}, \nu_{11}, \dots, \nu_{1k(1)}, \dots, \nu_{t0}, \nu_{t1}, \dots, \nu_{tk(t)}$$

of $C^{(\epsilon)}$. Since

$$\nu_{ij}(z) - \nu_{i,j-1}(z) = \begin{cases} 1 & \text{if } \ z = z_{ij}, \\ 0 & \text{otherwise} \end{cases}$$

for any $0 \le i \le t$ and $1 \le j \le k(i)$ and

$$\nu_{i0}(y_j) - \nu_{i-1,k(i-1)}(y_j) = \begin{cases} 1 & \text{if } \ j = t-i \\ 0 & \text{if } \ j \ne t-i \end{cases}$$

for any $1 \le i \le t$, we see that

$$\nu_{00}, \nu_{01}, \dots, \nu_{0k(0)}, \nu_{10}, \nu_{11}, \dots, \nu_{1k(1)}, \dots, \nu_{t0}, \nu_{t1}, \dots, \nu_{tk(t)}$$

are affinely independent, since $F_i \cap F_j = \emptyset$ for $i \ne j$ and $y_j \notin F_i$ for any i and j. Since $\nu_{ij} \in C^{(\epsilon)}$ for any $0 \le i \le t$ and $0 \le j \le k(i)$, we see that

$$\dim C^{(\epsilon)} \ge k(0) + k(1) + \cdots + k(t) + t. \tag{4.5}$$

Next we set $F = F_0 \cup F_1 \cup \cdots \cup F_t$ and state the following.

Lemma 4.5. *For $w \in P$ the following conditions are equivalent.*

(1) *$w \notin F$.*

(2) *$\nu_s^{\downarrow}(w) = \nu_s^{\uparrow}(w)$ for any s with $0 \le s \le t$.*

(3) *$w \in G$.*

PROOF. (1)⇒(2): Since $\nu_{00}=\nu_t^\downarrow$ and $w\notin F_0=\{z\in P\mid \nu_{00}(z)<\nu_t^\uparrow(z)\}$, we see that $\nu_t^\downarrow(w)=\nu_t^\uparrow(w)$. Further, $\nu_{0k(0)}(w)=\nu_{00}(w)=\nu_t^\downarrow(w)$, since $w\notin F_0$. Therefore, $\nu_{10}(w)=\max\{\nu_{t-1}^\downarrow(w),\nu_t^\downarrow(w)\}=\nu_{t-1}^\downarrow(w)$.

Since $w\notin F_1=\{z\in P\mid \nu_{10}(z)<\nu_{t-1}^\uparrow(z)\}$, we see that $\nu_{t-1}^\downarrow(w)=\nu_{10}(w)=\nu_{t-1}^\uparrow(w)$. Further, $\nu_{1k(1)}(w)=\nu_{10}(w)=\nu_{t-1}^\downarrow(w)$, since $w\notin F_1$. By repeating this argument, we see (2).

(2)⇒(3): By assumption, we see that $\nu_0^\downarrow(w)=\nu_0^\uparrow(w)$. By the definition of $\nu_0^\downarrow$ and $\nu_0^\uparrow$, we see that there are i and j such that $x_i\leq w\leq y_j$, $\nu_0^\downarrow(w)=\mu_0(y_j)+q^{(\epsilon)}\mathrm{dist}(w,y_j)$ and $\nu_0^\uparrow(w)=\mu_0(x_i)-q^{(\epsilon)}\mathrm{dist}(x_i,w)$. Take j maximal and i minimal. We shall show that $i=j$. Since $x_i\leq y_j$ and $(y_0,x_1,\ldots,y_{t-1},x_t)$ satisfies Condition N, we see that $j\geq i-1$.

First suppose that $j\geq i+1$. Then

$$\begin{aligned}
\nu_0^\uparrow(w)-\nu_0^\downarrow(w)&=\mu_0(x_i)-q^{(\epsilon)}\mathrm{dist}(x_i,w)-q^{(\epsilon)}\mathrm{dist}(w,y_j)-\mu_0(y_j)\\
&\geq \mu(x_i)-\mu(y_j)-q^{(\epsilon)}\mathrm{dist}(x_i,y_j)\\
&=q^{(\epsilon)}(x_i,y_i,\ldots,x_j,y_j)-q^{(\epsilon)}\mathrm{dist}(x_i,y_j)\\
&>0,
\end{aligned}$$

since $(y_0,x_1,\ldots,y_{t-1},x_t)$ is $q^{(\epsilon)}$-reduced. This contradicts the assumption.

Next suppose that $j=i-1$. Take ℓ and ℓ' such that $x_\ell\leq w\leq y_{\ell'}$, $\nu_i^\downarrow(w)=\mu_i(y_{\ell'})+q^{(\epsilon)}\mathrm{dist}(w,y_{\ell'})$ and $\nu_i^\uparrow(w)=\mu_i(x_\ell)-q^{(\epsilon)}\mathrm{dist}(x_\ell,w)$. If $\ell'\geq i=j+1$, then

$$\begin{aligned}
\nu_i^\downarrow(w)&=\mu_i(y_{\ell'})+q^{(\epsilon)}\mathrm{dist}(w,y_{\ell'})\\
&=\mu_0(y_{\ell'})+q^{(\epsilon)}\mathrm{dist}(w,y_{\ell'})\\
&<\nu_0^\downarrow(w),
\end{aligned}$$

since we took j maximal. If $\ell'<i$, then

$$\begin{aligned}
\nu_i^\downarrow(w)&=\mu_i(y_{\ell'})+q^{(\epsilon)}\mathrm{dist}(w,y_{\ell'})\\
&=\mu_0(y_{\ell'})-1+q^{(\epsilon)}\mathrm{dist}(w,y_{\ell'})\\
&<\nu_0^\downarrow(w).
\end{aligned}$$

On the other hand, if $\ell<i$,

$$\begin{aligned}
\nu_i^\uparrow(w)&=\mu_i(x_\ell)-q^{(\epsilon)}\mathrm{dist}(x_\ell,w)\\
&=\mu_0(x_\ell)-1-q^{(\epsilon)}\mathrm{dist}(x_\ell,w)\\
&\geq\nu_0^\uparrow(w),
\end{aligned}$$

since we took i minimal. If $\ell\geq i$, then

$$\begin{aligned}
\nu_i^\uparrow(w)&=\mu_i(x_\ell)-q^{(\epsilon)}\mathrm{dist}(x_\ell,w)\\
&=\mu_0(x_\ell)-q^{(\epsilon)}\mathrm{dist}(x_\ell,w)\\
&\geq\nu_0^\uparrow(w).
\end{aligned}$$

Thus, we see that

$$\nu_i^{\downarrow}(w) < \nu_0^{\downarrow}(w) = \nu_0^{\uparrow}(w) \le \nu_i^{\uparrow}(w).$$

This contradicts the assumption.

Therefore, $j = i$ and we see that

$$\begin{aligned}0 &= \nu_0^{\uparrow}(w) - \nu_0^{\downarrow}(w)\\ &= \mu(x_i) - q^{(\epsilon)}\mathrm{dist}(x_i, w) - q^{(\epsilon)}\mathrm{dist}(w, y_i) - \mu(y_i)\\ &= q^{(\epsilon)}\mathrm{dist}(x_i, y_i) - q^{(\epsilon)}\mathrm{dist}(x_i, w) - q^{(\epsilon)}\mathrm{dist}(w, y_i).\end{aligned}$$

This means $w \in G_i$.

