

Strongly singular bilinear Calderón–Zygmund operators and a class of bilinear pseudodifferential operators

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Abstract. Motivated by the study of kernels of bilinear pseudodifferential operators with symbols in a Hörmander class of critical order, we investigate boundedness properties of strongly singular Calderón–Zygmund operators in the bilinear setting. For such operators, whose kernels satisfy integral-type conditions, we establish boundedness properties in the setting of Lebesgue spaces as well as endpoint mappings involving the space of functions of bounded mean oscillations and the Hardy space. Assuming pointwise-type conditions on the kernels, we show that strongly singular bilinear Calderón–Zygmund operators satisfy pointwise estimates in terms of maximal operators, which imply their boundedness in weighted Lebesgue spaces.

1. Introduction.

Bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes $BS_{\rho,\delta}^m$ are a priori defined from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ and have the form

$$T_\sigma(f, g)(x) := \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta \quad \forall x \in \mathbb{R}^n, f, g \in \mathcal{S}(\mathbb{R}^n);$$

given $0 \leq \delta \leq \rho \leq 1$ and $m \in \mathbb{R}$, the corresponding symbol σ belongs to the class $BS_{\rho,\delta}^m$ if $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ is an infinitely differentiable function such that for any given multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0$ there exists $C_{\alpha,\beta,\gamma} > 0$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha,\beta,\gamma} (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+\gamma)} \quad \forall x, \xi, \eta \in \mathbb{R}^n. \quad (1.1)$$

Such operators have been studied extensively and their boundedness properties have been proved in a variety of settings; see the articles Bényi–Bernicot–Maldonado–Naibo–Torres [2], Bényi–Maldonado–Naibo–Torres [3], Bényi–Torres [4], [5], Brummer–Naibo [6], Herbert–Naibo [13], [14], Koezuka–Tomita [15], Michalowski–Rule–Staubach [16], Miyachi–Tomita [17], [18], [19], Naibo [20], [21], Naibo–Thomson [22], Rodríguez-López–Staubach [23] and references therein.

The work in this manuscript is motivated by bilinear pseudodifferential operators with symbols in the class $BS_{\rho,\delta}^{-n(1-\rho)}$ with $0 \leq \delta \leq \rho < 1$. As shown in [2, Theorem 2.2], if $\sigma \in BS_{\rho,\delta}^m$ with $m < -n(1-\rho)$, then T_σ is bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$;

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moreover, it was proved in [17, Theorem A.2] that if $\sigma \in BS_{\rho,\delta}^m$ with $m > -n(1 - \rho)$, then T_σ may fail to be bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. In view of this, $-n(1 - \rho)$ and $BS_{\rho,\delta}^{-n(1-\rho)}$ are referred to as a *critical order* and a *critical bilinear Hörmander class*, respectively. If σ is in the critical class $BS_{\rho,\delta}^{-n(1-\rho)}$, then T_σ is also bounded from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$; this was first proved for $0 \leq \rho < 1/2$ and $\delta = 0$ in [2, Theorem 2.4] and then extended to $0 \leq \delta \leq \rho < 1/2$ in [21, Theorem 1.1]. Finally, the results in the recent manuscript [18] settled the boundedness from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ for all operators with symbols in the critical classes of order $-n(1 - \rho)$ with $0 < \rho < 1$.

In many instances, it is convenient to consider the kernel representation of bilinear pseudodifferential operators with symbols in $BS_{\rho,\delta}^m$, formally

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} K_\sigma(x, y, z) f(y) g(z) dy dz,$$

since the estimates of the symbol σ translate into quantitative estimates on the kernel K_σ , see [2, Theorem E]. For example, if $\sigma \in BS_{1,\delta}^0$ with $0 \leq \delta < 1$ or $\sigma \in BS_{\rho,\delta}^m$ with $m < -2n(1 - \rho)$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $\rho > 0$, then T_σ is a bilinear Calderón–Zygmund operator; that is, T_σ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and the kernel K_σ is a locally integrable function away from the diagonal $\Delta = \{(x, x, x) : x \in \mathbb{R}^n\}$ that, for some $\varepsilon > 0$, satisfies estimates of the form

$$\begin{aligned} |K_\sigma(x, y, z)| &\lesssim (|x - y| + |x - z|)^{-2n}, \\ |K_\sigma(x + h, y, z) - K_\sigma(x, y, z)| + |K_\sigma(x, y + h, z) - K_\sigma(x, y, z)| + \\ &|K_\sigma(x, y, z + h) - K_\sigma(x, y, z)| \lesssim |h|^\varepsilon (|x - y| + |x - z|)^{-2n-\varepsilon}, \end{aligned}$$

for $(x, y, z) \in \mathbb{R}^{3n} \setminus \Delta$ and $|h| \leq (|x - y| + |x - z|)/2$. In particular, the bilinear Calderón–Zygmund theory established in Coifman–Meyer [7] and Grafakos–Torres [12] gives that T_σ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ with $1 < p, q < \infty$ and $1/r = 1/p + 1/q$, and from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. However, except for some ranges of indices as those mentioned above, the classes $BS_{\rho,\delta}^m$ do not necessarily give rise to bilinear Calderón–Zygmund operators and boundedness properties for the corresponding operators have been studied in the references cited at the beginning of this section.

In this article, we show that bilinear pseudodifferential operators with symbols in the critical classes $BS_{\rho,\delta}^{-n(1-\rho)}$ possess kernels that lie barely outside the scope of the bilinear Calderón–Zygmund theory. More specifically, the kernels of such operators are shown to be *strongly singular bilinear Calderón–Zygmund kernels*, and when $0 < \rho < 1/2$, the operators are shown to be *strongly singular bilinear Calderón–Zygmund operators*. For the latter operators, whose kernels satisfy integral-type conditions, we establish boundedness properties in the setting of Lebesgue spaces as well as the endpoint mappings from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$, from $\mathcal{H}^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n) \times \mathcal{H}^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where $\mathcal{H}^1(\mathbb{R}^n)$ denotes the Hardy space. Assuming pointwise conditions on the kernels, we also show that strongly singular bilinear Calderón–Zygmund operators satisfy pointwise estimates in terms of the sharp maximal operator

and the Hardy–Littlewood maximal operator, which imply their boundedness in weighted Lebesgue spaces.

The article is organized as follows. In Section 2, we define strongly singular bilinear Calderón–Zygmund operators and state and prove their boundedness properties in unweighted Lebesgue spaces as well as endpoint mappings involving $BMO(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)$. Pointwise estimates in terms of maximal operators and weighted estimates in Lebesgue spaces for such singular operators are presented in Section 3. Section 4 discusses the realization of bilinear pseudodifferential operators with symbols in the critical classes $BS_{\rho,\delta}^{-n(1-\rho)}$ as strongly singular bilinear Calderón–Zygmund operators.

We end this section with definitions and notation used throughout the manuscript.

DEFINITIONS AND NOTATION. The Hardy–Littlewood maximal operator will be denoted by \mathcal{M} , that is,

$$\mathcal{M}(f)(x) = \sup_{x \in Q} \int_Q |f(y)| \, dy \quad \forall x \in \mathbb{R}^n, f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x and $\int_Q |f(y)| \, dy = (1/|Q|) \int_Q |f(y)| \, dy$.

