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Positive factorizations of symmetric mapping classes

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Abstract. We study a question of Etnyre and Van Horn-Morris whether a symmetric mapping class admitting a positive factorization is a lift of a quasipositive braid. We answer the question affirmatively for mapping classes satisfying certain cyclic conditions.

1. Introduction.

In [**10**, Question 7.9] Etnyre and Van Horn-Morris ask the following question:

Question 1.1. If a symmetric mapping class $\phi \in \mathrm{Mod}(S)$ admits a positive factorization, then is ϕ a lift of a quasipositive braid?

This is a profound question connecting three important objects in topology: (1) symmetric mapping classes, (2) positive factorizations, and (3) quasipositive braids. We describe each here.

Let S be a compact, oriented surface with non-empty boundary, which is a special (see Section 2) cyclic branched covering of a disk D branched at n-points. A mapping class $\phi \in \mathrm{Mod}(S)$ is called *symmetric* (fiber-preserving) [**4**, p.65] if ϕ is a lift of an element of the braid group $B_n = \mathrm{Mod}(D^2 \setminus \{n \text{ points}\})$. Symmetric mapping class groups were introduced and studied by Birman and Hilden in a series of papers culminating in [**3**]. As Margalit and Winarski say in [**26**], the Birman–Hilden theory has had influence on many areas of mathematics, from low-dimensional topology, to geometric group theory, to representation theory, to algebraic geometry and more.

A *positive factorization* of a mapping class $\phi \in \mathrm{Mod}(S)$ is a factorization of ϕ into positive (right-handed) Dehn twists about simple closed curves. In contact and symplectic geometry, positive factorizations of mapping classes play an important role due to the following fact: A contact 3-manifold is Stein fillable if and only if it is supported by an open book whose monodromy admits a positive factorization [**2**], [**14**], [**25**].

A *quasipositive* braid in B_n is a braid which factorizes into positive half twists about proper simple arcs in the n-punctured disk. Quasipositive knots and links are introduced and studied by Rudolph in a series of papers. Rudolph showed [**28**] that a quasipositive knot can be realized as an intersection (transverse $\mathbb{C}$-link) of the unit sphere in $\mathbb{C}^2$ with

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an algebraic complex curve in $\mathbb{C}^2$. Boileau and Orevkov [**6**] proved the converse that any knot arising as such an intersection must be quasipositive.

In this paper, we give partial answers to the question of Etnyre and Van Horn-Morris.

Let S be a compact, oriented surface with non-empty boundary. Let D be a disk. Suppose $\pi : S \to D$ is a special k-fold cyclic branched covering of the disk D branched at n points. The meaning of "special" will be made clear in Section 2. In [**3**] Birman and Hilden show that there is a well-defined injective homomorphism $\Psi : B_n \to \mathrm{Mod}(S)$ whose image is the symmetric mapping class group $\mathrm{SMod}(S)$ which is defined in Section 2.

Using the homomorphism $\Psi : B_n \to \mathrm{SMod}(S)$, Etnyre and Van Horn-Morris in [**10**] consider various submonoids in B_n.

$$\begin{aligned}
P(n) &:= \{b \in B_n \mid b \text{ is a positive braid}\} \\
QP(n) &:= \{b \in B_n \mid b \text{ is a quasipositive braid}\} \\
\mathrm{Dehn}^+(n,k) &:= \Psi^{-1}(\mathrm{Dehn}^+(S)) \\
\mathrm{Tight}(n,k) &:= \Psi^{-1}(\mathrm{Tight}(S)) \\
RV(n) &:= \{b \in B_n \mid b \text{ is a right-veering braid}\} \\
\mathrm{Veer}^+(n,k) &:= \Psi^{-1}(\mathrm{Veer}^+(S))
\end{aligned}$$

Here, a braid $b \in B_n$ is *positive* if it is a product of standard generators $\sigma_1, \dots, \sigma_{n-1}$, and is *quasipositive* if it is a product of conjugates of σ_1. We have $P(2) = QP(2)$ and $P(n) \subsetneq QP(n)$ for $n > 2$. The set $\mathrm{Dehn}^+(S) \subset \mathrm{Mod}(S)$ is a monoid generated by positive Dehn twists, $\mathrm{Tight}(S) \subset \mathrm{Mod}(S)$ is a monoid consisting of monodromies supporting tight contact structures, and $\mathrm{Veer}^+(S) \subset \mathrm{Mod}(S)$ is a monoid consisting of right-veering mapping classes. One can see that $\Psi(b)$ is right-veering if and only if b is right-veering (see [**20**, Section 3] for the definition(s) of right-veering braids).

Proposition 1.2 ([**21**, Corollary 4.4]). *We have* $\mathrm{Veer}^+(n,k) = RV(n)$ *for all* n *and* k.

Etnyre and Van Horn-Morris observe that [**18**, Lemma 3.1] implies the following:

Proposition 1.3 ([**10**, p.355]). *For all* $n \geq 2$ *and* $k \geq 2$ *we have* $QP(n) \subset \mathrm{Dehn}^+(n,k)$.

In summary, we have;

$$P(n) \subset QP(n) \subset \mathrm{Dehn}^+(n,k) \subset \mathrm{Tight}(n,k) \subset \mathrm{Veer}^+(n,k) = RV(n) \subsetneq B_n.$$

In [**19**, Example 2.9], the strictness of the inclusion $QP(3) \subset \mathrm{Veer}^+(3,2)$ is shown. Moreover, we have:

Proposition 3.3. *For general* n *and* k, *both the inclusions* $\mathrm{Dehn}^+(n,k) \subset \mathrm{Tight}(n,k) \subset \mathrm{Veer}^+(n,k)$ *are strict.*

Thus, the unique inclusion whose strictness is unknown is $QP(n) \subset \mathrm{Dehn}^+(n,k)$, and with these terminologies, Question 1.1 of Etnyre and Van Horn-Morris is equivalent to the following:

QUESTION 1.4. Do we have $QP(n) = \mathrm{Dehn}^+(n,k)$?

In [**10**] they say "the answer is almost certainly no". Thus our goal can be set to find sufficient conditions for $QP(n) \supset \mathrm{Dehn}^+(n,k)$.