(3)⇒(1): Suppose that $w \in G_i$. Let s be an arbitrary integer with $0 \le s \le t$. Since $\nu_s^{\uparrow}(w) \le \mu_s(x_i) - q^{(\epsilon)}\mathrm{dist}(x_i, w)$ and $\nu_s^{\downarrow}(w) \ge \mu_s(y_i) + q^{(\epsilon)}\mathrm{dist}(w, y_i)$ by the definition of $\nu_s^{\downarrow}$ and $\nu_s^{\uparrow}$, we see that

$$\begin{aligned}\nu_s^{\uparrow}(w) - \nu_s^{\downarrow}(w) &\le \mu_s(x_i) - q^{(\epsilon)}\mathrm{dist}(x_i, w) - q^{(\epsilon)}\mathrm{dist}(w, y_i) - \mu_s(y_i)\\ &= \mu_s(x_i) - \mu_s(y_i) - q^{(\epsilon)}\mathrm{dist}(x_i, y_i)\\ &= 0,\end{aligned}$$

since $w \in G_i$. Since $\nu_s^{\downarrow}(w) \le \nu_{t-s,0}(w) \le \nu_s^{\uparrow}(w)$, we see that $\nu_{t-s,0}(w) = \nu_s^{\uparrow}(w)$. Therefore, $w \notin F_{t-s}$. Since s is an arbitrary integer with $0 \le s \le t$, we see that $w \notin F$. □

By the above lemma, we see that $P \setminus G = F$ and therefore

$$\#(P \setminus G) = \#F = k(0) + k(1) + \cdots + k(t),$$

since $F_i \cap F_j = \emptyset$ if $i \neq j$. Therefore, by inequalities (4.2) and (4.5), we see the following.

Theorem 4.6. *It holds that* $\dim C^{(\epsilon)} = \#(P \setminus G) + t$.

Remark 4.7. By equation (4.1), $\nu(z) = \nu(y_i) + q^{(\epsilon)}\mathrm{dist}(z, y_i)$ for any $\nu \in C^{(\epsilon)}$ and $z \in G_i$. Therefore, by Theorem 4.6, we see that $C^{(\epsilon)}$ is essentially a full dimensional convex polytope in $\mathbb{R}^{(P \setminus G) \cup \{y_0, \ldots, y_{t-1}\}}$.

By considering the case where the N-sequence under consideration in this section is the empty sequence, we see by Theorem 4.6, the following.

Corollary 4.8. *If* $t = 0$, *then* $\dim C^{(1)} = \#P_{\mathrm{nonmax}}$, *where*

$$P_{\mathrm{nonmax}} := \{z \in P \mid z \text{ is not in any chain of } P \text{ of maximal length}\}$$

(*resp.* $\dim C^{(-1)} = \#P_{\mathrm{nonmin}}$, *where* $P_{\mathrm{nonmin}} := \{z \in P \mid z$ *is not in any maximal chain of* P *of minimal length*$\}$).

5. Canonical and anticanonical analytic spreads.

In this section, we describe the fiber cones $\bigoplus_{n\geq 0}\omega^n/\mathfrak{m}\omega^n$ (resp. $\bigoplus_{n\geq 0}(\omega^{(-1)})^n/\mathfrak{m}(\omega^{(-1)})^n$) and the analytic spread $\dim\bigoplus_{n\geq 0}\omega^n/\mathfrak{m}\omega^n$ (resp. $\dim\bigoplus_{n\geq 0}(\omega^{(-1)})^n/\mathfrak{m}(\omega^{(-1)})^n$) of the canonical (resp. anticanonical) ideal of the Hibi ring $\mathcal{R}_{\mathbb{K}}[H]$ in terms of the notation introduced in the previous section, where $\mathfrak{m}$ is the irrelevant maximal ideal of $\mathcal{R}_{\mathbb{K}}[H]$. By Theorem 2.9, we see that $\omega^n=\omega^{(n)}$ and $(\omega^{(-1)})^n=\omega^{(-n)}$ for a positive integer n. Therefore, we consider the ring

$$\bigoplus_{n\geq 0}\omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)},$$

where $\epsilon=\pm 1$.

Set

$$N^{(\epsilon)}:=\left\{(y_0,x_1,\ldots,y_{t-1},x_t)\;\middle|\;\begin{array}{l}(y_0,x_1,\ldots,y_{t-1},x_t)\text{ is a }q^{(\epsilon)}\text{-reduced}\\ \text{N-sequence}\end{array}\right\}.$$

For $(y_0,x_1,\ldots,y_{t-1},x_t)\in N^{(\epsilon)}$, we denote by $C^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ the convex polytope in $\mathbb{R}^P$ defined by $(y_0,x_1,\ldots,y_{t-1},x_t)$ in the previous section and $R^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}\subset\mathbb{K}[T_x^{\pm 1}\mid x\in P][Y]$ the Ehrhart ring defined by $C^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ over $\mathbb{K}$. Further, we denote by $G^{(\epsilon)}_{i,(y_0,x_1,\ldots,y_{t-1},x_t)}$ (resp. $G^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$) the sets denoted by G_i (resp. G) in the previous section. Then by Theorem 4.6, we see that

$$\dim C^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}=\#(P\setminus G^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)})+t$$

for any $(y_0,x_1,\ldots,y_{t-1},x_t)\in N^{(\epsilon)}$. Moreover, for any positive integer n, $nC^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}=C^{(n\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$. Further, we see by Lemma 3.5, that any $\nu\in C^{(n\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}\cap\mathbb{Z}^P$ is a minimal element of $\mathcal{T}^{(n\epsilon)}(P)$ and therefore T^ν is a generator of $\omega^{(n\epsilon)}$, i.e., the residue class of T^ν is a basis element of $\omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)}$. Thus, $R^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ is embedded in $\bigoplus_{n\geq 0}\omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)}$.

Conversely, assume that T^ν is a generator of $\omega^{(n\epsilon)}$, where n is a positive integer. Then by Lemma 3.6, we see that there is a $q^{(n\epsilon)}$-reduced N-sequence $(y_0,x_1,\ldots,y_{t-1},x_t)$ such that $\nu(x_i)-\nu(y_i)=q^{(n\epsilon)}\mathrm{dist}(x_i,y_i)$ for $0\leq i\leq t$, where we set $y_t=\infty$. Since an N-sequence is $q^{(\epsilon)}$-reduced if and only if $q^{(n\epsilon)}$-reduced, we see that $(y_0,x_1,\ldots,y_{t-1},x_t)\in N^{(\epsilon)}$. Further, by the choice of $(y_0,x_1,\ldots,y_{t-1},x_t)$, we see that

$$\nu\in C^{(n\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}\cap\mathbb{Z}^P=nC^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}\cap\mathbb{Z}^P$$

and we can consider that $T^\nu Y^n$ is an element of $R^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ with degree n. Thus, we see that

$$\bigoplus_{n\geq 0}\omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)}=\sum_{(y_0,x_1,\ldots,y_{t-1},x_t)\in N^{(\epsilon)}}R^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}.\tag{5.1}$$

Since there are only finitely many N-sequences, we see, by considering the Hilbert function of $\bigoplus_{n\geq 0}\omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)}$, that

$$\begin{aligned}\dim\left(\bigoplus_{n\geq 0}\omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)}\right) &= \max_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(\epsilon)}} \dim R^{(\epsilon)}_{(y_0,x_1,\dots,y_{t-1},x_t)} \\ &= \max_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(\epsilon)}} \dim C^{(\epsilon)}_{(y_0,x_1,\dots,y_{t-1},x_t)} + 1 \\ &= \max_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(\epsilon)}} \#(P\setminus G^{(\epsilon)}_{(y_0,x_1,\dots,y_{t-1},x_t)}) + t + 1.\end{aligned}$$

Therefore, we see the following.