A weight on \mathbb{R}^n is a non-negative locally integrable function defined on \mathbb{R}^n . Given $1 < p < \infty$, the Muckenhoupt class A_p is defined as the family of weights w on \mathbb{R}^n such that

$$\sup_Q \left(\int_Q w(y) \, dy \right) \left(\int_Q w(y)^{1-p'} \, dy \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and p' is the conjugate exponent of p . We set $A_\infty := \bigcup_{p>1} A_p$.

Given a weight on \mathbb{R}^n and $0 < p \leq \infty$, the notation $L^p_w(\mathbb{R}^n)$ means the Lebesgue space with respect to the measure $w(x) \, dx$; if $w \equiv 1$ we simply write $L^p(\mathbb{R}^n)$. We recall that for $1 < p < \infty$, $w \in A_p$ if and only if \mathcal{M} is bounded on $L^p_w(\mathbb{R}^n)$.

The sharp maximal operator $\mathcal{M}^\#$ is defined by

$$\mathcal{M}^\#(f)(x) = \sup_{x \in Q} \int_Q |f(y) - f_Q| \, dy \quad \forall x \in \mathbb{R}^n, f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

where $f_Q = \int_Q f(y) \, dy$ and the supremum is taken over cubes $Q \subset \mathbb{R}^n$ containing x .

The space of functions of bounded mean oscillations, $BMO(\mathbb{R}^n)$, consists of all measurable functions defined on \mathbb{R}^n (identified modulo constants) such that

$$\|f\|_{BMO} := \|\mathcal{M}^\#(f)\|_{L^\infty} < \infty.$$

We recall that the dual of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$.

The Schwartz class of smooth rapidly decreasing functions on \mathbb{R}^n will be denoted by $\mathcal{S}(\mathbb{R}^n)$ and its dual, the class of tempered distributions, by $\mathcal{S}'(\mathbb{R}^n)$. The notation $C^\infty_c(\mathbb{R}^n)$ will mean the space of compactly supported infinitely differentiable functions

defined on \mathbb{R}^n . The space of bounded measurable functions defined on \mathbb{R}^n that have compact support will be indicated by $L_c^\infty(\mathbb{R}^n)$.

The Fourier transform of a tempered distribution f is denoted by \widehat{f} .

The notation $A \lesssim B$ means $A \leq cB$, where c is a constant that may depend on some of the parameters and weights used but not on the functions involved.

2. Strongly singular bilinear Calderón–Zygmund operators.

In this section we first present the definitions of strongly singular bilinear Calderón–Zygmund kernels and operators, which are inspired by the work in the linear setting of Fefferman [10], Fefferman–Stein [11] and Alvarez–Milman [1]. We then state and prove boundedness properties of these operators in the context of Lebesgue spaces, $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$.

Let $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a continuous bilinear operator and \mathcal{K} be a complex-valued locally integrable function defined on $\mathbb{R}^{3n} \setminus \Delta$. We say that T is associated to \mathcal{K} if for any $f, g \in C_c^\infty(\mathbb{R}^n)$,

$$T(f, g)(x) = \int_{\mathbb{R}^{2n}} \mathcal{K}(x, y, z) f(y) g(z) dy dz \quad \forall x \notin \text{supp}(f) \cap \text{supp}(g). \tag{2.2}$$

The formal transposes of the operator T will be denoted by T^{*1} and T^{*2} and are defined by

$$\langle T(f, g), h \rangle = \langle T^{*1}(h, g), f \rangle = \langle T^{*2}(f, h), g \rangle \quad \forall f, g, h \in \mathcal{S}(\mathbb{R}^n).$$

It follows that the kernels \mathcal{K}^{*j} of T^{*j} for $j = 1, 2$ are given by

$$\mathcal{K}^{*1}(x, y, z) = \mathcal{K}(y, x, z) \quad \text{and} \quad \mathcal{K}^{*2}(x, y, z) = \mathcal{K}(z, y, x).$$

DEFINITION 2.1. Let $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a continuous bilinear operator associated to a complex-valued locally integrable function \mathcal{K} defined on $\mathbb{R}^{3n} \setminus \Delta$. We say that T is a *strongly singular bilinear Calderón–Zygmund operator* if the following conditions hold:

- (C1) T can be extended to a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$;
- (C2) there exists $0 < \varepsilon < 1$ such that

$$\sup_{x, x' \in \mathbb{R}^n} \int_{|x-x'|^\varepsilon \lesssim |x-y|+|x-z|} |\mathcal{K}(x, y, z) - \mathcal{K}(x', y, z)| dy dz < \infty, \tag{2.3}$$

$$\sup_{y, y' \in \mathbb{R}^n} \int_{|y-y'|^\varepsilon \lesssim |y-x|+|y-z|} |\mathcal{K}(x, y, z) - \mathcal{K}(x, y', z)| dx dz < \infty, \tag{2.4}$$

$$\sup_{z, z' \in \mathbb{R}^n} \int_{|z-z'|^\varepsilon \lesssim |z-x|+|z-y|} |\mathcal{K}(x, y, z) - \mathcal{K}(x, y, z')| dx dy < \infty; \tag{2.5}$$

- (C3) T, T^{*1} and T^{*2} can be extended to bounded operators from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{1/\varepsilon}(\mathbb{R}^n)$ with ε as given above.

A straightforward example of a kernel that satisfies (2.3) is the following: Let \mathcal{K} be a locally integrable function defined on $\mathbb{R}^{3n} \setminus \Delta$ such that there exist $0 < s \leq 1$ and $0 < \varepsilon < 1$ for which

$$|\mathcal{K}(x, y, z) - \mathcal{K}(x', y, z)| \lesssim \frac{|x - x'|^s}{(|x - y| + |x - z|)^{2n+s/\varepsilon}} \quad \text{if } |x - x'|^\varepsilon \lesssim |x - y| + |x - z|. \tag{2.6}$$

Similar statements follow in relation to (2.4) and (2.5).

REMARK 2.1. We note that if $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is a bilinear continuous operator associated to a kernel \mathcal{K} that can be extended to a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for some $1 \leq r < \infty$ then (2.2) holds for $f, g \in L_c^\infty(\mathbb{R}^n)$ and for almost every $x \notin \text{supp}(f) \cap \text{supp}(g)$. This follows from a limiting argument and will be implicitly used throughout the proofs.

We next state and prove the main results in this section.

THEOREM 2.1. *Let T be a strongly singular bilinear Calderón–Zygmund operator. Then T can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$, from $\mathcal{H}^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, from $L^\infty(\mathbb{R}^n) \times \mathcal{H}^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for all p, q, r satisfying $1 \leq p, q \leq \infty$, $1 \leq r < \infty$, $1/r = 1/p + 1/q$, $(1/p, 1/q, 1/r) \neq (1, 0, 1)$ and $(1/p, 1/q, 1/r) \neq (0, 1, 1)$.*

The proof of Theorem 2.1, detailed at the end of this section, will be a consequence of the following theorem, duality and bilinear interpolation.