1.1. Motivation.

Our particular branched covering $\pi : S \to D$ is closely related to the cyclic branched covering of the standard contact 3-sphere $(S^3, \xi_{std} = \ker \alpha_{std})$. Let $K = \widehat{b}$ be a transverse knot in (S^3, ξ_{std}) represented by the closure $\widehat{b}$ of an n-braid $b \in B_n$ with respect to the open book (D^2, id). Let $p : M_{K,k} \to S^3$ be the k-fold cyclic branched covering, branched along K. Then $M_{K,k}$ is equipped with a contact structure $\xi_{K,k}$ that is a perturbation of the kernel of the pull-back $p^*(\alpha_{std})$. Such a contact structure is supported by the open book $(S, \Psi(b))$. Thus, $(M_{K,k}, \xi_{K,k}) \simeq (M_{(S,\Psi(b))}, \xi_{(S,\Psi(b))})$.

Let $B^4 (\subset \mathbb{C}^2)$ be the unit complex ball giving a Stein filling of (S^3, ξ_{std}). If the braid $b \in B_n$ is quasipositive, a factorization of b as a product of positive half twists gives rise to an immersed Seifert surface of $K = \widehat{b}$ with ribbon intersections as shown in [**10**, Figure 9]. Pushing this surface into the interior of B^4 we have a properly embedded symplectic surface Σ in B^4 such that $\Sigma \cap \partial B^4 = \partial\Sigma = K$. Let W be the k-fold cyclic branched cover of B^4 branched along Σ. Then W gives a Stein filling of $(M_{K,k}, \xi_{K,k})$.

On the other hand, a factorization of $b \in B_n$ into positive half twists induces a factorization of $\Psi(b) \in \mathrm{Mod}(S)$ into positive Dehn twists [**18**, Lemma 3.1], to which one can associate a Legendrian surgery diagram (cf. [**18**, Figure 12]). Let X be the 4-dimensional handlebody obtained by attaching 2-handles to B^4 according to the Legendrian surgery diagram. By [**8**], [**15**, Proposition 2.3] the manifold X is a Stein filling of $(M_{(S,\Psi(b))}, \xi_{(S,\Psi(b))})$.

In fact, these two constructions of Stein fillings give the same manifold. Namely, the two 4-manifolds X and W are diffeomorphic, which follows from the proof of [**22**, Claim 2.1]. Thus, the branched covering construction behaves nicely not only for contact structures but also for Stein fillings.

With the above discussion in mind, we may extend Question 1.1 to the following question about Stein filling and $\mathbb{Z}/k\mathbb{Z}$-action:

Assume that $b \in \mathrm{Dehn}^+(n,k)$. Then $(M_{K,k}, \xi_{K,k}) \simeq (M_{(S,\Psi(b))}, \xi_{(S,\Psi(b))})$ is Stein fillable because $\Psi(b) \in \mathrm{Dehn}^+(S)$ and $\mathrm{Dehn}^+(S)$ is contained in the monoid $\mathrm{Stein}(S)$ of Stein fillable open books [**8**], [**14**]. The contact manifold $(M_{K,k}, \xi_{K,k})$ also admits a $\mathbb{Z}/k\mathbb{Z}$-action as a contactomorphism with the quotient space (S^3, ξ_{std}). Our question is: *Can we find a Stein filling X coming from the above construction*; *namely, can the $\mathbb{Z}/k\mathbb{Z}$-action on $(M_{K,k}, \xi_{K,k})$ extend to a holomorphic $\mathbb{Z}/k\mathbb{Z}$-action on some X with the quotient space B^4?*

This new question suggests that, even if Question 1.1 may have negative answer in general as Etnyre and Van Horn-Morris expect in [**10**], it is worth trying to find sufficient conditions for $QP(n) \supset \mathrm{Dehn}^+(n,k)$ to hold.

1.2. Main results.

The following are our main results that give sufficient conditions for $QP(n) \supset \mathrm{Dehn}^+(n,k)$.

Theorem 3.2. *For $n \leq 4$, $QP(n) = \mathrm{Dehn}^+(n,2)$.*

Theorem 5.1. *Let $\iota : S \to S$ be a deck transformation. Let C be a simple closed geodesic curve in S such that $C, \iota(C), \dots, \iota^{e-1}(C)$ are pairwise disjoint and $\iota^e(C) = C$ for some $e \in \{1, \dots, k\}$ that divides k. Let $d, j \in \mathbb{N}$. For an n-braid $b \in B_n$, suppose that $b^d \in \mathrm{Dehn}^+(n,k)$ with*

$$\Psi(b^d) = \left(T_C T_{\iota(C)} T_{\iota^2(C)} \dots T_{\iota^{e-1}(C)}\right)^j .$$

Then $b \in QP(n)$ (and so $b^d \in QP(n)$).

This theorem states that Question 1.1 has an affirmative answer for a symmetric mapping class which is a root of symmetric product of positive Dehn twists (see also Corollary 5.2).

Let us choose a hyperbolic metric on S so that the deck translation $\iota : S \to S$ is an isometry. Let $S' \subsetneq S$ be a subsurface of S which is either (a) an annular neighborhood of a geodesic simple closed curve or (b) a hyperbolic surface with geodesic boundary such that the inclusion map is an isometry. For $i = 1, \dots, k$ define $S'_i := \iota^{i-1}(S') \subset S$. For a homeomorphism $f : S' \to S'$ fixing $\partial S'$ pointwise, we put $f_i := \iota^{i-1} \circ f \circ \iota^{-(i-1)} : S'_i \to S'_i$. We may view f_i as a homeomorphism $f_i : S \to S$ by extending it as identity on $S \setminus S'_i$.

Theorem 5.4. *Suppose that the subsurfaces $S'_1, \dots, S'_k \subset S$ are pairwise non-isotopic. Assume that $[f] \in \mathrm{Dehn}^+(S')$ is either*

- *a non-negative power of a single Dehn twist (when S' is an annulus which is a neighborhood of a simple closed geodesic curve), or*
- *a pseudo-Anosov map (when S' is a hyperbolic surface).*

Suppose that $b \in \mathrm{Dehn}^+(n,k)$ satisfies

$$\Psi(b) = [f_1 \circ f_2 \circ \cdots \circ f_k]$$

then $b \in QP(n)$.