THEOREM 5.1. *The fiber cone of the canonical (resp. anticanonical) ideal of the Hibi ring $\mathcal{R}_{\mathbb{K}}[H]$ is*

$$\sum_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(1)}} R^{(1)}_{(y_0,x_1,\dots,y_{t-1},x_t)}$$

(resp. $\sum_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(-1)}} R^{(-1)}_{(y_0,x_1,\dots,y_{t-1},x_t)}$) and the canonical (resp. anticanonical) analytic spread is

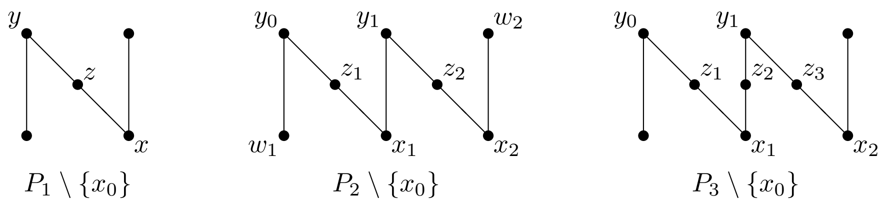
$$\max_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(1)}} \dim C^{(1)}_{(y_0,x_1,\dots,y_{t-1},x_t)} + 1$$

(resp. $\max_{(y_0,x_1,\dots,y_{t-1},x_t)\in N^{(-1)}} \dim C^{(-1)}_{(y_0,x_1,\dots,y_{t-1},x_t)} + 1$).

As a special case, we see by Theorems 3.12, 5.1 and Corollary 4.8, the following fact whose anticanonical level part is [**Pag**, Theorem 4.6].

COROLLARY 5.2. *If $\mathcal{R}_{\mathbb{K}}[H]$ is level (resp. anticanonical level), then the canonical (resp. anticanonical) analytic spread of $\mathcal{R}_{\mathbb{K}}[H]$ is $\#P_{\mathrm{nonmax}}+1$ (resp. $\#P_{\mathrm{nonmin}}+1$).*

EXAMPLE 5.3. Let $P_1\setminus\{x_0\}$, $P_2\setminus\{x_0\}$ and $P_3\setminus\{x_0\}$ be the poset with the following Hasse diagram respectively.



As for P_1, there are two $q^{(-1)}$-reduced N-sequences: (y,x) and the empty sequence. Since $P_1\setminus G^{(-1)}_{(y,x)} = P_1\setminus G^{(-1)}_{()} = \{z\}$, $\dim C^{(-1)}_{(y,x)} = 2$ and $\dim C^{(-1)}_{()} = 1$ and the anticanonical analytic spread is 3 and it comes from the $q^{(-1)}$-reduced N-sequence (y,x). By the definition of $C^{(-1)}_{(y,x)}$ and $C^{(-1)}_{()}$, we see that $C^{(-1)}_{()}\subset C^{(-1)}_{(y,x)}$.

As for P_2, there are four $q^{(-1)}$-reduced N-sequences: (y_0,x_1,y_1,x_2); (y_0,x_1); (y_1,x_2)

and the empty sequence. $P_2 \setminus G^{(-1)}_{(y_0,x_1,y_1,x_2)} = \{z_1, z_2\}$, $P_2 \setminus G^{(-1)}_{(y_0,x_1)} = \{z_1, z_2, x_2, w_2\}$, $P_2 \setminus G^{(-1)}_{(y_1,x_2)} = \{w_1, y_0, z_1, z_2\}$ and $P_2 \setminus G^{(-1)}_{()} = \{z_1, z_2\}$. Therefore, we see that $\dim C^{(-1)}_{(y_0,x_1,y_1,x_2)} = 4$, $\dim C^{(-1)}_{(y_0,x_1)} = \dim C^{(-1)}_{(y_1,x_2)} = 5$ and $\dim C^{(-1)}_{()} = 2$, the anticanonical analytic spread is 6 and it comes from the $q^{(-1)}$-reduced N-sequences (y_0, x_1) and (y_1, x_2). Further, we see that $C^{(-1)}_{(y_0,x_1)} \cap C^{(-1)}_{(y_1,x_2)} = C^{(-1)}_{()}$. Moreover, $C^{(-1)}_{(y_0,x_1)} \cap C^{(-1)}_{(y_0,x_1,y_1,x_2)} = \{\nu \colon P_2^+ \to \mathbb{R} \mid \nu \in C^{(-1)}_{(y_0,x_1,y_1,x_2)},\ \nu(y_1) = -1\} = \{\nu \colon P_2^+ \to \mathbb{R} \mid \nu \in C^{(-1)}_{(y_0,x_1)},\ \nu(w_2) = -1,\ \nu(x_2) = -2\}$. This is a 3-dimensional face of both $C^{(-1)}_{(y_0,x_1,y_1,x_2)}$ and $C^{(-1)}_{(y_0,x_1)}$. A similar fact holds for $C^{(-1)}_{(y_1,x_2)} \cap C^{(-1)}_{(y_0,x_1,y_1,x_2)}$.

As for P_3, there are two $q^{(-1)}$-reduced N-sequences: (y_0, x_1, y_1, x_2) and the empty sequence. $P_3 \setminus G^{(-1)}_{(y_0,x_1,y_1,x_2)} = \{z_1, z_3\}$ and $P_3 \setminus G^{(-1)}_{()} = \{z_1, z_2, z_3, x_1, y_1\}$. Therefore, $\dim C^{(-1)}_{(y_0,x_1,y_1,x_2)} = 4$, $\dim C^{(-1)}_{()} = 5$, the anticanonical analytic spread is 6 and it comes from the empty sequence. Further, $C^{(-1)}_{(y_0,x_1,y_1,x_2)} \cap C^{(-1)}_{()} = \{\nu \colon P_3^+ \to \mathbb{R} \mid \nu \in C^{(-1)}_{()}$, $\nu(y_1) = \nu(z_2) + 1 = \nu(x_1) + 2\} = \{\nu \colon P_3^+ \to \mathbb{R} \mid \nu \in C^{(-1)}_{(y_0,x_1,y_1,x_2)},\ \nu(y_0) = -1\}$, which is a 3-dimensional face of both $C^{(-1)}_{(y_0,x_1,y_1,x_2)}$ and $C^{(-1)}_{()}$.

Problem 5.4. Determine the polytopal complex structure of

$$\{C \mid \text{there is } (y_0, \ldots, x_t) \in N^{(\epsilon)} \text{ such that } C \text{ is a face of } C^{(\epsilon)}_{(y_0,\ldots,x_t)}\}.$$

Suppose that $T^{\nu_1}T^{\nu_2} \neq 0$ in $\bigoplus_{n\geq 0} \omega^{(n\epsilon)}/\mathfrak{m}\omega^{(n\epsilon)}$. Set $T^{\nu_j} \in \omega^{(n_j\epsilon)}$ for $j = 1, 2$, $n = n_1 + n_2$ and $\nu = \nu_1 + \nu_2$. Then by Lemma 3.6, we see that there is a $q^{(n\epsilon)}$-reduced N-sequence $(y_0, x_1, \ldots, y_{t-1}, x_t)$ such that $\nu(x_i) - \nu(y_i) = q^{(n\epsilon)}\operatorname{dist}(x_i, y_i) = nq^{(\epsilon)}\operatorname{dist}(x_i, y_i)$ for $0 \leq i \leq t$, where $y_t := \infty$. Since $\nu_j(x_i) - \nu_j(y_i) \geq n_j q^{(\epsilon)}\operatorname{dist}(x_i, y_i)$ for any i and j, we see that

$$\nu_j(x_i) - \nu_j(y_i) = n_j q^{(\epsilon)}\operatorname{dist}(x_i, y_i)$$

for any i and j. Thus, we see that ν_1 and ν_2 are elements of $C^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ and the product $T^{\nu_1}T^{\nu_2} = T^{\nu}$ is the one in $R^{(\epsilon)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$.

In other words, if we set

$$\Gamma := \{C \mid \text{there is } (y_0, \ldots, x_t) \in N^{(\epsilon)} \text{ such that } C \text{ is a face of } C^{(\epsilon)}_{(y_0,\ldots,x_t)}\},$$

then the product of the elements T^{ν} and $T^{\nu'}$ in the right hand side of the equation (5.1) is zero if there is no facet of Γ containing both ν and ν' and the one in the Ehrhart ring of C if there is a facet C containing both ν and ν'.

A Stanley–Reisner ring is a ring of this kind over a simplicial complex whose facets have normalized volume 1. On account of this fact, we propose the following.