THEOREM 2.2. *Let $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a bilinear continuous operator associated to a complex-valued locally integrable function \mathcal{K} defined on $\mathbb{R}^{3n} \setminus \Delta$ that verifies condition (2.3) for some $0 < \varepsilon < 1$. Assume also that T can be extended to a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ (or from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ or from $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$) and from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{1/\varepsilon}(\mathbb{R}^n)$. Then T can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.*

PROOF. Note that T is well-defined on $L_c^\infty(\mathbb{R}^n) \times L_c^\infty(\mathbb{R}^n)$ since T is defined on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and recall that, as pointed out in Remark 2.1, (2.2) holds for $f, g \in L_c^\infty(\mathbb{R}^n)$ and for almost every $x \notin \text{supp}(f) \cap \text{supp}(g)$. We will prove that for all cubes $Q \subset \mathbb{R}^n$ there is $C_Q \in \mathbb{C}$ such that

$$\int_Q |T(f, g)(x) - C_Q| \, dx \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty} \quad \forall f, g \in L_c^\infty(\mathbb{R}^n),$$

where the implicit constant is independent of Q . This gives that T is bounded from $L_c^\infty(\mathbb{R}^n) \times L_c^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. We refer the reader to Appendix A regarding how T can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.

Fix a cube Q contained in \mathbb{R}^n of side length $d > 0$ and center x_Q . Define \tilde{Q} as the cube with center x_Q and side length $2d^\varepsilon$ if $d \leq 1$ or side length $2d$ if $d > 1$; note that $Q \subset \tilde{Q}$. For $f, g \in L_c^\infty(\mathbb{R}^n)$, we write

$$T(f, g) = T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}}) + T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c}) + T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}}) + T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c}).$$

We first estimate the term $T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})$. If $d \leq 1$ apply Hölder’s inequality, the boundedness of T from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{1/\varepsilon}(\mathbb{R}^n)$ and the fact that $|\tilde{Q}| \sim |Q|^\varepsilon$ to get that

$$\begin{aligned} \int_Q |T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})(x)| \, dx &\leq \left(\int_Q |T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})(x)|^{1/\varepsilon} \, dx \right)^\varepsilon \\ &\lesssim \left(\int_{\tilde{Q}} |f(y)|^2 \, dy \right)^{1/2} \left(\int_{\tilde{Q}} |g(z)|^2 \, dz \right)^{1/2} \\ &\lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}. \end{aligned}$$

When $d > 1$ we proceed similarly using the boundedness of T from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ (or from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ or from $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$) and that $|\tilde{Q}| \sim |Q|$, which implies

$$\int_Q |T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})(x)| \, dx \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}.$$

We will next prove that

$$\int_Q |T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})(x) - C_{Q,1}| \, dx \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}, \tag{2.7}$$

$$\int_Q |T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}})(x) - C_{Q,2}| \, dx \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}, \tag{2.8}$$

$$\int_Q |T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c})(x) - C_{Q,3}| \, dx \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}, \tag{2.9}$$

with the implicit constants independent of f, g and Q , and where

$$\begin{aligned} C_{Q,1} &:= \int_{\mathbb{R}^{2n}} \mathcal{K}(x_Q, y, z) f(y) \chi_{\tilde{Q}}(y) g(z) \chi_{\tilde{Q}^c}(z) \, dydz, \\ C_{Q,2} &:= \int_{\mathbb{R}^{2n}} \mathcal{K}(x_Q, y, z) f(y) \chi_{\tilde{Q}^c}(y) g(z) \chi_{\tilde{Q}}(z) \, dydz, \\ C_{Q,3} &:= \int_{\mathbb{R}^{2n}} \mathcal{K}(x_Q, y, z) f(y) \chi_{\tilde{Q}^c}(y) g(z) \chi_{\tilde{Q}^c}(z) \, dydz. \end{aligned}$$

We will show (2.7), with the estimates (2.8) and (2.9) following in the same way. Note that if $x \in Q$, $y \in \tilde{Q}$ and $z \in \tilde{Q}^c$ then

$$\begin{aligned} |x - x_Q|^\varepsilon &\lesssim d^\varepsilon \lesssim |x_Q - z| \lesssim |x_Q - z| + |x_Q - y| \quad \text{if } d \leq 1, \\ |x - x_Q|^\varepsilon &\lesssim d^\varepsilon \leq d \lesssim |x_Q - z| \lesssim |x_Q - z| + |x_Q - y| \quad \text{if } d > 1. \end{aligned}$$

Therefore, for $x \in Q$ and using (2.3), we have

$$|T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})(x) - C_{Q,1}|$$

$$\begin{aligned} &\lesssim \int_{y \in \tilde{Q}, z \in \tilde{Q}^c} |\mathcal{K}(x, y, z) - \mathcal{K}(x_Q, y, z)| |f(y)| |g(z)| \, dydz \\ &\lesssim \int_{|x-x_Q|^\varepsilon \lesssim |x_Q-z|+|x_Q-y|} |\mathcal{K}(x, y, z) - \mathcal{K}(x_Q, y, z)| |f(y)| |g(z)| \, dydz \\ &\lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}. \end{aligned}$$

Averaging in x over Q we obtain (2.7). □

We end this section with the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Since T can be extended to a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, a duality argument implies that T^{*1} and T^{*2} can be extended to bounded operators from $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ and from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, respectively. Theorem 2.2 then gives that T, T^{*1} and T^{*2} can be extended to bounded operators from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. As a consequence, by duality, T can be extended to a bounded operator from $\mathcal{H}^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n) \times \mathcal{H}^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Bilinear complex interpolation implies that T is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for all p, q, r such that $(1/p, 1/q, 1/r)$ is in the convex hull of the points $(0, 0, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. That is, $1 \leq p, q \leq \infty$, $1 \leq r < \infty$, $1/r = 1/p + 1/q$, $(1/p, 1/q, 1/r) \neq (1, 0, 1)$ and $(1/p, 1/q, 1/r) \neq (0, 1, 1)$. □

3. Pointwise inequalities and weighted estimates for strongly singular bilinear Calderón–Zygmund operators.

In this section we show that Theorem 2.2 can be improved if the stronger condition (2.6) is assumed, by proving a pointwise inequality in terms of the sharp maximal operator and the Hardy–Littlewood maximal operator. In particular, such result will imply weighted estimates for strongly singular bilinear Calderón–Zygmund operators that satisfy (2.6).

THEOREM 3.1. *Let $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a bilinear continuous operator associated to a complex-valued locally integrable function \mathcal{K} defined on $\mathbb{R}^{3n} \setminus \Delta$ that verifies condition (2.6) for some $0 < \varepsilon < 1$ and $0 < s \leq 1$. Assume also that T can be extended to a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{1/\varepsilon}(\mathbb{R}^n)$. Then T satisfies*

$$\mathcal{M}^\#(T(f, g))(x) \lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall x \in \mathbb{R}^n, f, g \in L_c^\infty(\mathbb{R}^n), \tag{3.10}$$

where $\mathcal{M}_2(h) = (\mathcal{M}(|h|^2))^{1/2}$.