We say that a simple closed curve C is *symmetric* if it is invariant under some deck translation. Question 1.1 can be understood as a question of the existence of factorizations of elements of $\mathrm{Dehn}^+(S) \cap \mathrm{SMod}(S)$ into positive Dehn twists about symmetric simple closed curves.

A well-known example where a positive factorization coincides with a product of Dehn twists about symmetric simple closed curves may be the daisy relation [**9**]. In Example 5.7 we see that the technical looking assumptions in Theorem 5.4 can be understood as a generalization of the setting of the daisy relation, and view Theorem 5.4 as a generalization of the daisy relation.

1.3. Organization of the paper.

In Section 2, we review results of Birman and Hilden that we need in this paper.

In Section 3, we show that the answer to the question is affirmative when the number of branch points n is two (Theorem 3.1) or the degree of the branched covering is two with $n \leq 4$ branch points (Theorem 3.2).

In Section 4, we discuss roots of quasipositive braids and find conditions that a root of a quasipositive braid is also quasipositive. We obtain results that are used in Section 5.

In Section 5, we prove our main results (Theorems 5.1 and 5.4).

2. Birman–Hilden theory.

Throughout the paper, unless otherwise stated, we always assume that the boundary of a surface is non-empty, any homeomorphism of a surface with marked points (punctures) fixes the boundary pointwise and permutes the marked points, and any isotopy of homeomorphisms $\{f_t\}$ pointwise fixes the boundary and the marked points. We denote by $[f] \in \mathrm{Mod}(S)$ the isotopy class of a homeomorphism $f \in \mathrm{Homeo}^+(S)$. For a simple closed curve C in a surface S, we denote by $t_C \in \mathrm{Homeo}^+(S)$ a right-handed Dehn twist homeomorphism about C which is supported in a neighborhood of C, and denote its isotopy class by $T_C \in \mathrm{Mod}(S)$. A simple closed curve in a surface is called *essential* if it is not homotopic to a point, a puncture, or a boundary component of the surface.

Let $P = \{p_1, \dots, p_n\} \subset \mathrm{Int}(D)$ be n points in the interior of a 2-disk D. We may identify the braid group B_n with the mapping class group $\mathrm{Mod}(D \setminus P)$ of the n-punctured disk $D \setminus P$. The fundamental group $\pi_1(D \setminus P)$ is the free group of rank n and generated by $\{x_1, \dots, x_n\}$, where x_i is a loop winding once around p_i counter-clockwise. For $k \in \mathbb{N}$ let $e_k : \pi_1(D \setminus P) \to \mathbb{Z}/k\mathbb{Z}$ be the homomorphism defined by $e_k(x_i) = 1$ for all $i = 1, \dots, n$. For $n \geq 2$ and $k \geq 2$, let

$$\pi : S \approx S_{n,k} \to D$$

be the k-fold cyclic branched covering branched at P such that the associated regular covering $S \setminus \pi^{-1}(P) \to D \setminus P$ is the k-fold cyclic cover corresponding to the subgroup $\ker(e_k)$ of $\pi_1(D \setminus P)$. We denote by $\widetilde{P} = \{\widetilde{p_1}, \dots, \widetilde{p_n}\} = \pi^{-1}(P) \subset S$ the set of branch points in S. Let $\iota = \iota_k : S \to S$ be a generator of the deck transformation group $\mathrm{Aut}(S, \pi) \simeq \mathbb{Z}/k\mathbb{Z}$.

We visualize S and ι as follows. The covering S can be viewed as the canonical Seifert surface of the (n,k)-torus link represented as the closure of the n-braid $(\sigma_1 \cdots \sigma_{n-1})^k$ (see Figure 1). The deck transformation ι_k is the $2\pi/k$ rotation of the surface S about the braid axis through the branch points $\widetilde{P}$, and ∂S is not pointwise fixed by ι_k.

Let $\beta \in B_n \approx \mathrm{Mod}(D \setminus P)$ and $f_\beta : (D,P) \to (D,P)$ be a homeomorphism representing the braid β. Since $\beta(\ker(e_k)) = \ker(e_k)$ via the action of B_n on $\pi_1(D \setminus P)$, there is a lift $\widetilde{f_\beta} : S \to S$ of f_β (see [**3**, Lemma 5.1] and [**13**, Theorem 1.1]). Among the k possible lifts which are related to each other by deck transformations, $\widetilde{f_\beta}$ is the unique lift that fixes ∂S pointwise.

When two homeomorphisms f_β and $f'_\beta \in \mathrm{Homeo}^+(D,P)$ represent the same braid $\beta \in B_n$ we note that an isotopy between f_β and f'_β lifts to an isotopy between their lifts

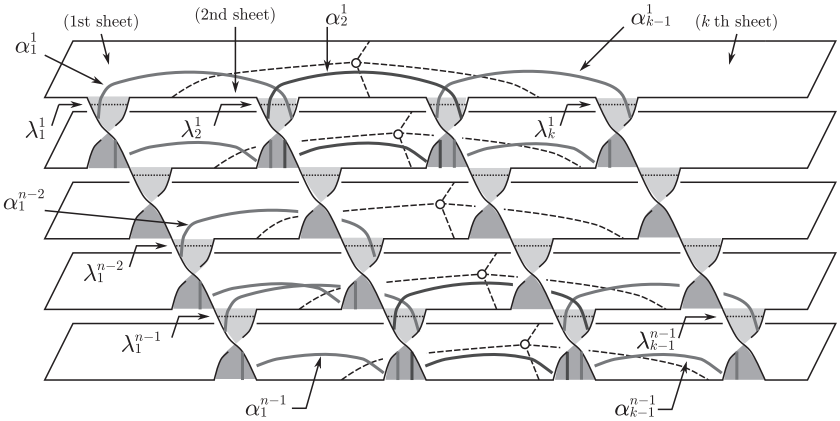

Figure 1. The Seifert surface S of an (n,k) torus link, where $n = 5$ and $k = 4$. Hollowed circles are the lifted branch points $\widetilde{P}$. Cutting S along the dashed arcs gives k sheets of disks. The deck transformation ι_k takes ith disk to $(i+1)$st disk.