Problem 5.5. Establish a theory of polytopal complex version of Stanley–Reisner rings.

Ishida [**Ish**] studied Cohen–Macaulay and Gorenstein properties of this kind of rings defined by subcomplexes of boundary complex of convex polytopes.

6. Complexity of a graded ring and Frobenius complexity.

From now on, we use the term ring to express a not necessarily commutative ring with identity.

First we define the complexity of a graded ring.

Definition 6.1. Let $A = \bigoplus_{n\geq 0} A_n$ be an $\mathbb{N}$-graded ring. For $e \geq 0$, we denote by $G_e(A)$ the subring of A generated by homogeneous elements with degree at most e over A_0. For $e \geq 1$, we denote by $c_e(A)$ the minimal number of elements which generate $A_e/G_{e-1}(A)_e$ as a two sided A_0-module. If $c_e(A)$ is finite for any e, we say that A is degree-wise finitely generated. Suppose that A is degree-wise finitely generated. We define the complexity $\mathrm{cx}(A)$ of A by

$$\mathrm{cx}(A) := \inf\{n \in \mathbb{R}_{>0} \mid c_e(A) = O(n^e)\ (e \to \infty)\}$$

if $\{n \in \mathbb{R}_{>0} \mid c_e(A) = O(n^e)\ (e \to \infty)\} \neq \emptyset$. We define $\mathrm{cx}(A) := \infty$ if $\{n \in \mathbb{R}_{>0} \mid c_e(A) = O(n^e)\ (e \to \infty)\} = \emptyset$.

O in the above definition is the Landau symbol, i.e., $g(x) = O(f(x))\ (x \to \infty)$ means that there is a positive real number K such that $|g(x)| < K|f(x)|$ for $x \gg 0$. We denote by $g(x) \neq O(f(x))\ (x \to \infty)$ if $g(x) = O(f(x))\ (x \to \infty)$ does not hold. We state over which variable the limit is taken when using Landau symbol, except the case that there is no fear of confusion.

Enescu–Yao [**EY1**, Definition 2.9] defined the notion of left R-skew algebra. We refine their definition and define the notion of strong left R-skew algebra.

Definition 6.2. Let R be a commutative ring and $A = \bigoplus_{n\geq 0} A_n$ a graded ring. Suppose that a ring homomorphism $R \to A_0$ is fixed. We say that A is a strong left R-skew algebra if $aI \subset Ia$ for any homogeneous element $a \in A$ and any ideal $I \subset R$.

Remark 6.3. Let R be a commutative ring, I an ideal of R and $A = \bigoplus_{n\geq 0} A_n$ a strong left R-skew algebra. Then $IA = \bigoplus_{n\geq 0} IA_n$ is a two sided ideal of A and $A/IA = \bigoplus_{n\geq 0} A_n/IA_n$ has naturally a graded ring structure.

Remark 6.4. Suppose that A_0 is commutative and $A = \bigoplus_{n\geq 0} A_n$ is a degree-wise finitely generated strong left A_0-skew algebra. Then $c_e(A)$ is equal to the minimal number of generators of $A_e/G_{e-1}(A)_e$ as a left A_0-module. Moreover, if A_0 is a local ring with maximal ideal $\mathfrak{m}$, then $A/\mathfrak{m}A = \bigoplus_{n\geq 0} A_n/\mathfrak{m}A_n$ and $c_e(A) = c_e(A/\mathfrak{m}A)$. In particular, $c_e(A)$ is equal to the dimension of $A_e/(G_{e-1}(A)_e + \mathfrak{m}A_e) = (A/\mathfrak{m}A)_e/G_{e-1}(A/\mathfrak{m}A)_e$ as a vector space over $A_0/\mathfrak{m}$.

Next we recall the definition of Frobenius complexity. Let R be a commutative ring with prime characteristic p and M an R-module. We denote by ${}^eM$ the R-module whose additive group structure is that of M and the action of R is defined by e times iterated Frobenius map, i.e., $r \cdot m = r^{p^e}m$ for $r \in R$ and $m \in M$, where the R-action of right hand side is the original R-module structure of M. Note that for $\varphi \in \mathrm{Hom}_R(M, {}^eM)$ and $\psi \in \mathrm{Hom}_R(M, {}^{e'}M)$, $\psi \circ \varphi \in \mathrm{Hom}_R(M, {}^{e+e'}M)$. On account of this fact, we state the following.

Definition 6.5. Let R and M be as above. We set $\mathcal{F}^e(M) := \mathrm{Hom}_R(M, {}^eM)$ and

$$\mathcal{F}(M) := \bigoplus_{e \geq 0} \mathcal{F}^e(M).$$

We call $\mathcal{F}(M)$ the ring of Frobenius operators on M. The multiplication on $\mathcal{F}(M)$ is defined by composition of maps.

Definition 6.6 ([**EY1**, Definition 2.13]). Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring and E the injective hull of k. Then the Frobenius complexity $\mathrm{cx}_F(R)$ of R is defined by

$$\mathrm{cx}_F(R) := \log_p(\mathrm{cx}(\mathcal{F}(E)))$$

if $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E)$. If $\mathcal{F}(E)$ is finitely generated over $\mathcal{F}^0(E)$, we define $\mathrm{cx}_F(R) := -\infty$. For an $\mathbb{N}$-graded commutative Noetherian ring $R = \bigoplus_{n \geq 0} R_n$ with R_0 a field, we define the Frobenius complexity of R to be that of $\mathfrak{m}$-adic completion of R, where $\mathfrak{m} = \bigoplus_{n > 0} R_n$.

Remark 6.7 (cf. [**LS**, Proposition 3.3]). Let $(R, \mathfrak{m}, k)$ be as above and $\hat{R}$ the $\mathfrak{m}$-adic completion of R. Then $(\hat{R}, \mathfrak{m}\hat{R}, k)$ is a local ring, $E_R(k) = E_{\hat{R}}(k)$ and $\mathrm{Hom}_R(E_R(k), {}^eE_R(k)) = \mathrm{Hom}_{\hat{R}}(E_{\hat{R}}(k), {}^eE_{\hat{R}}(k))$. Thus, we see that Frobenius complexity does not vary by taking completion.

7. T-construction and T-complexity.

Katzman et al. introduced an important graded ring construction method from a commutative graded ring with prime characteristic. We first recall their definition.

Definition 7.1 ([**KSSZ**, Definition 2.1]). Let $R = \bigoplus_{n \geq 0} R_n$ be an $\mathbb{N}$-graded commutative ring with characteristic p. We set $T(R)_e := R_{p^e - 1}$ for $e \geq 0$ and

$$T(R) := \bigoplus_{e \geq 0} T(R)_e.$$

The multiplication in $T(R)$ is defined by $a * b := ab^{p^e}$ for $a \in T(R)_e$ and $b \in T(R)_{e'}$ (the right hand side is the original product in R).

Note that since R is a commutative ring with characteristic p, the product $*$ satisfies distributive law. Thus, $T(R)$ is an $\mathbb{N}$-graded ring.

Next we make the following.

Definition 7.2. In the situation of Definition 7.1, we set

$$\mathrm{Tcx}(R) := \log_p \mathrm{cx}(T(R))$$

if $T(R)$ is not finitely generated over $T(R)_0$ and $\mathrm{Tcx}(R) := -\infty$ if $T(R)$ is finitely generated over $T(R)_0$. We call $\mathrm{Tcx}(R)$ the T-complexity of R.

Next we recall the following.

Definition 7.3 ([**KSSZ**, Definition 3.2]). Let R be a commutative Noetherian normal ring that is either complete local or $\mathbb{N}$-graded and finitely generated over a field R_0. Let ω denote the canonical ideal of R and for $m \in \mathbb{Z}$, let $\omega^{(m)}$ be the m-th power of ω in $\mathrm{Div}(R)$. Then

$$\mathcal{R} := \bigoplus_{n\geq 0} \omega^{(-n)}$$

is called the anticanonical symbolic Rees algebra of R (Katzman et al. called $\mathcal{R}$ the anticanonical cover of R).