COROLLARY 3.2. *If $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ satisfies the hypothesis of Theorem 3.1, the following statements hold true:*

- (a) T can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.

(b) Let $2 < p, q < \infty$ and r be such that $1/r = 1/p + 1/q$; consider $v \in A_{p/2}$, $w \in A_{q/2}$ and define $u := v^{r/p}w^{r/q}$. Then T can be extended to a bounded operator from $L_v^p(\mathbb{R}^n) \times L_w^q(\mathbb{R}^n)$ into $L_u^r(\mathbb{R}^n)$.

REMARK 3.1. If T is a strongly singular bilinear Calderón–Zygmund operator that satisfies (2.6), then, besides the results from Theorem 2.1, T satisfies (3.10) and the weighted estimates stated in item (b) of Corollary 3.2. If T satisfies (2.6) and its symmetric counterparts, then (3.10) and item (b) of Corollary 3.2 also hold for T^{*1} and T^{*2} .

We briefly comment on the proof of Corollary 3.2 and then prove Theorem 3.1. Note that item (a) of Corollary 3.2 follows from Theorem 2.2, but of course, it can be inferred from (3.10): such inequality gives that T_σ is bounded from $L_c^\infty(\mathbb{R}^n) \times L_c^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$, which in turn implies that T_σ can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ as shown in Appendix A. For (b) we use the weighted Fefferman–Stein inequality (see Fefferman–Stein [11] and Cruz–Uribe–Martell–Pérez [8, Theorem 1.3]) to get that

$$\|T_\sigma(f, g)\|_{L_u^r} \leq \|\mathcal{M}(T_\sigma(f, g))\|_{L_u^r} \lesssim \|\mathcal{M}^\#(T_\sigma(f, g))\|_{L_u^r};$$

we then use (3.10), Hölder’s inequality and the boundedness properties in weighted Lebesgue spaces of \mathcal{M}_2 to obtain

$$\|T_\sigma(f, g)\|_{L_u^r} \lesssim \|f\|_{L_v^p} \|g\|_{L_w^q} \quad \forall f, g \in L_c^\infty(\mathbb{R}^n).$$

T_σ can then be extended by density to a bounded operator from $L_v^p(\mathbb{R}^n) \times L_w^q(\mathbb{R}^n)$ into $L_u^r(\mathbb{R}^n)$.

PROOF OF THEOREM 3.1. We follow ideas from [21, Theorem 2.2]. We have to prove that for all cubes $Q \subset \mathbb{R}^n$ there is a constant $C_Q \in \mathbb{C}$ such that

$$\int_Q |T(f, g)(y) - C_Q| dy \lesssim \mathcal{M}_2(f)(x)\mathcal{M}_2(g)(x) \quad \forall x \in Q, f, g \in L_c^\infty(\mathbb{R}^n),$$

where the implicit constant is independent of Q .

Fix a cube $Q \subset \mathbb{R}^n$ of side length $d > 0$ and center x_Q ; let \tilde{Q} be as in the proof of Theorem 2.2. For $f, g \in L_c^\infty(\mathbb{R}^n)$ consider

$$T(f, g) = T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}}) + T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c}) + T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}}) + T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c}).$$

The term $T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})$ is controlled using the boundedness of T in the same way done in the proof of Theorem 2.2. If $d \leq 1$, we apply Hölder’s inequality and the boundedness of T from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{1/\varepsilon}(\mathbb{R}^n)$ to get that for all $x \in Q$

$$\int_Q |T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})(v)| dv \lesssim \left(\int_{\tilde{Q}} |f(y)|^2 dy\right)^{1/2} \left(\int_{\tilde{Q}} |g(z)|^2 dz\right)^{1/2} \lesssim \mathcal{M}_2(f)(x)\mathcal{M}_2(g)(x).$$

When $d > 1$ we proceed similarly using the boundedness of T from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, which implies

$$\int_Q \left| T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})(v) \right| dv \lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall x \in Q.$$

For the terms $T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})$, $T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}})$ and $T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c})$, we will prove that

$$\begin{aligned} \int_Q \left| T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})(v) - C_{Q,1} \right| dv &\lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall x \in Q, \\ \int_Q \left| T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}})(v) - C_{Q,2} \right| dv &\lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall x \in Q, \\ \int_Q \left| T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c})(v) - C_{Q,3} \right| dv &\lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall x \in Q, \end{aligned}$$

where $C_{Q,1}, C_{Q,2}, C_{Q,3}$ are as in the proof of Theorem 2.2.

Define $\mathcal{L}_Q(v, y, z) = \mathcal{K}(v, y, z) - \mathcal{K}(x_Q, y, z)$; set $t = \varepsilon$ if $d < 1$ and $t = 1$ if $d > 1$. For $v, x \in Q$ we have that

$$\begin{aligned} \left| T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})(v) - C_{Q,1} \right| &\lesssim \sum_{k=1}^{\infty} \int_{\substack{y \in \tilde{Q} \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)| |f(y)| |g(z)| dydz \\ &\lesssim \sum_{k=1}^{\infty} \left(\int_{\substack{y \in \tilde{Q} \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)|^2 dydz \right)^{1/2} \left(\int_{\tilde{Q}} |f(y)|^2 dy \right)^{1/2} \left(\int_{|z-x_Q| \sim (2^k d)^t} |g(z)|^2 dz \right)^{1/2} \\ &\lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \sum_{k=1}^{\infty} d^{nt} 2^{k(n/2)t} \left(\int_{\substack{y \in \tilde{Q} \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)|^2 dydz \right)^{1/2}. \end{aligned}$$

We next show that the sum above is bounded by a constant independent of d, v and Q . Indeed, we have

$$\begin{aligned} \int_{\substack{y \in \tilde{Q} \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)|^2 dydz &\lesssim \int_{\substack{y \in \tilde{Q} \\ |z-x_Q| \sim (2^k d)^t}} \left(\frac{|v-x_Q|^s}{(|x_Q-y|+|x_Q-z|)^{2n+s/\varepsilon}} \right)^2 dydz \\ &\sim |v-x_Q|^{2s} |\tilde{Q}| (2^k d)^{-2t(2n+s/\varepsilon)+tn} \\ &\lesssim d^{2s} d^{tn} (2^k d)^{-2t(2n+s/\varepsilon)+tn}, \end{aligned}$$

where in the first inequality we have used that $|z-x_Q| \sim (2^k d)^t$ implies that $|v-x_Q|^\varepsilon \lesssim |x_Q-y|+|x_Q-z|$ since $v \in Q$ and in the third inequality we have used that $v \in Q$ and the definition of \tilde{Q} . Then, the sum above is controlled by

$$\begin{aligned} \sum_{k=1}^{\infty} d^{nt} 2^{k(n/2)t} (d^{2s} d^{tn} (2^k d)^{-2t(2n+s/\varepsilon)+tn})^{1/2} &= \sum_{k=1}^{\infty} d^{s(1-t/\varepsilon)} 2^{-kt(n+s/\varepsilon)} \\ &\leq \sum_{k=1}^{\infty} 2^{-kt(n+s/\varepsilon)} < \infty. \end{aligned}$$