$\widetilde{f_\beta}$ and $\widetilde{f'_\beta}$. Thus, we have a well-defined homomorphism

$$\Psi : B_n \to \mathrm{Mod}(S)$$

defined by $\Psi(\beta) = [\widetilde{f_\beta}]$. Let

$$\Theta = \Theta_k : \mathrm{Mod}(S) \to \mathrm{Mod}(S)$$

be an automorphism defined by $\Theta([f]) = [\iota_k \circ f \circ \iota_k^{-1}]$.

Definition 2.1. Define

$$\mathrm{SMod}(S) := \bigl\{ [f] \in \mathrm{Mod}(S) \bigm| f \in \mathrm{Homeo}^+(S) \text{ and } \Theta([f]) = [f] \bigr\}$$

and call it the *symmetric mapping class group*. An element of $\mathrm{SMod}(S)$ is called *symmetric mapping class*.

The following fact is attributed to Birman and Hilden.

Proposition 2.2. *The homomorphism Ψ is injective and onto* $\mathrm{SMod}(S)$*.*

For the sake of completeness, we include a proof of Proposition 2.2.

Proof of Proposition 2.2. The injectivity of Ψ follows from [**3**]. A homeomorphism $f : S \to S$ is called *fiber-preserving* if $\pi \circ f(p) = \pi \circ f(p')$ for any $p, p' \in S$ with $\pi(p) = \pi(p')$. Suppose that $f_0, f_1 \in \mathrm{Homeo}^+(D,P)$ satisfy $\Psi([f_0]) = \Psi([f_1])$. According to [**3**, Theorem 1], since the lifts $\widetilde{f_0}, \widetilde{f_1} \in \mathrm{Homeo}^+(S)$ represent the same element of $\mathrm{SMod}(S)$ and are fiber-preserving, there exists a fiber-preserving isotopy $g_t \in \mathrm{Homeo}^+(S)$

between $\widetilde{f_0}$ and $\widetilde{f_1}$. Since g_t is fiber-preserving the composition $\pi \circ g_t \circ \pi^{-1}$ is a well-defined homeomorphism of (D, P) and it gives an isotopy between f_0 and f_1; hence, $[f_0] = [f_1] \in B_n$.

To see $\mathrm{Im}(\Psi) \supset \mathrm{SMod}(S)$, assume that $f \in \mathrm{Homeo}^+(S)$ satisfies $[f] = [\iota \circ f \circ \iota^{-1}]$. The same argument as in the proof of [**3**, Theorem 3] with Teichmüller's theorem for bordered surfaces [**1**, p.59] shows that $[f]$ can be represented by a homeomorphism $f' \in \mathrm{Homeo}^+(S)$ such that $f' = \iota \circ f' \circ \iota^{-1}$.

For $x \in D$ let $\tilde{x} \in \pi^{-1}(x) \subset S$ be a pre-image of x under the branched covering map $\pi : S \to D$. Define a homeomorphism $b \in \mathrm{Homeo}^+(D)$ by

$$b(x) = \pi(f'(\tilde{x})).$$

Since $\pi \circ f' \circ \iota(\tilde{x}) = \pi \circ \iota \circ f'(\tilde{x}) = \pi \circ f'(\tilde{x})$, the image $b(x)$ is independent of the choice of the pre-image $\tilde{x}$.

We observe that b acts on the branch set P as a permutation. Suppose that $p \in P$ and $\tilde{p} \in \tilde{P}$ satisfy $\pi(\tilde{p}) = p$. Since ι fixes the branch set $\tilde{P}$ point-wise we have $\iota(f'(\tilde{p})) = f'(\iota(\tilde{p})) = f'(\tilde{p})$. That is, $f'(\tilde{p})$ is a fixed point of ι and $f'(\tilde{p}) \in \tilde{P}$. We get $b(p) = \pi(f'(\tilde{p})) \in \pi(\tilde{P}) = P$ which shows $b(P) = P$.

Since $\pi \circ f' = b \circ \pi$ the map $f' \in \mathrm{Homeo}^+(S)$ is the unique lift of $b \in \mathrm{Homeo}^+(D, P)$. We obtain $[f] = [f'] = \Psi([b]) \in \mathrm{Im}(\Psi)$.

To see $\mathrm{Im}(\Psi) \subset \mathrm{SMod}(S)$, let α_l^i be a simple closed curve that goes through ith and $(i+1)$st sheets and lth and $(l+1)$st twisted bands as depicted in Figure 1. Let $t_{i,l} := t_{\alpha_l^i} \in \mathrm{Homeo}^+(S)$ be a positive Dehn twist about α_l^i. It is shown in [**18**, Lemma 3.1] that for the standard braid generators $\sigma_1, \ldots, \sigma_{n-1}$ of the Artin braid group B_n we have

$$\Psi(\sigma_i) = [t_{i,1} \circ t_{i,2} \circ \cdots \circ t_{i,k-1}] \in \mathrm{Mod}(S).$$

For $j = 1, \ldots, n-1$ and $l = 1, \ldots, k$ let λ_l^j (thin dashed line in Figure 1) be properly embedded arcs on the lth band between jth and $(j+1)$st sheets such that $\iota(\lambda_l^j) = \lambda_{l+1}^j$. Let $[\lambda_l^j]$ denote the isotopy class of λ_l^j relative to ∂S. We get

$$\left[\iota^{-1} \circ (t_{i,1} \circ t_{i,2} \circ \cdots \circ t_{i,k-1}) \circ \iota\right] \left[\lambda_l^j\right] = [t_{i,1} \circ t_{i,2} \circ \cdots \circ t_{i,k-1}] \left[\lambda_l^j\right]$$

for all $j = 1, \ldots, n-1$ and $l = 1, \ldots, k$. Knowing that the surface S cut along the union of arcs $\bigcup_{j,l} \lambda_l^j$ yields n disjoint disks, the Alexander trick [**11**, Proposition 2.8] implies that $\left[\iota^{-1} \circ (t_{i,1} \circ t_{i,2} \circ \cdots \circ t_{i,k-1}) \circ \iota\right] = [t_{i,1} \circ t_{i,2} \circ \cdots \circ t_{i,k-1}]$. This shows that $\Psi(\sigma_i) \in \mathrm{SMod}(S)$. □

3. Positive answers for simple cases.

THEOREM 3.1. *We have* $P(2) = QP(2) = \mathrm{Dehn}^+(2, k) = \mathrm{Tight}(2, k) = RV(2)$ *for all* $k \geq 2$.