Note that $\mathcal{R}$ is a commutative graded ring with $\mathcal{R}_0 = R$. In particular, $\mathcal{R}$ and R have the same characteristic.

Now we recall a crucially important result of Katzman et al.

Fact 7.4 ([**KSSZ**, Theorem 3.3]). *Let $(R, \mathfrak{m})$ be a commutative Cohen–Macaulay normal complete local ring of characteristic p, E the injective hull of $R/\mathfrak{m}$, $\mathcal{R}$ the anticanonical symbolic Rees algebra of R. Then there is an isomorphism of graded rings*

$$\mathcal{F}(E) \cong T(\mathcal{R}).$$

Note in the setting of Fact 7.4, $T(\mathcal{R})_0 = R$ and $T(\mathcal{R})$ is a strong left R-skew algebra.

Remark 7.5. If R is a normal excellent ring (e.g., a finitely generated commutative ring over a field) and $\mathfrak{m}$ a maximal ideal of R, then the $\mathfrak{m}$-adic completion $\hat{R}$ of R is a normal complete local ring with maximal ideal $\mathfrak{m}\hat{R}$. See e.g., [**Mat**, (33.I) Theorem 79]. In particular, if $R = \bigoplus_{n\geq 0} R_n$ is a commutative normal graded ring which is finitely generated over a field R_0, then the $\mathfrak{m}$-adic completion of R is a normal complete local ring, where $\mathfrak{m} = \bigoplus_{n>0} R_n$.

Further, if $(R, \mathfrak{m})$ is an excellent Cohen–Macaulay normal local ring with canonical module or finitely generated $\mathbb{N}$-graded Cohen–Macaulay normal ring over a field R_0 and $\mathfrak{m} = \bigoplus_{n>0} R_n$, then

$$\omega_R^{(-n)}/\mathfrak{m}\omega_R^{(-n)} = \omega_{\hat{R}}^{(-n)}/\mathfrak{m}\omega_{\hat{R}}^{(-n)}$$

for any $n \geq 0$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.

In view of Remark 7.5, Fact 7.4, Remark 6.4 and equation (5.1), we consider the T-complexity of Ehrhart rings. First we note the following fact (cf. [**Pag**, Proposition 2.6]).

Lemma 7.6. *Let $R = \bigoplus_{n\geq 0} R_n$ be a commutative Noetherian $\mathbb{N}$-graded ring with R_0 a field of characteristic p. Then*

$$\mathrm{Tcx}(R) \leq \dim R - 1.$$

Proof. Set $d = \dim R$. Since R is Noetherian, there is a polynomial $f(n)$ of n with degree $d-1$ such that

$$\dim_{R_0} R_n \leq f(n) \quad \text{for } n \gg 0.$$

Therefore, $c_e(T(R)) \leq \dim_{R_0} R_{p^e-1} \leq f(p^e-1) = O(p^{(d-1)e})$ $(e \to \infty)$. Thus, $\mathrm{Tcx}(R) \leq d-1$. $\square$

Now we state the following.

Lemma 7.7. *Let d be an integer with $d \geq 2$, Δ an integral convex polytope in $\mathbb{R}^d$ such that* $\dim \Delta = d$ *and*

$$\Delta \subset \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; x_i \geq 0 (1 \leq i \leq d),\ \sum_{i=1}^{d} x_i \leq d-1 \right\}$$

and R the Ehrhart ring defined by Δ with base field $\mathbb{K}$ of characteristic p. Then $\lim_{p\to\infty} \mathrm{Tcx}(R) = d$.

Proof. By Lemma 7.6, we see that $\mathrm{Tcx}(R) \leq d$ for any p.

Now we prove that $\liminf_{p\to\infty} \mathrm{Tcx}(R) \geq d$. Let e' be a positive integer and $x = (x_1, \ldots, x_d) \in \mathbb{N}^d$. We set y'_i the remainder when x_i is divided by $p^{e'}$ for $1 \leq i \leq d$ and $y' = (y'_1, \ldots, y'_d)$. If $y'_i \geq p^{e'} - \lfloor p^{e'}/d \rfloor$ for any i, then $y' \notin (p^{e'}-1)\Delta$ since $\sum_{i=1}^{d} y'_i \geq (d-1)p^{e'}$ and

$$(p^{e'}-1)\Delta \subset \left\{ (w_1, \ldots, w_d) \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} w_i \leq (d-1)(p^{e'}-1) \right\}.$$

Thus, there are no $y = (y_1, \ldots, y_d) \in (p^{e'}-1)\Delta \cap \mathbb{Z}^d$ and $z \in \mathbb{Z}^d$ such that

$$x = y + p^{e'}z,$$

since $y_i \equiv y'_i \pmod{p^{e'}}$ for $1 \leq i \leq d$. Therefore for $e \geq 2$ and $x = (x_1, \ldots, x_d) \in (p^e-1)\Delta \cap \mathbb{Z}^d$, if each digit of the position $1, p, p^2, \ldots, p^{e-2}$ in base p expansion of x_i is greater than or equal to $p - \lfloor p/d \rfloor$, then there are no e', x' and x'' such that $0 < e' < e$, $x = x' + p^{e'}x''$, $x' \in (p^{e'}-1)\Delta \cap \mathbb{Z}^d$ and $x'' \in (p^{e-e'}-1)\Delta \cap \mathbb{Z}^d$. In fact, since the digit of x_i of the position $p^{e'-1}$ is greater than or equal to $p - \lfloor p/d \rfloor$, we see that the remainder when x_i is divided by $p^{e'}$ is greater than or equal to $p^{e'-1}(p - \lfloor p/d \rfloor) \geq p^{e'} - \lfloor p^{e'}/d \rfloor$ for $1 \leq i \leq d$. This contradicts the fact proved above.

Since Δ has an interior point, there is $N > d$ such that if $p > N$, then there are positive integers $a_1, \ldots, a_d$ such that

$$\left[\frac{a_1}{p}, \frac{a_1+2}{p}\right] \times \cdots \times \left[\frac{a_d}{p}, \frac{a_d+2}{p}\right] \subset \Delta.$$

Let e be an integer with $e \geq 2$. Since $a_i p^{e-1} > (p^e-1)a_i/p$ and $(a_i+1)p^{e-1} - 1 < (p^e-1)(a_i+2)/p$, we see that

$$[a_1p^{e-1}, (a_1+1)p^{e-1}-1] \times \cdots \times [a_dp^{e-1}, (a_d+1)p^{e-1}-1] \subset (p^e-1)\Delta.$$

If $x = (x_1, \ldots, x_d) \in [a_1p^{e-1}, (a_1+1)p^{e-1}-1] \times \cdots \times [a_dp^{e-1}, (a_d+1)p^{e-1}-1]$ and each digit of the position 1, p, $p^2, \ldots, p^{e-2}$ of base p expansion of x_i is greater than or equal to $p - \lfloor p/d \rfloor$ for any i, then $x \in (p^e-1)\Delta$ and there are no e', x' and x'' such that $0 < e' < e$, $x = x' + p^{e'}x''$, $x' \in (p^{e'}-1)\Delta \cap \mathbb{Z}^d$ and $x'' \in (p^{e-e'}-1)\Delta \cap \mathbb{Z}^d$.

Since there are $(\lfloor p/d \rfloor)^{d(e-1)}$ choices of x, we see that $c_e(T(R)) \geq (\lfloor p/d \rfloor)^{d(e-1)}$ if $p > N$. Thus $\mathrm{cx}(T(R)) \geq (\lfloor p/d \rfloor)^d$ and $\liminf_{p\to\infty} \mathrm{Tcx}(R) = \liminf_{p\to\infty} \log_p \mathrm{cx}(T(R)) \geq d$. $\square$

Next we state a lemma which is crucial to apply Lemma 7.7 to more general polytope. We first state the definition of symbols.