We have then obtained that

$$\left| T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})(v) - C_{Q,1} \right| \lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall v, x \in Q.$$

Integrating in v over Q it follows that

$$\int_Q \left| T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}^c})(v) - C_{Q,1} \right| dv \lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall x \in Q,$$

as desired. The term $T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}})(v)$ is treated analogously. For the term $T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c})$ we proceed similarly as above and get that if $v, x \in Q$ then

$$\begin{aligned} \left| T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c})(v) - C_{Q,3} \right| &\lesssim \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{\substack{|y-x_Q| \sim (2^j d)^t \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)| |f(y)| |g(z)| dydz \\ &\lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} d^{nt} 2^{(j+k)(n/2)t} \left(\int_{\substack{|y-x_Q| \sim (2^j d)^t \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)|^2 dydz \right)^{1/2}. \end{aligned}$$

This last sum is finite and controlled by a constant independent of d, v and Q as the following computation shows. We have

$$\begin{aligned} \int_{\substack{|y-x_Q| \sim (2^j d)^t \\ |z-x_Q| \sim (2^k d)^t}} |\mathcal{L}_Q(v, y, z)|^2 dydz &\lesssim \int_{\substack{|y-x_Q| \sim (2^j d)^t \\ |z-x_Q| \sim (2^k d)^t}} \left(\frac{|v-x_Q|^s}{(|x_Q-y|+|x_Q-z|)^{2n+s/\varepsilon}} \right)^2 dydz \\ &\sim |v-x_Q|^{2s} \left(\int_{|y-x_Q| \sim (2^j d)^t} \frac{1}{|x_Q-y|^{2n+s/\varepsilon}} dy \right) \left(\int_{|z-x_Q| \sim (2^k d)^t} \frac{1}{|x_Q-z|^{2n+s/\varepsilon}} dz \right) \\ &\lesssim d^{2s} (2^j d)^{-t(n+s/\varepsilon)} (2^k d)^{-t(n+s/\varepsilon)}, \end{aligned}$$

which gives that the sum above is controlled by

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} d^{nt} 2^{(j+k)(n/2)t} (d^{2s} (2^j d)^{-t(n+s/\varepsilon)} (2^k d)^{-t(n+s/\varepsilon)})^{1/2} \\ = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} d^{s(1-t/\varepsilon)} 2^{-j(ts/(2\varepsilon))} 2^{-k(ts/(2\varepsilon))} \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} 2^{-j(ts/(2\varepsilon))} 2^{-k(ts/(2\varepsilon))} < \infty. \end{aligned}$$

We then obtain that

$$\left| T(f\chi_{\tilde{Q}^c}, g\chi_{\tilde{Q}^c})(v) - C_{Q,3} \right| \lesssim \mathcal{M}_2(f)(x) \mathcal{M}_2(g)(x) \quad \forall v, x \in Q,$$

and integrating in v over Q gives the expected inequality. □

4. The critical bilinear Hörmander class $BS_{\rho,\delta}^{-n(1-\rho)}$.

In this section, we show that if $\sigma \in BS_{\rho,\delta}^{-n(1-\rho)}$ with $0 \leq \delta \leq \rho$ and $0 < \rho < 1$ then the kernel of T_σ satisfies condition (2.3) with $\varepsilon = \rho$. By the symbolic calculus of the bilinear Hörmander classes proved in [3, Theorem 2.1], (2.4) and (2.5) also follow with $\varepsilon = \rho$. As a consequence, T_σ has a strongly singular bilinear Calderón–Zygmund kernel.

The operator T_σ is known to be bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ (see [2] and references therein); it was shown in [21, Lemma 2.1] that if the condition $0 < \rho < 1/2$ is further imposed, then T_σ and its transposes are also bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^{1/\rho}(\mathbb{R}^n)$. Therefore, T_σ with such a symbol becomes a strongly singular bilinear Calderón–Zygmund operator. In particular, Theorem 2.1 applied to T_σ recovers [21, Theorem 1.1]. Note that T_σ does not necessarily satisfies the hypothesis of Theorem 3.1 since its kernel may fail to satisfy the stronger condition (2.6); however, the pointwise inequality (3.10) does hold for T_σ as proved in [21, Theorem 2.2]. Therefore, not only does T_σ satisfy all boundedness properties stated in Theorem 2.1 but it also verifies the weighted estimated from part (b) of Corollary 3.2.

Fix $\sigma \in BS_{\rho,\delta}^{-n(1-\rho)}$ with $0 \leq \delta \leq \rho$ and $0 < \rho < 1$. We next proceed to show that the kernel of T_σ , which we denote by K , satisfies (2.3) with $\varepsilon = \rho$.

We begin by showing that

$$\sup_{|x-x'| \geq 1} \int_{R_\rho} |K(x, y, z) - K(x', y, z)| \, dydz < \infty, \tag{4.11}$$

where for fixed x, x' , we denote

$$R_\rho := \{(y, z) \in \mathbb{R}^{2n} : 3|x - x'|^\rho \leq |x - y| + |x - z|\};$$

let us also write

$$R := \{(y, z) \in \mathbb{R}^{2n} : |x - x'| \leq |x - y| + |x - z|\}.$$

Fix $x, x' \in \mathbb{R}^n$ so that $|x - x'| \geq 1$. Using [2, Theorem E (v)], we obtain that for $(y, z) \in R_\rho$ we have

$$|K(x, y, z)| \lesssim (|x - y| + |x - z|)^{-n(1+1/\rho)}.$$

Furthermore, if $(y, z) \in R$, by [2, Theorem E (vi)] we also get

$$|K(x, y, z) - K(x', y, z)| \lesssim |x - x'|^\rho (|x - y| + |x - z|)^{-n(1+1/\rho)-1}.$$

Note that $n(1 + 1/\rho) = 2n + n(1 - \rho)/\rho > 2n$. We break down the left hand side of (4.11) into two cases. In the first case, let us further assume that $|x - x'| \leq 3^{1/(1-\rho)}$. Then, $R_\rho \subset R$ and we can write

$$\begin{aligned} \int_{R_\rho} |K(x, y, z) - K(x', y, z)| \, dydz &\lesssim \int_{R_\rho} |x - x'|^\rho (|x - y| + |x - z|)^{-n(1+1/\rho)-1} \, dydz \\ &\lesssim \int_{R_\rho} (|x - y| + |x - z|)^{-n(1+1/\rho)} \, dydz \\ &\lesssim |x - x'|^{-n(1-\rho)} \lesssim 1. \end{aligned}$$

Secondly, let us now assume that $|x - x'| > 3^{1/(1-\rho)}$; in this case $R \subset R_\rho$. Write

$$\int_{R_\rho} |K(x, y, z) - K(x', y, z)| \, dydz = I + II,$$

with $I = \int_R |K(x, y, z) - K(x', y, z)| \, dydz$ and $II = \int_{R_\rho \setminus R} |K(x, y, z) - K(x', y, z)| \, dydz$. We estimate I as follows:

$$\begin{aligned} I &\lesssim \int_R |x - x'|^\rho (|x - y| + |x - z|)^{-n(1+1/\rho)-1} \, dydz \\ &\lesssim \int_R (|x - y| + |x - z|)^{-n(1+1/\rho)-1+\rho} \, dydz \\ &\lesssim |x - x'|^{n(1-1/\rho)-1+\rho} \lesssim 1. \end{aligned}$$