PROOF. By the definition of right-veering, $\sigma_1^m \in RV(2)$ if and only if $m \geq 0$. This shows that $P(2) = RV(2)$. The remaining equalities follow from Proposition 1.3. □

Theorem 3.2. *For $n \leq 4$, $QP(n) = \mathrm{Dehn}^+(n,2)$.*

The equality $QP(3) = \mathrm{Dehn}^+(3,2)$ has been known and used in the literature. However for the sake of completeness, we give a proof of this case along with the case of $n = 4$.

Proof. Since the $n = 2$ case is shown in Theorem 3.1 we may assume $n = 3$ or 4.
Let a_1, a_2, δ (for the $n = 3$ case) and $a_1, a_2, a_3, \delta_1, \delta_2$ (for the $n = 4$ case) be simple closed curves on S as depicted in Figure 2. The Dehn twists about these curves generate $\mathrm{Mod}(S)$, and $\Psi(\sigma_i) = T_{a_i}$ holds.

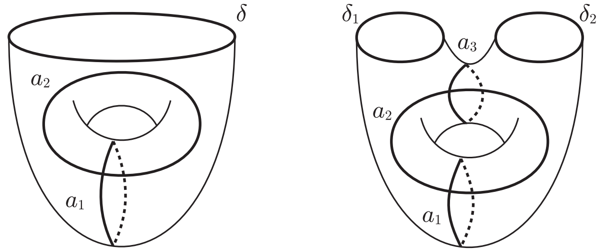

Figure 2.

If $b \in \mathrm{Dehn}^+(n,2)$ then there exist non-boundary parallel simple closed curves $\Gamma_1, \ldots, \Gamma_m \subset S$ (possibly $\Gamma_i = \Gamma_j$ for $i \neq j$) and $x, y \geq 0$ such that

$$\Psi(b) = \begin{cases} T_{\Gamma_1} \cdots T_{\Gamma_m} T_\delta^x & (n = 3) \\ T_{\Gamma_1} \cdots T_{\Gamma_m} T_{\delta_1}^x T_{\delta_2}^y & (n = 4). \end{cases}$$

We may assume that all of the curves Γ_i are non-separating since the positive Dehn twist about a separating curve can be written as a product of the positive Dehn twists about non-separating curves. Since a_1 is non-separating $\Gamma_i = f_i(a_1)$ for some $[f_i] \in \mathrm{Mod}(S)$. We can write $[f_i]$ as

$$[f_i] = T_{a_{j_l}}^{\varepsilon_l} \cdots T_{a_{j_1}}^{\varepsilon_1}$$

for some $j_p \in \{1, \ldots, n-1\}$ and $\varepsilon_p \in \{\pm 1\}$. Since the Dehn twist along the boundary does not affect simple closed curves in S, the words T_δ, T_{δ_1} and T_{δ_2} are not needed in the expression. Note that $[f_i] = \Psi(b_i)$ for $b_i = \sigma_{j_l}^{\varepsilon_l} \cdots \sigma_{j_1}^{\varepsilon_1} \in B_n$. Therefore, we have

$$T_{\Gamma_i} = T_{f(a_1)} = [f_i \circ t_{a_1} \circ f_i^{-1}] = \Psi(b_i)\Psi(\sigma_1)\Psi(b_i^{-1}) = \Psi(b_i \sigma_1 b_i^{-1})$$

and $T_{\Gamma_i} \in \Psi(QP(n))$ for all i. In particular, $\Theta(T_{\Gamma_1} \cdots T_{\Gamma_m}) = T_{\Gamma_1} \cdots T_{\Gamma_m}$ by Proposition 2.2.

For the case $n = 4$, since $\iota(\delta_1) = \delta_2$ we get $\Theta(T_{\delta_1}) = T_{\delta_2}$ and $\Theta(T_{\delta_2}) = T_{\delta_1}$ which give

$$T_{\Gamma_1} \cdots T_{\Gamma_m} T_{\delta_1}^x T_{\delta_2}^y = \Theta(T_{\Gamma_1} \cdots T_{\Gamma_m} T_{\delta_1}^x T_{\delta_2}^y)$$

$$= \Theta(T_{\Gamma_1} \cdots T_{\Gamma_m})\Theta(T_{\delta_1}^x)\Theta(T_{\delta_2}^y)$$
$$= T_{\Gamma_1} \cdots T_{\Gamma_m} T_{\delta_2}^x T_{\delta_1}^y.$$

Therefore, $x = y$.

With the chain relations [**11**, Proposition 4.12] $T_\delta = \Psi((\sigma_1\sigma_2)^6)$ and $T_{\delta_1}^x T_{\delta_2}^x = \Psi((\sigma_1\sigma_2\sigma_3)^{4x})$, we can conclude $b \in QP(n)$. □

Finally, we observe that the following inclusions are strict.

PROPOSITION 3.3. *For general n and k, both the inclusions* $\mathrm{Dehn}^+(n,k) \subset \mathrm{Tight}(n,k) \subset \mathrm{Veer}^+(n,k)$ *are strict.*

PROOF. Let $\beta = (\sigma_1\sigma_2)^6\sigma_1^{-13} \in B_3$. By [**19**, Theorem 1.2] the braid β is in $\mathrm{Tight}(3,2)$ since the fractional Dehn twist coefficient of $\Psi(\beta) \in \mathrm{Mod}(S)$ is one. However, β is not in $\mathrm{Dehn}^+(3,2)$ since its exponent sum is negative which means $b \notin QP(3) = \mathrm{Dehn}^+(3,2)$. (See [**19**, Corollary 3.6] for a better criterion for $b \notin QP(n)$ than a naive exponent sum argument.)

To see $\mathrm{Tight}(n,k) \subsetneq \mathrm{Veer}^+(n,k)$, recall [**27**, Proposition 3.1] (for the open book (D^2, id)) and [**20**, Proposition 3.22] (for general open books) that every braid with respect to an open book is transversely isotopic to a right-veering braid after suitable positive stabilizations. Let $\beta \in B_n$ be a right-veering braid that is a stabilization of $\sigma_1^{-1} \in B_2$. Then for each $k \geq 2$, β is in $\mathrm{Veer}^+(n,k)$ but not in $\mathrm{Tight}(n,k)$. □

4. Roots in quasipositive braids.

In this section we address the following question. Among the results, Corollary 4.7 plays an important role to prove our main theorems in Section 5.