Definition 7.8. Let Δ be a convex polytope in $\mathbb{R}^d$ with $\dim \Delta = d$ and δ a positive real number. We denote by $\partial\Delta$ the boundary of Δ. We set

$$\begin{aligned}
\partial'_\delta(\Delta) &:= \{P \in \Delta \mid \text{the distance between } P \text{ and } \partial\Delta \text{ is less than } \delta\},\\
\partial''_\delta(\Delta) &:= \{P \in \Delta \mid \text{the distance between } P \text{ and } \partial\Delta \text{ is equal to } \delta\} \text{ and}\\
\mathrm{int}'_\delta(\Delta) &:= \{P \in \Delta \mid \text{the distance between } P \text{ and } \partial\Delta \text{ is greater than } \delta\}.
\end{aligned}$$

Now we state the following.

Lemma 7.9. *Let Δ' be a $d+1$ dimensional integral convex polytope in $\mathbb{R}^{d+1}$. Set $\Delta = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \exists y \in \mathbb{R}; (x_1, \ldots, x_d, y) \in \Delta'\}$ and let R (resp. R') be the Ehrhart ring of Δ (resp. Δ') in a Laurent polynomial ring $\mathbb{K}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, T]$ (resp. $\mathbb{K}[X_1^{\pm 1}, \ldots, X_{d+1}^{\pm 1}, T]$), where $\mathbb{K}$ is a field of characteristic p. Then $\lim_{p\to\infty} \mathrm{Tcx}(R') = d+1$ if $\lim_{p\to\infty} \mathrm{Tcx}(R) = d$.*

Proof. First we note that by Lemma 7.6, $\mathrm{Tcx}(R') \leq d+1$ for any p.

Next we prove that $\liminf_{p\to\infty} \mathrm{Tcx}(R') \geq d+1$. Let ϵ be an arbitrary real number with $0 < \epsilon < 1$. Since $\lim_{p\to\infty} \mathrm{Tcx}(R) = d$, we see that there exists N such that if $p > N$, then $\mathrm{Tcx}(R) > d - \epsilon$. Let p be such a prime number. For a positive integer e, we set

$$H_e(R) = \{P \in (p^e-1)\Delta \cap \mathbb{Z}^d \mid X^P T^{p^e-1} \notin G_{e-1}(T(R))\},$$

where $X^P := X_1^{p_1} \cdots X_d^{p_d}$ for $P = (p_1, \ldots, p_d)$. We define $H_e(R')$ similarly. Then

$$H_e(R) = \left\{ P \in (p^e-1)\Delta \cap \mathbb{Z}^d \;\middle|\; \begin{array}{l} \text{there are no } e', P_1 \text{ and } P_2 \text{ such that } 0 < e' < e, \\ P_1 \in (p^{e'}-1)\Delta \cap \mathbb{Z}^d,\ P_2 \in (p^{e-e'}-1)\Delta \cap \mathbb{Z}^d \\ \text{and } P = P_1 + p^{e'}P_2 \end{array} \right\}.$$

In particular, $(H_e(R) \times \mathbb{Z}) \cap (p^e-1)\Delta' \subset H_e(R')$.

Set $r = 1 - \epsilon$. Then $\#(\partial'_{p^{re}}((p^e-1)\Delta) \cap \mathbb{Z}^d) = O(p^{(r+d-1)e}) = O(p^{(d-\epsilon)e})$ $(e \to \infty)$, since $\dim \partial\Delta = d-1$. On the other hand, since $\mathrm{Tcx}(R) > d - \epsilon$, we see that $\#H_e(R) \neq O(p^{(d-\epsilon)e})$ $(e \to \infty)$. Therefore,

$$\#(H_e(R) \setminus (\partial'_{p^{re}}(p^e-1)\Delta)) \neq O(p^{(d-\epsilon)e}) \quad (e \to \infty). \tag{7.1}$$

For $x = (x_1, \dots, x_d) \in \Delta$, set

$$h(x) := \max\{y - z \mid (x_1, \dots, x_d, y), (x_1, \dots, x_d, z) \in \Delta'\}.$$

Then, since Δ' is the intersection of finite number of halfspaces, we see that there exist positive real numbers a and δ such that if $0 < \delta' \leq \delta$ and $x \in \partial''_{\delta'}(\Delta)$, then $h(x) \geq a\delta'$. For these a and δ, we see that $h(P) \geq a\delta'$ for any $0 < \delta' \leq \delta$ and $P \in \mathrm{int}'_{\delta'}(\Delta)$, since Δ' is convex. Thus, we see by (7.1) that

$$\begin{aligned} c_e(T(R')) &= \# H_e(R') \\ &\geq \#((H_e(R) \setminus \partial'_{p^{re}}(p^e - 1)\Delta) \times \mathbb{Z} \cap (p^e - 1)\Delta') \\ &\geq \#(H_e(R) \setminus \partial'_{p^{re}}(p^e - 1)\Delta)(ap^{re} - 1) \\ &\neq O(p^{(d-\epsilon+r)e}) \ (e \to \infty), \end{aligned}$$

since $p^{re}/(p^e - 1) \leq \delta$ for $e \gg 0$. Therefore, $\mathrm{Tcx}(R') \geq d - \epsilon + r = d + 1 - 2\epsilon$.

Since ϵ is an arbitrary real number with $0 < \epsilon < 1$, we see that $\liminf_{p\to\infty} \mathrm{Tcx}(R') \geq d + 1$. $\square$

8. T-complexities of fiber cones and limit Frobenius complexities of Hibi rings.

In this section, we consider the limit of Frobenius complexities of Hibi rings as $p \to \infty$, where p is the characteristic of the base field. Recall that H is a finite distributive lattice with minimal element x_0, P the set of join-irreducible elements of H, $\mathcal{R}_{\mathbb{K}}[H]$ the Hibi ring over a field $\mathbb{K}$ on H and ω the canonical ideal of $\mathcal{R}_{\mathbb{K}}[H]$. In this section, we assume that $\mathbb{K}$ is a field of characteristic p.

In view of Fact 7.4, Remarks 6.4 and 7.5, we consider the T-complexity of fiber cones of the anticanonical ideal of $\mathcal{R}_{\mathbb{K}}[H]$. We use the notation of Section 5. First we state the following.

Lemma 8.1. *It holds that*

$$c_e\left(T\left(\bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)}\right)\right) \geq c_e\left(T\left(R^{(-1)}_{(y_0,x_1,\dots,y_{t-1},x_t)}\right)\right)$$

for any $e > 0$ and any $(y_0, x_1, \dots, y_{t-1}, x_t) \in N^{(-1)}$.

Proof. Set $R = R^{(-1)}_{(y_0,x_1,\dots,y_{t-1},x_t)}$. Suppose that $T^\nu \in T(R)_e \setminus G_{e-1}(T(R))_e$. Since $T^\nu \in T(\bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)})$ by equation (5.1), it is enough to show that $T^\nu \notin G_{e-1}(T(\bigoplus_{n\geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)}))_e$. Assume the contrary. Then there exist $e' \in \mathbb{N}$, ν' and ν'' such that $0 < e' < e$, $T^{\nu'} \in \omega^{(1-p^{e'})}/\mathfrak{m}\omega^{(1-p^{e'})}$, $T^{\nu''} \in \omega^{(1-p^{e-e'})}/\mathfrak{m}\omega^{(1-p^{e-e'})}$ and $T^\nu = T^{\nu'} * T^{\nu''} = T^{\nu'}(T^{\nu''})^{p^{e'}}$. Since

$$\nu'(y_i) - \nu'(x_i) \geq q^{(1-p^{e'})}\mathrm{dist}(x_i, y_i),$$

$$\nu''(y_i) - \nu''(x_i) \geq q^{(1-p^{e-e'})}\mathrm{dist}(x_i, y_i),$$
$$\nu(y_i) - \nu(x_i) = q^{(1-p^e)}\mathrm{dist}(x_i, y_i) \text{ and}$$
$$\nu = \nu' + p^{e'}\nu'',$$

we see that $\nu'(y_i)-\nu'(x_i) = q^{(1-p^{e'})}\mathrm{dist}(x_i, y_i)$ and $\nu''(y_i)-\nu''(x_i) = q^{(1-p^{e-e'})}\mathrm{dist}(x_i, y_i)$ for any i. Therefore, $T^{\nu'} \in T(R)_{e'}$, $T^{\nu''} \in T(R)_{e-e'}$ and $T^{\nu} = T^{\nu'} * T^{\nu''}$. This contradicts the assumption that $T^{\nu} \notin G_{e-1}(T(R))_e$. $\square$

Before going further, we note that, by Remark 4.7, $C^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ is essentially a full dimensional convex polytope in $\mathbb{R}^{(P\setminus G^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)})\cup\{y_0,\ldots,y_{t-1}\}}$.