We estimate II as follows:

$$II \leq \int_{R_\rho \setminus R} |K(x, y, z)| \, dydz + \int_{R_\rho \setminus R} |K(x', y, z)| \, dydz.$$

Now,

$$\int_{R_\rho \setminus R} |K(x, y, z)| \, dydz \lesssim \int_{R_\rho} (|x - y| + |x - z|)^{-n(1+1/\rho)} \, dydz \lesssim |x - x'|^{-n(1-\rho)} \lesssim 1.$$

Note also that, if $(y, z) \in R_\rho \setminus R$, we have $|x - y| + |x - z| < |x - x'|$, hence

$$|x' - y| + |x' - z| \geq 2|x - x'| - (|x - y| + |x - z|) > |x - x'|.$$

Thus, by appealing again to [2, Theorem E (v)], we can write

$$\begin{aligned} \int_{R_\rho \setminus R} |K(x', y, z)| \, dydz &\lesssim \int_{|x-x'| < |x'-y|+|x'-z|} (|x' - y| + |x' - z|)^{-n(1+1/\rho)} \, dydz \\ &\lesssim |x - x'|^{n(1-1/\rho)} \lesssim 1. \end{aligned}$$

This finishes the proof of (4.11).

It remains to show that the following estimate also holds:

$$\sup_{0 < |x-x'| < 1} \int_{R_\rho} |K(x, y, z) - K(x', y, z)| \, dydz < \infty. \tag{4.12}$$

Note that the approach used for the case $|x - x'| > 1$ using the pointwise estimates for K does not work anymore since this leads to negative powers of $|x - x'|$. In order to prove

(4.12) we follow the idea in [1]; see also [24, Chapter VII, p.322]. Let $\varphi(\xi, \eta)$ and $\psi(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be infinitely differentiable functions so that $\text{supp } \varphi \subset \{(\xi, \eta) : |\xi| + |\eta| \leq 2\}$, $\text{supp } \psi \subset \{(\xi, \eta) : 1/2 \leq |\xi| + |\eta| \leq 2\}$ and $\varphi(\xi, \eta) + \sum_{j=1}^\infty \psi(2^{-j}\xi, 2^{-j}\eta) = 1$ for all $\xi, \eta \in \mathbb{R}^n$. Fix $x, x' \in \mathbb{R}^n$ so that $|x - x'| < 1$. For $j \geq 0$, we write

$$K_j(x, y, z) = \int_{\mathbb{R}^{2n}} \sigma_j(x, \xi, \eta) e^{i(x-y)\cdot\xi} e^{i(x-z)\cdot\eta} d\xi d\eta,$$

where

$$\sigma_0(x, \xi, \eta) = \sigma(x, \xi, \eta)\varphi(\xi, \eta) \quad \text{and} \quad \sigma_j(x, \xi, \eta) = \sigma(x, \xi, \eta)\psi(2^{-j}\xi, 2^{-j}\eta) \text{ for } j \geq 1.$$

It suffices to estimate

$$\int_{R_\rho} |K_j(x, y, z) - K_j(x', y, z)| dydz \lesssim c_j(x - x'),$$

where the functions c_j satisfy $\sum_{j=0}^\infty c_j(x - x') \lesssim 1$ uniformly in x, x' .

As a first instance, we estimate crudely

$$\int_{R_\rho} |K_j(x, y, z) - K_j(x', y, z)| dydz \leq \int_{R_\rho} |K_j(x, y, z)| dydz + \int_{R_\rho} |K_j(x', y, z)| dydz.$$

Note that, since $|x - x'|^\rho \geq |x - x'|$, if $(y, z) \in R_\rho$, then

$$|x' - y| + |x' - z| \geq |x - y| + |x - z| - 2|x - x'| \geq 3|x - x'|^\rho - 2|x - x'| \geq |x - x'|^\rho.$$

Thus, it suffices to estimate $\int_{R_\rho} |K_j(x, y, z)| dydz$.

Let $N \in 2\mathbb{N}$ be so that $N > n$. By using Hölder's inequality, the integral of $|K_j(x, y, z)|$ over R_ρ is less than

$$\|(|x - y|^2 + |x - z|^2)^{N/2} K_j(x, y, z)\|_{L^2(dydz)} \|(|x - y|^2 + |x - z|^2)^{-N/2} \chi_{R_\rho}\|_{L^2(dydz)}.$$

Using polar coordinates immediately gives that

$$\|(|x - y|^2 + |x - z|^2)^{-N/2} \chi_{R_\rho}\|_{L^2(dydz)} \sim |x - x'|^{\rho(n-N)}.$$

We now estimate the first L^2 norm above. Integration by parts gives

$$\begin{aligned} (|x - y|^2 + |x - z|^2)^{N/2} K_j(x, y, z) &= (-1)^{N/2} \int_{\mathbb{R}^{2n}} \Delta_{\xi, \eta}^{N/2} \sigma_j(x, \xi, \eta) e^{i(x-y)\cdot\xi + i(x-z)\cdot\eta} d\xi d\eta \\ &= \sum_{|\alpha_1 + \alpha_2| = N} c_{\alpha_1, \alpha_2} \int_{\mathbb{R}^{2n}} \partial_{\xi, \eta}^{\alpha_1} \sigma(x, \xi, \eta) 2^{-j|\alpha_2|} (\partial_{\xi, \eta}^{\alpha_2} \psi)(2^{-j}\xi, 2^{-j}\eta) e^{i(x-y)\cdot\xi + i(x-z)\cdot\eta} d\xi d\eta, \end{aligned}$$

where ψ should be replaced by φ if $j = 0$. Now, using Plancherel's theorem, the conditions on the symbol σ and the supports of φ and ψ , we can estimate

$$\begin{aligned} \|(|x - y|^2 + |x - z|^2)^{N/2} K_j(x, y, z)\|_{L^2(dydz)} &\lesssim \sum_{|\alpha_1 + \alpha_2| = N} (2^j)^{n(\rho-1) - \rho|\alpha_1|} 2^{-j\rho|\alpha_2|} 2^{jn} \\ &\sim 2^{j\rho(n-N)}. \end{aligned}$$

We have thus obtained that

$$\int_{R_\rho} |K_j(x, y, z) - K_j(x', y, z)| \, dydz \lesssim (2^j|x - x'|)^{\rho(n-N)} \quad \forall j \in \mathbb{N}_0. \tag{4.13}$$

It turns out that we can improve the estimate (4.13) if we further require $2^j|x - x'| \leq 1$; we show this next. As in the above calculation, we let $N \in 2\mathbb{N}$ be so that $N > n$ and apply Hölder’s inequality to get

$$\int_{R_\rho} |K_j(x, y, z) - K_j(x', y, z)| \, dydz \leq I_1 I_2,$$

where

$$I_1 := \|(1 + 2^{j\rho N}(|x - y|^2 + |x - z|^2)^{N/2})[K_j(x, y, z) - K_j(x', y, z)]\|_{L^2(dydz)}$$

and

$$I_2 := \|(1 + 2^{j\rho N}(|x - y|^2 + |x - z|^2)^{N/2})^{-1} \chi_{R_\rho}\|_{L^2(dydz)}.$$