QUESTION 4.1. Are $QP(n)$ and $\mathrm{Dehn}^+(n,k)$ closed under taking roots? That is,

(1) Does $b = x^d \in QP(n)$ for an integer $d \geq 2$ imply $x \in QP(n)$?

(2) Does $x^d \in \mathrm{Dehn}^+(n,k)$ for an integer $d \geq 2$ imply $x \in \mathrm{Dehn}^+(n,k)$?

DEFINITION 4.2 (Property (QP-root)). We say that a quasipositive braid $b \in QP(n)$ has the property (QP-root) if the following condition is satisfied.

$$\text{If } b = x^d \text{ for } x \in B_n \text{ and } d \geq 2 \text{ then } x \in QP(n).$$

It is shown in [**16**, Theorem 1.1] that the d-th root of a braid (if it exists) is unique up to conjugacy; namely, $x^d = y^d \in B_n$ implies that x and y are conjugate to each other. This leads to the following observation.

PROPOSITION 4.3. *For $x, x' \in B_n$ assume that $x^d = x'^d$. Then $x \in QP(n)$ if and only if $x' \in QP(n)$.*

Let $b \in B_n$ be a non-periodic reducible braid. According to [**17**, Definition 5.1], up to conjugacy, b is in the following *regular form*.

Let C be an essential multi-curve in the n-punctured disk D_n such that

- $b(C)$ is isotopic to C, and
- any simple closed curve which has non-zero geometric intersection with C must not be preserved by any power of b.

Such a multi-curve C is called a *canonical reduction system* for b [**5**] and it is unique up to conjugacy. It always exists for a non-periodic reducible braid [**23**].

Let $\mathcal{A}'$ be the set of outermost curves of C. Define a set of curves $\mathcal{A}$ to be the union of $\mathcal{A}'$ and one circle around each such puncture of D_n that is not enclosed by any circles of $\mathcal{A}'$. We may enumerate the elements of $\mathcal{A}$ as

$$\mathcal{A} = \{a_{1,1}, \dots, a_{1,r_1}, a_{2,1}, \dots, a_{2,r_2}, \dots, a_{c,1}, \dots, a_{c,r_c}\}$$

so that $b(a_{i,j}) = a_{i,j+1}$ $(j = 1, \dots, r_i - 1)$ and $b(a_{i,r_i}) = a_{i,1}$. This action of b on $\mathcal{A}$ gives $m = r_1 + r_2 + \cdots + r_c$ disjoint braided tubes (see Figure 3 (i)). Identifying each tube with a string, we get an m-braid $\widehat{b} \in B_m$ which we call the *tubular braid* associated to $\mathcal{A}$. By nature of canonical reduction systems, this braid $\widehat{b}$ is unique up to conjugacy.

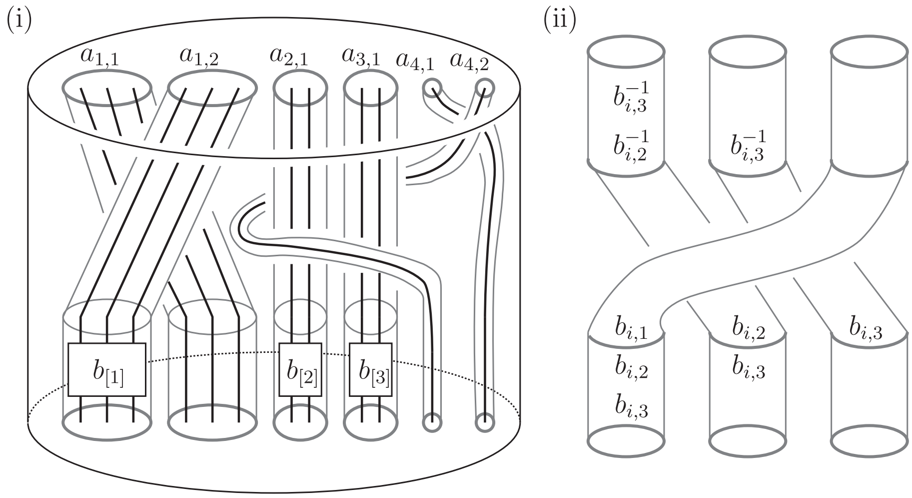

Figure 3. (i) Non-periodic and reducible braid in regular form. (ii) Taking a conjugate the condition (4.1) is satisfied.

The braid closure of $\widehat{b}$ gives a c-component link. The braid closure of the original braid b can be viewed as a satellite of the c-component link. Let $b_{i,j}$ denote the braiding inside the tube which starts at $a_{i,j}$ and ends at $a_{i,j+1}$ (where the indices j are considered up to modulo r_i), which we call the *interior braids*.

We say that b is in *regular form* if

$$\text{the only non-trivial interior braids are } b_{1,r_1}, \dots, b_{c,r_c}. \tag{4.1}$$

We denote these non-trivial interior braids by $b_{[1]}, \dots, b_{[c]}$. The condition (4.1) can be realized by shifting non-trivial interior braid $b_{i,j}$ $(j \neq r_i)$ along the closure of the tubular braid $\widehat{b}$, which is equivalent to taking a conjugate (see Figure 3 (ii)). In this process, the tubular braid $\widehat{b}$ remains the same and

$$b_{[i]} = b_{i,1}b_{i,2}\cdots b_{i,r_i} \text{ (up to conjugacy).} \tag{4.2}$$

Remark 4.4. In [**17**] a regular form requires one more property that if $b_{[i]}$ and $b_{[j]}$ are conjugate then $b_{[i]} = b_{[j]}$. But we do not use this property in this paper.

Here is a simple observation.