Next we state the following.

LEMMA 8.2. *Assume that P is not pure and let $(y_0, x_1, \ldots, y_{t-1}, x_t)$ be a $q^{(-1)}$-reduced N-sequence such that*

$$\dim C^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)} = \max_{(y'_0,x'_1,\ldots,y'_{t'-1},x'_{t'})\in N^{(-1)}} \dim C^{(-1)}_{(y'_0,x'_1,\ldots,y'_{t'-1},x'_{t'})}.$$

Then $\lim_{p\to\infty} \mathrm{Tcx}(R^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}) = \dim C^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$.

PROOF. By Lemmas 7.7 and 7.9, it is enough to show that there is a projection of Euclidean space to a coordinate subspace whose image Δ of $C^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ satisfies the condition of Lemma 7.7 under unimodular transformation. We set $y_t = \infty$.

First consider the case where $t = 0$. We first show that $P \setminus G^{(-1)}_{()} \neq \emptyset$. Assume the contrary. Since P is not pure, there is a saturated chain

$$x_0 = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_{v-1} \lessdot w_v = \infty$$

in P^+ with $v > \mathrm{dist}(x_0, \infty)$. Take j with $\mathrm{dist}(x_0, w_j) = j$ and $\mathrm{dist}(x_0, w_{j+1}) \leq j$. Then $j \geq 2$. Since $P \setminus G^{(-1)}_{()} = \emptyset$, we see that

$$\mathrm{dist}(x_0, w_j) + \mathrm{dist}(w_j, \infty) = \mathrm{dist}(x_0, \infty) < v.$$

In particular, $j \leq v - 2$. Therefore, (w_{j+1}, w_j) is an N-sequence. Further, since

$$\begin{aligned} &q^{(-1)}(x_0, w_{j+1}, w_j, \infty) \\ &\quad = -\mathrm{dist}(x_0, w_{j+1}) + \mathrm{dist}(w_j, w_{j+1}) - \mathrm{dist}(w_j, \infty) \\ &\quad \geq -j + 1 - (\mathrm{dist}(x_0, \infty) - j) \\ &\quad > -\mathrm{dist}(x_0, \infty) \\ &\quad = q^{(-1)}\mathrm{dist}(x_0, \infty), \end{aligned}$$

we see that (w_{j+1}, w_j) is $q^{(-1)}$-reduced. Therefore, $(w_{j+1}, w_j) \in N^{(-1)}$ and

$$\dim C^{(-1)}_{(w_{j+1},w_j)} = \#(P \setminus G^{(-1)}_{(w_{j+1},w_j)}) + 1 > 0 = \#(P \setminus G^{(-1)}_{()}) = \dim C^{(-1)}_{()}.$$

This contradicts the maximality of $\dim C^{(-1)}_{()}$.

Thus, $P \setminus G^{(-1)}_{()} \neq \emptyset$ and there is a sequence of elements $z_0, z_1, \dots, z_u$ in P^+ such that $u \geq 2$, $z_0 \lessdot z_1 \lessdot \cdots \lessdot z_u$, $z_0, z_u \in G^{(-1)}_{()}$ and $z_i \notin G^{(-1)}_{()}$ for $1 \leq i \leq u-1$. Here, we claim that

$$\operatorname{dist}(z_0, \infty) - \operatorname{dist}(z_u, \infty) \geq 2.$$

Assume the contrary. Then, since $\operatorname{dist}(z_0, z_u) \geq 2$ and $\operatorname{dist}(x_0, z_0) + \operatorname{dist}(z_0, \infty) = \operatorname{dist}(x_0, \infty)$, we see that

$$\begin{aligned}
&\operatorname{dist}(x_0, z_0) + \operatorname{dist}(z_0, z_u) + \operatorname{dist}(z_u, \infty) \\
&\quad = \operatorname{dist}(x_0, \infty) - \operatorname{dist}(z_0, \infty) + \operatorname{dist}(z_0, z_u) + \operatorname{dist}(z_u, \infty) \\
&\quad = \operatorname{dist}(x_0, \infty) - (\operatorname{dist}(z_0, \infty) - \operatorname{dist}(z_u, \infty)) + \operatorname{dist}(z_0, z_u) \\
&\quad > \operatorname{dist}(x_0, \infty).
\end{aligned}$$

Since $\operatorname{dist}(x_0, z_0) + \operatorname{dist}(z_0, \infty) = \operatorname{dist}(x_0, z_u) + \operatorname{dist}(z_u, \infty) = \operatorname{dist}(x_0, \infty)$, we see by the above inequality that $z_0, z_u \notin \{x_0, \infty\}$, i.e., (z_u, z_0) is an N-sequence. Further, since

$$\begin{aligned}
&q^{(-1)}(x_0, z_u, z_0, \infty) \\
&\quad = -\operatorname{dist}(x_0, z_u) + \operatorname{dist}(z_0, z_u) - \operatorname{dist}(z_0, \infty) \\
&\quad = \operatorname{dist}(z_u, \infty) - \operatorname{dist}(x_0, \infty) + \operatorname{dist}(z_0, z_u) + \operatorname{dist}(x_0, z_0) - \operatorname{dist}(x_0, \infty) \\
&\quad > -\operatorname{dist}(x_0, \infty) \\
&\quad = q^{(-1)}\operatorname{dist}(x_0, \infty),
\end{aligned}$$

(z_u, z_0) is $q^{(-1)}$-reduced. Since $z_0, z_u \in G^{(-1)}_{()}$, we see that $G^{(-1)}_{(z_u, z_0)} \subset G^{(-1)}_{()}$. Thus, we see that

$$\dim C^{(-1)}_{(z_u, z_0)} = \#(P \setminus G^{(-1)}_{(z_u, z_0)}) + 1 > \#(P \setminus G^{(-1)}_{()}) = \dim C^{(-1)}_{()}.$$

This contradicts the maximality of $\dim C^{(-1)}_{()}$. Therefore,

$$\operatorname{dist}(z_0, \infty) - \operatorname{dist}(z_u, \infty) \geq 2.$$

Consider the image of $C^{(-1)}_{()}$ of composition of the projection (restriction) $\mathbb{R}^P \to \mathbb{R}^{\{z_1, \dots, z_{u-1}\}}$ and the transformation $\xi(z_i) = \nu(z_{i-1}) - \nu(z_i) + 1$ for $1 \leq i \leq u-1$. This transformation $\nu \mapsto \xi$ is unimodular since it is a composition of a parallel translation and a linear transformation whose representation matrix is an upper triangular matrix with diagonal entries -1. Note that $\nu(z_0)$ is independent of ν and therefore $\xi(z_1), \dots, \xi(z_{u-1})$ are defined by $\nu(z_1), \dots, \nu(z_{u-1})$. Further, $\xi(z_i) \geq 0$ for $1 \leq i \leq u-1$, since by the definition of $C^{(-1)}_{()}$, $\nu(z_{i-1}) - \nu(z_i) \geq -1$. Moreover,

$$\sum_{i=1}^{u-1} \xi(z_i) = \nu(z_0) - \nu(z_{u-1}) + u - 1$$

$$\begin{aligned}&\leq \nu(z_0) - \nu(z_{u-1}) + u - 1 + \nu(z_{u-1}) - \nu(z_u) + 1\\&= q^{(-1)}\mathrm{dist}(z_0, \infty) - q^{(-1)}\mathrm{dist}(z_u, \infty) + u\\&\leq u - 2,\end{aligned}$$

since $\nu(z_0) = q^{(-1)}\mathrm{dist}(z_0, \infty)$, $\nu(z_u) = q^{(-1)}\mathrm{dist}(z_u, \infty)$ and $\mathrm{dist}(z_0, \infty) - \mathrm{dist}(z_u, \infty) \geq 2$. Therefore, the image of $C^{(-1)}_{()}$ by the composition of the projection and the above unimodular transformation satisfies the assumption of Lemma 7.7.