Using a change of variables we see that

$$I_2 \lesssim \left(\int_{\mathbb{R}^{2n}} (1 + |y|^2 + |z|^2)^{-N} 2^{-2jn\rho} \, dydz \right)^{1/2} \sim 2^{-jn\rho}.$$

We estimate I_1 next. Observe first that

$$K_j(x, y, z) - K_j(x', y, z) = I + II,$$

where

$$I = \int_{\mathbb{R}^{2n}} \sigma_j(x', \xi, \eta) e^{i(x-y)\cdot\xi + i(x-z)\cdot\eta} (1 - e^{i(x'-x)\cdot(\xi+\eta)}) \, d\xi d\eta$$

and

$$II = \int_{\mathbb{R}^{2n}} (\sigma_j(x, \xi, \eta) - \sigma_j(x', \xi, \eta)) e^{i(x-y)\cdot\xi + i(x-z)\cdot\eta} \, d\xi d\eta.$$

Therefore, we have $I_1 \leq I_{1,1} + I_{1,2}$, with

$$I_{1,1} = \|(1 + 2^{j\rho N}(|x - y|^2 + |x - z|^2)^{N/2})I\|_{L^2(dydz)}$$

and

$$I_{1,2} = \|(1 + 2^{j\rho N}(|x - y|^2 + |x - z|^2)^{N/2})II\|_{L^2(dydz)}.$$

Integration by parts gives that $(1 + 2^{j\rho N}(|x - y|^2 + |x - z|^2)^{N/2})I$ equals

$$\int_{\mathbb{R}^{2n}} (1 + (-1)^{N/2} 2^{j\rho N} \Delta_{\xi, \eta}^{N/2}) [\sigma_j(x', \xi, \eta) (1 - e^{i(x'-x)\cdot(\xi+\eta)})] e^{i(x-y)\cdot\xi + i(x-z)\cdot\eta} d\xi d\eta.$$

Now, since $2^j|x - x'| \leq 1$ and $|\xi| + |\eta| \sim 2^j$ ($\lesssim 1$ if $j = 0$) in the support of σ_j , we have

$$|1 - e^{i(x'-x)\cdot(\xi+\eta)}| \leq |x - x'| |\xi + \eta| \lesssim 2^j |x - x'| \leq (2^j |x - x'|)^\rho.$$

The above and the fact that $\sigma \in BS_{\rho, \delta}^{-n(1-\rho)}$ leads to

$$|\sigma_j(x', \xi, \eta) (1 - e^{i(x'-x)\cdot(\xi+\eta)})| \lesssim 2^{-jn(1-\rho)} (2^j |x - x'|)^\rho.$$

Moreover, $2^{j\rho N} \Delta_{\xi, \eta}^{N/2} [\sigma_j(x', \xi, \eta) (1 - e^{i(x'-x)\cdot(\xi+\eta)})]$ is given by

$$\begin{aligned} & 2^{j\rho N} \sum_{\substack{|\alpha_1 + \alpha_2 + \alpha_3| = N \\ |\alpha_3| > 0}} c_{\alpha_1, \alpha_2, \alpha_3} \partial_{\xi, \eta}^{\alpha_1} \sigma(x', \xi, \eta) 2^{-j|\alpha_2|} (\partial_{\xi, \eta}^{\alpha_2} \psi)(2^{-j}\xi, 2^{-j}\eta) (x - x')^{\alpha_3} e^{i(x'-x)\cdot(\xi+\eta)} \\ & + 2^{j\rho N} \sum_{|\alpha_1 + \alpha_2| = N} c_{\alpha_1, \alpha_2, \alpha_3} \partial_{\xi, \eta}^{\alpha_1} \sigma(x', \xi, \eta) 2^{-j|\alpha_2|} (\partial_{\xi, \eta}^{\alpha_2} \psi)(2^{-j}\xi, 2^{-j}\eta) (1 - e^{i(x'-x)\cdot(\xi+\eta)}), \end{aligned}$$

where ψ should be replaced by φ if $j = 0$. Since $0 < \rho < 1$ and $2^j|x - x'| \leq 1$ we can control

$$\begin{aligned} & |2^{j\rho N} \Delta_{\xi, \eta}^{N/2} [\sigma_j(x', \xi, \eta) (1 - e^{i(x'-x)\cdot(\xi+\eta)})]| \\ & \lesssim 2^{j\rho N} \sum_{\substack{|\alpha_1 + \alpha_2 + \alpha_3| = N \\ |\alpha_3| > 0}} (2^j)^{-n(1-\rho) - \rho|\alpha_1|} 2^{-j\rho|\alpha_2|} (2^j |x - x'|)^{|\alpha_3|} 2^{-j\rho|\alpha_3|} \\ & \quad + 2^{j\rho N} \sum_{|\alpha_1 + \alpha_2| = N} (2^j)^{-n(1-\rho) - \rho|\alpha_1|} 2^{-j\rho|\alpha_2|} (2^j |x - x'|)^\rho \\ & \lesssim (2^j |x - x'|)^\rho 2^{-jn(1-\rho)}. \end{aligned}$$

The previous estimates imply that

$$|(1 + (-1)^{N/2} 2^{j\rho N} \Delta_{\xi, \eta}^{N/2}) [\sigma_j(x', \xi, \eta) (1 - e^{i(x'-x)\cdot(\xi+\eta)})]| \lesssim (2^j |x - x'|)^\rho 2^{-jn(1-\rho)};$$

thus, by using Plancherel's theorem, we conclude that

$$I_{1,1} \lesssim (2^j |x - x'|)^\rho 2^{-jn(1-\rho)} 2^{jn} \sim (2^j |x - x'|)^\rho 2^{jn\rho}.$$

A similar argument shows that $I_{1,2} \lesssim (2^j |x - x'|)^\rho 2^{jn\rho}$; the difference is that now we control

$$\begin{aligned} & |(1 + (-1)^{N/2} 2^{j\rho N} \Delta_{\xi, \eta}^{N/2}) (\sigma_j(x, \xi, \eta) - \sigma_j(x', \xi, \eta))| \lesssim |x - x'| (2^j)^{-n(1-\rho) + \delta} \\ & \leq (2^j |x - x'|)^\rho 2^{-jn(1-\rho)}. \end{aligned}$$

All in all, we have shown that $I_1 \lesssim (2^j|x - x'|)^\rho 2^{jn\rho}$, which gives, for $2^j|x - x'| < 1$, the improved estimate

$$\int_{R_\rho} |K_j(x, y, z) - K_j(x', y, z)| \, dydz \lesssim (2^j|x - x'|)^\rho 2^{jn\rho} 2^{-jn\rho} \sim (2^j|x - x'|)^\rho. \tag{4.14}$$

Letting $c_j(x - x') := \min\{(2^j|x - x'|)^\rho, (2^j|x - x'|)^{\rho(n-N)}\}$ and combining (4.13) and (4.14) gives

$$\int_{R_\rho} |K_j(x, y, z) - K_j(x', y, z)| \, dydz \lesssim c_j(x - x') \quad \forall j \in \mathbb{N}_0.$$

Now, let $j_0 \geq 0$ be so that $2^{j_0}|x - x'| < 1$ and $2^{j_0+1}|x - x'| \geq 1$ and notice that

$$\sum_{j=0}^\infty c_j(x - x') \leq \sum_{j=0}^{j_0} 2^{(j-j_0)\rho} + \sum_{j=j_0+1}^\infty 2^{(j-j_0-1)\rho(n-N)} \lesssim 1,$$

with the implicit constant independent of x and x' , finishing the proof of (4.12).