Lemma 4.5. *If a braid b is in regular form with quasipositive tubular braid and interior braids then b is also quasipositive.*

The converse of Lemma 4.5 is not true. The reducible 4-braid $(\sigma_2\sigma_3\sigma_2^{-1})(\sigma_1\sigma_2\sigma_1^{-1}) = (\sigma_2\sigma_1\sigma_3\sigma_2)\sigma_1^{-2}$ is quasipositive and in regular form with the tubular braid $\sigma_1 \in B_2$ and the interior braid $\sigma_1^{-2} \in B_2$.

The next proposition gives a criterion of the property (QP-root).

Proposition 4.6. *Let b be a non-periodic reducible, quasipositive braid. If b has a regular form such that all of its tubular and interior braids are quasipositive and having the property* (QP-root)*, then b also has the property* (QP-root).

Proof. Let b be a non-periodic reducible, quasipositive braid. Assume that $b = x^d$ for some $x \in B_n$ and $d \geq 1$. We will show that x is quasipositive. Suppose that b is in regular form with quasipositive tubular braid $\widehat{b} \in B_m$ and quasipositive interior braids $b_{[1]}, \ldots, b_{[c]}$ all of which have the property (QP-root).

Since b is non-periodic and reducible, the root x is also non-periodic and reducible. We may assume that x is in regular form with tubular braid $\widehat{x} \in B_m$ and interior braids $x_{[1]}, \ldots, x_{[c']}$. Since $b = x^d$ we see that $c' \leq c$ and $\widehat{b}$ is conjugate to $(\widehat{x})^d$. That is, there exists $y \in B_m$ such that $\widehat{b} = y(\widehat{x})^d y^{-1} = (y\widehat{x}y^{-1})^d$. By the property (QP-root) of $\widehat{b}$, the tubular braid $y\widehat{x}y^{-1}$ (hence, $\widehat{x}$) is quasipositive.

As for the interior braids, each $b_{[i]}$ is conjugate to a power of some single interior braid $x_{[i']}$. Moreover, for each $i' \in \{1, \ldots, c'\}$ there exists $i \in \{1, \ldots, c\}$ such that $b_{[i]}$ is conjugate to a power of $x_{[i']}$. Therefore, by the property (QP-root) of $b_{[1]}, \ldots, b_{[c]}$, all the interior braids $x_{[1]}, \ldots, x_{[c']}$ are quasipositive.

By Lemma 4.5 we conclude that x is quasipositive. $\square$

A classical theorem of Kerékjártó and Eilenberg [**24**], [**7**] states that any root of a periodic n-braid is conjugate to either $(\sigma_1\sigma_2\sigma_3\cdots\sigma_{n-1})^i$ or $(\sigma_1^2\sigma_2\sigma_3\cdots\sigma_{n-1})^i$ for some i. Thus, every periodic braid has the property (QP-root). This fact and Proposition 4.6 imply the following corollary.

Corollary 4.7. *Let $b \in QP(n)$ admitting a factorization $b = T_{a_1}^{N_1}\cdots T_{a_p}^{N_p} \in \mathrm{Mod}(D_n) = B_n$ where $a_1, \ldots, a_p$ are pairwise disjoint simple closed curves in D_n and $N_1, \ldots, N_p > 0$. If $b = x^d$ for some $d \geq 2$ then $x \in QP(n)$.*

The above results show that Question 4.1, whether a quasipositive braid has the property (QP-root), is reduced to pseudo-Anosov braids.

The next proposition gives a potential negative answer to Question 4.1.

Proposition 4.8. *Let $b \in QP(n)$ be a pseudo-Anosov quasipositive braid. Let $Ab : B_n \to \mathbb{Z}$ denote the abelianization (exponent sum) map. If $b = x^d$ for some $d > 1$ and $x \in B_n$ with $Ab(x) = 1$ then b does not have the property* (QP-root).

Proof. Since $Ab(x) = 1$, if x is quasipositive then x must be the positive half twist about a simple arc connecting two distinct punctures of D_n. Such a mapping class x is reducible. Hence, x^d is reducible and $b = x^d$ cannot be pseudo-Anosov. □

At this time of writing, we do not know whether a pseudo-Anosov quasipositive braid that satisfies the assumption of Proposition 4.8 exists or not.

5. Sufficient conditions for quasipositivity.

In this section, we study quasipositive braids admitting certain symmetric conditions. Throughout this section, we fix a hyperbolic structure on S so that the deck transformation $\iota = \iota_k : S \to S$ is an isometry (see [**12**, Theorem 11.6]). Let $S' \subsetneq S$ be a connected subsurface of S that satisfies one of the following conditions.

- S' is an annular neighborhood of a geodesic simple closed curve in S.
- S' is a hyperbolic surface with geodesic boundary and the inclusion $S' \hookrightarrow S$ is an isometry.

Note that the surface $S'_i := \iota^{i-1}(S') \subset S$ also satisfies the same property. For $f \in \mathrm{Homeo}^+(S')$ let $\widehat{f} \in \mathrm{Homeo}^+(S)$ be a homeomorphism extending f such that $\widehat{f}(x) = x$ for $x \in S \setminus S'$. For $i = 1, \ldots, k$ let

$$f_i := \iota^{i-1} \circ \widehat{f} \circ \iota^{-i+1} \in \mathrm{Homeo}^+(S).$$

Our goal is to study elements $\Psi(b) = [\phi] \in \mathrm{SMod}(S) \cap \mathrm{Dehn}^+(S)$ of the form

$$\phi = f_1 \circ f_2 \circ \cdots \circ f_k \in \mathrm{Homeo}^+(S)$$

and find sufficient conditions that guarantees $b \in QP(n)$.

We first study the following special case.

Theorem 5.1. *Let C be a simple closed geodesic curve in S such that $C, \iota(C), \ldots, \iota^{e-1}(C)$ are pairwise disjoint with $\iota^e(C) = C$ for some $e \in \{1, \ldots, k\}$ that divides k. Let $d, j \in \mathbb{N}$. For an n-braid $b \in B_n$, suppose that $b^d \in \mathrm{Dehn}^+(n, k)$ with*

$$\Psi(b^d) = \left(T_C T_{\iota(C)} T_{\iota^2(C)} \ldots T_{\iota^{e-1}(C)}\right)^j.$$

Then $b \in QP(n)$ (and so $b^d \in QP(n)$).

When $e = 1$ we get the following.