Next consider the case where $t > 0$. First note that

$$\mathrm{dist}(x_t, y_{t-1}) + \mathrm{dist}(y_{t-1}, \infty) > \mathrm{dist}(x_t, \infty).$$

In fact, since $(y_0, x_1, \ldots, y_{t-1}, x_t)$ is a $q^{(-1)}$-reduced N-sequence, we see that

$$q^{(-1)}(x_{t-1}, y_{t-1}, x_t, \infty) > q^{(-1)}\mathrm{dist}(x_{t-1}, \infty),$$

i.e.,

$$\mathrm{dist}(x_{t-1}, y_{t-1}) - \mathrm{dist}(x_t, y_{t-1}) + \mathrm{dist}(x_t, \infty) < \mathrm{dist}(x_{t-1}, \infty).$$

Since $\mathrm{dist}(x_{t-1}, \infty) \leq \mathrm{dist}(x_{t-1}, y_{t-1}) + \mathrm{dist}(y_{t-1}, \infty)$, we see the inequality above. In particular, $y_{t-1} \notin G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$.

Now set $\ell = \mathrm{dist}(x_t, y_{t-1})$ and take elements $z_0, z_1, \ldots, z_\ell$ such that

$$x_t = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_\ell = y_{t-1}.$$

We claim that $z_i \notin G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$ for $1 \leq i \leq \ell$. Assume the contrary and take i with $z_i \in G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$. Then $i < \ell$ since $y_{t-1} \notin G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$. Therefore, it is easily verified that $(y_0,\ x_1, \ldots, x_{t-1},\ y_{t-1},\ z_i)$ is a $q^{(-1)}$-reduced N-sequence. It is also verified that

$$G^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},z_i)} \subsetneq G^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}.$$

This contradicts the maximality of $\dim C^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$. Thus $z_i \notin G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$ for $1 \leq i \leq \ell$. We see that $z_i \notin G^{(-1)}_{t-1,(y_0,x_1,\ldots,y_{t-1},x_t)}$ for $0 \leq i \leq \ell - 1$ by the same way. Therefore,

$$z_i \notin G^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)} \quad \text{for } 1 \leq i \leq \ell - 1,$$

since $(y_0, x_1, \ldots, y_{t-1}, x_t)$ satisfies Condition N.

Now take elements $z_{\ell+1}, \ldots, z_{u-1}$, z_u such that

$$y_{t-1} = z_\ell \lessdot z_{\ell+1} \lessdot \cdots \lessdot z_{u-1} \lessdot z_u,$$

$z_i \notin G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$ for $\ell + 1 \leq i \leq u - 1$ and $z_u \in G^{(-1)}_{t,(y_0,x_1,\ldots,y_{t-1},x_t)}$ (u may be equal to $\ell + 1$). Then $z_i \notin G^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ for $\ell + 1 \leq i \leq u - 1$, since $(y_0, x_1, \ldots, y_{t-1}, x_t)$

satisfies Condition N.

Since $z_0 = x_t$, $\mathrm{dist}(x_t, z_u) + \mathrm{dist}(z_u, \infty) = \mathrm{dist}(x_t, \infty)$ and z_0 is not covered by z_u, we see that

$$\mathrm{dist}(z_0, \infty) - \mathrm{dist}(z_u, \infty) \geq 2.$$

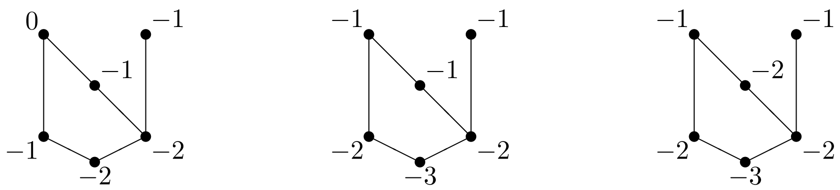
Therefore, we see by the same argument as in the case where $t = 0$, that the image of $C^{(-1)}_{(y_0,x_1,\ldots,y_{t-1},x_t)}$ of the composition of projection $\mathbb{R}^P \to \mathbb{R}^{\{z_1,\ldots,z_{u-1}\}}$ and the same unimodular transformation as in the case where $t = 0$ satisfies the condition of Lemma 7.7. □

Remark 8.3. Consider P_1 of Example 5.3. $x, y \in G^{(-1)}_{()}$, $\mathrm{dist}(x, \infty) - \mathrm{dist}(y, \infty) = 1$ and $\mathrm{dist}(x, y) = 2$. Therefore, $\mathrm{dist}(x, \infty) - \mathrm{dist}(y, \infty) = \mathrm{dist}(x, y)$ does not hold in general for $x, y \in G^{(-1)}_{()}$ with $x < y$. Thus, we need to prove $\mathrm{dist}(z_0, \infty) - \mathrm{dist}(z_u, \infty) \geq 2$ in the case of $t = 0$ of the proof of Lemma 8.2. In fact,

$$\dim C^{(-1)}_{()} = \max_{(y'_0,x'_1,\ldots,y'_{t'-1},x'_{t'}) \in N^{(-1)}} \dim C^{(-1)}_{(y'_0,x'_1,\ldots,y'_{t'-1},x'_{t'})}$$

is essential.

Example 8.4. Consider P_1 of Example 5.3. There are following 3 minimal elements



of $\mathcal{T}^{(-1)}(P)$. These are the vertices of $C^{(-1)}_{(y,x)}$ and the image of projection of $C^{(-1)}_{(y,x)}$ to $\mathbb{R}^{\{y,z\}}$ is a rectangular equilateral triangle with normalized volume 1.

Now we state the following.

Theorem 8.5. *If $\mathcal{R}_{\mathbb{K}}[H]$ is not Gorenstein, then*

$$\lim_{p \to \infty} \mathrm{cx}_F(\mathcal{R}_{\mathbb{K}}[H]) = \dim \left(\bigoplus_{n \geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)} \right) - 1.$$

Proof. By Fact 7.4, Remarks 6.4 and 7.5, we see that

$$\mathrm{cx}_F(\mathcal{R}_{\mathbb{K}}[H]) = \mathrm{Tcx}\left(T\left(\bigoplus_{n \geq 0} \omega^{(-n)}/\mathfrak{m}\omega^{(-n)} \right)\right).$$

On the other hand, by Lemma 7.6, we see that

$$\mathrm{Tcx}\left(T\left(\bigoplus_{n\geq 0}\omega^{(-n)}/\mathfrak{m}\omega^{(-n)}\right)\right)\leq \dim\left(\bigoplus_{n\geq 0}\omega^{(-n)}/\mathfrak{m}\omega^{(-n)}\right)-1$$

for any p. Further, by Theorem 5.1, Lemmas 8.1 and 8.2, we see that

$$\lim_{p\to\infty}\mathrm{Tcx}\left(T\left(\bigoplus_{n\geq 0}\omega^{(-n)}/\mathfrak{m}\omega^{(-n)}\right)\right)\geq \dim\left(\bigoplus_{n\geq 0}\omega^{(-n)}/\mathfrak{m}\omega^{(-n)}\right)-1. \qquad \square$$

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