Appendix A. Defining T on $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$.

The purpose of this appendix is to indicate how an operator T satisfying the hypothesis of Theorem 2.2 can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into BMO . Our discussion is inspired by the one in Duoandikoetxea [9, 119–120].

Fix $f, g \in L^\infty(\mathbb{R}^n)$. Let $Q \subset \mathbb{R}^n$ be a cube centered at the origin which contains x and let \tilde{Q} be the cube containing Q introduced in the proof of Theorem 2.2. We define

$$T_Q(f, g)(x) = T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})(x) + I_{1,Q}(f, g)(x) + I_{2,Q}(f, g)(x) + I_{3,Q}(f, g)(x), \tag{A.15}$$

where

$$\begin{aligned} I_{1,Q}(f, g)(x) &= \int_{\mathbb{R}^{2n}} (\mathcal{K}(x, y, z) - \mathcal{K}(0, y, z))f(y)\chi_{\tilde{Q}}(y)g(z)\chi_{\tilde{Q}^c}(z) \, dydz, \\ I_{2,Q}(f, g)(x) &= \int_{\mathbb{R}^{2n}} (\mathcal{K}(x, y, z) - \mathcal{K}(0, y, z))f(y)\chi_{\tilde{Q}^c}(y)g(z)\chi_{\tilde{Q}}(z) \, dydz, \\ I_{3,Q}(f, g)(x) &= \int_{\mathbb{R}^{2n}} (\mathcal{K}(x, y, z) - \mathcal{K}(0, y, z))f(y)\chi_{\tilde{Q}^c}(y)g(z)\chi_{\tilde{Q}^c}(z) \, dydz. \end{aligned}$$

We note that $T(f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}})$ is well-defined since $f\chi_{\tilde{Q}}, g\chi_{\tilde{Q}} \in L_c^\infty(\mathbb{R}^n)$ and that the terms $I_{j,Q}, j = 1, 2, 3$, are defined through absolutely convergent integrals. Indeed, for $0 < \varepsilon < 1$ as in (2.3), we have

$$\begin{aligned} |I_{j,Q}(f, g)(x)| &\lesssim \int_{|x|^\varepsilon \lesssim |x-y|+|x-z|} |\mathcal{K}(x, y, z) - \mathcal{K}(0, y, z)| \, dydz \, \|f\|_{L^\infty} \|g\|_{L^\infty} \\ &\lesssim \|f\|_{L^\infty} \|g\|_{L^\infty}. \end{aligned}$$

We show next that given cubes Q, R centered at the origin and containing x , the

definitions of $T_Q(f, g)(x)$ and $T_R(f, g)(x)$ are the same modulo a constant independent of x . Without loss of generality, let us assume that $Q \subset R$; in particular, this gives $\tilde{Q} \subset \tilde{R}$ as well. Using Remark 2.1, a straightforward calculation now shows that $T_Q(f, g)(x) - T_R(f, g)(x)$ equals

$$\begin{aligned} & -T(f\chi_{\tilde{R}\setminus\tilde{Q}}, g\chi_{\tilde{Q}}) - T(f\chi_{\tilde{R}}, g\chi_{\tilde{R}\setminus\tilde{Q}}) \\ & + \int_{\mathbb{R}^{2n}} (\mathcal{K}(x, y, z) - \mathcal{K}(0, y, z))f(y)g(z)\chi_{(\tilde{Q}\times\tilde{Q})^c}(y, z) dydz \\ & - \int_{\mathbb{R}^{2n}} (\mathcal{K}(x, y, z) - \mathcal{K}(0, y, z))f(y)g(z)\chi_{(\tilde{R}\times\tilde{R})^c}(y, z) dydz \\ & = - \int_{\mathbb{R}^{2n}} \mathcal{K}(x, y, z)f(y)g(z)(\chi_{\tilde{R}\setminus\tilde{Q}}(y)\chi_{\tilde{Q}}(z) + \chi_{\tilde{R}}(y)\chi_{\tilde{R}\setminus\tilde{Q}}(z)) dydz \\ & + \int_{\mathbb{R}^{2n}} \mathcal{K}(x, y, z)f(y)g(z)(\chi_{(\tilde{Q}\times\tilde{Q})^c} - \chi_{(\tilde{R}\times\tilde{R})^c})(y, z) dydz \\ & - \int_{\mathbb{R}^{2n}} \mathcal{K}(0, y, z)f(y)g(z)(\chi_{(\tilde{Q}\times\tilde{Q})^c} - \chi_{(\tilde{R}\times\tilde{R})^c})(y, z) dydz \\ & = - \int_{\tilde{R}\times\tilde{R}\setminus\tilde{Q}\times\tilde{Q}} \mathcal{K}(0, y, z)f(y)g(z) dydz. \end{aligned}$$

This gives the desired result since the last integral is independent of x and, because \mathcal{K} is locally integrable on $\mathbb{R}^{3n}\setminus\Delta$ and $f, g \in L^\infty(\mathbb{R}^n)$, we can assume without loss of generality that it is absolutely convergent. Thus, as a function in BMO , we can define $T(f, g)$ for $f, g \in L^\infty(\mathbb{R}^n)$ via the right hand-side of (A.15).

It remains to show that $\|T(f, g)\|_{BMO} \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty}$. Let $Q \subset \mathbb{R}^n$ be an arbitrary cube and let $R \subset \mathbb{R}^n$ be some cube centered at the origin such that $Q \subset R$. For $x \in Q$, we can then write $T(f, g)(x) = T_R(f, g)(x)$. By the proof of Theorem 2.2, since $f\chi_{\tilde{R}}, g\chi_{\tilde{R}} \in L^\infty_c(\mathbb{R}^n)$, there is some constant C_Q such that

$$\int_Q |T(f\chi_{\tilde{R}}, g\chi_{\tilde{R}})(x) - C_Q| dx \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty},$$

with the implicit constant independent of Q . Moreover, it was shown above that $|I_{j,R}(f, g)(x)| \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty}$ for $x \in R$ and $j = 1, 2, 3$; then we have

$$\int_Q |I_{j,R}(f, g)(x)| dx \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty} \quad \text{for } j = 1, 2, 3.$$

Altogether, we have that

$$\int_Q |T(f, g)(x) - C_Q| dx \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty},$$

with the implicit constant independent of Q . This gives the desired conclusion.

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