COROLLARY 5.2. *Let $j, d \in \mathbb{N}$. If $b^d \in \mathrm{Dehn}^+(n,k)$ with $\Psi(b^d) = T_C^j$ then $b \in QP(n)$.*

By Proposition 2.2 it is easy to see that $\Psi(b^d) = T_C^j$ implies $\iota(C) = C$.

PROOF OF THEOREM 5.1. Since C is simple and $\pi^{-1}(\pi(C)) = C \sqcup \iota(C) \sqcup \cdots \sqcup \iota^{e-1}(C)$, the projection $\pi(C)$ is also simple.

First, we treat an exceptional case where the projection $\pi(C)$ is a simple proper arc in the n-punctured disk $D_n := D \setminus P$ connecting two distinct punctures. This can be realized only if $k = 2$, $e = 1$ and C is a non-separating simple closed curve in S. Let $h \in QP(n)$ be the braid represented by a positive half twist about the arc $\pi(C)$. We have $\Psi(h) = T_C$. Thus, $\Psi(b^d) = T_C^j = \Psi(h^j)$. Since Ψ is injective (Proposition 2.2) $b^d = h^j \in QP(n)$. Let $Ab : B_n \to \mathbb{Z}$ be the abelianization map defined by $Ab(\sigma_i^{\pm 1}) = \pm 1$. Since $Ab(h) = 1$ we get $Ab(b) \cdot d = j$ and $\Psi(b^d) = \Psi(h^j) = \Psi((h^{Ab(b)})^d)$. Proposition 4.3 implies that $b \in QP(n)$.

Next, suppose that the projection $\pi(C)$ is a simple closed curve in the punctured disk D_n. Let $k' = k/e$. Since the map π restricted to each connected component of $\pi^{-1}(\pi(C))$ is a $k' : 1$ cover we have

$$\Psi((T_{\pi(C)})^{k'}) = T_C T_{\iota(C)} T_{\iota^2(C)} \cdots T_{\iota^{e-1}(C)}. \tag{5.1}$$

Hence,

$$\Psi(b^d) = (T_C T_{\iota(C)} T_{\iota^2(C)} \cdots T_{\iota^{e-1}(C)})^j = \Psi((T_{\pi(C)})^{k'j}).$$

Proposition 2.2 gives $b^d = (T_{\pi(C)})^{k'j}$. By Corollary 4.7, $b \in QP(n)$. □

Next, we study a more general case. Recall that $f \in \mathrm{Homeo}^+(S')$, $S'_i := \iota^{i-1}(S') \subset S$, and $f_i := \iota^{i-1} \circ \widehat{f} \circ \iota^{-i+1} \in \mathrm{Homeo}^+(S)$ for $i = 1, \ldots, k$.

LEMMA 5.3. *Suppose that $[f] \in \mathrm{Mod}(S')$ is pseudo-Anosov. Any centralizer $[g] \in Z([f]) \subset \mathrm{Mod}(S')$ of $[f]$ is either pseudo-Anosov or periodic.*

PROOF. Suppose that $[f] \in \mathrm{Mod}(S')$ is pseudo-Anosov and $[g] \in Z([f])$. Let $\mathcal{F}$ be the stable foliation of $[f]$. Since $\mathcal{F}$ is preserved under $[f]$ we have $[f][g](\mathcal{F}) = [g][f](\mathcal{F}) = [g](\mathcal{F})$, which means that the foliation $[g](\mathcal{F})$ is either $\mathcal{F}$ itself or the unstable foliation of $[f]$. In either way, the homeomorphism $g \in \mathrm{Homeo}^+(S')$ is freely isotopic to a pseudo-Anosov map or a periodic map. □

Here is our main result. Later in Example 5.7 we see that it generalizes the so called *daisy relation* [**9**] in mapping class group theory.

THEOREM 5.4. *Suppose that the surfaces $S'_1, \ldots, S'_k$ are pairwise non-isotopic. Assume that $[f] \in \mathrm{Dehn}^+(S')$ is either*

- *a non-negative power of a single Dehn twist (when S' is an annular neighborhood of a simple closed geodesic curve), or*

- *a pseudo-Anosov map (when S' is a hyperbolic surface).*

Suppose that $b \in \mathrm{Dehn}^+(n,k)$ *satisfies*

$$\Psi(b) = [f_1 \circ f_2 \circ \cdots \circ f_k]$$

then $b \in QP(n)$.

PROOF. There are two cases to consider.

Case 1: $S'_1, S'_2, \dots, S'_k$ are pairwise disjoint.

Since $[f] \in \mathrm{Dehn}^+(S')$ we may write $[f_1] = T_{C_1} \cdots T_{C_l}$ for some simple closed curves $C_1, \dots, C_l \subset S'$. We get $[f_j] = [\iota^{j-1}\widehat{f}\iota^{-j+1}] = T_{\iota^{j-1}(C_1)} \cdots T_{\iota^{j-1}(C_l)}$. Since $S'_1, S'_2, \dots, S'_k$ are pairwise disjoint, if $j \neq j'$ then $T_{\iota^j(C_i)}$ and $T_{\iota^{j'}(C_{i'})}$ commute for every i, i' and we have

$$\begin{aligned}\Psi(b) &= [f_1 \circ \cdots \circ f_k]\\ &= (T_{C_1} \cdots T_{C_l})(T_{\iota(C_1)} \cdots T_{\iota(C_l)}) \cdots (T_{\iota^{k-1}(C_1)} \cdots T_{\iota^{k-1}(C_l)})\\ &= (T_{C_1} T_{\iota(C_1)} \cdots T_{\iota^{k-1}(C_1)}) \cdots (T_{C_l} T_{\iota(C_l)} \cdots T_{\iota^{k-1}(C_l)}).\end{aligned}$$

Since $S'_1, S'_2, \dots, S'_k$ are pairwise disjoint, the projection $\pi(C_i)$ is a simple closed curve in D_n, and we can use the second-half argument in the proof of Theorem 5.1. By the formula (5.1) we get $\Psi(T_{\pi(C_i)}) = T_{C_i} T_{\iota(C_i)} \cdots T_{\iota^{k-1}(C_i)}$. Proposition 2.2 gives $b = T_{\pi(C_1)} \cdots T_{\pi(C_l)} \in QP(n)$.

Case 2: $S'_1 \cap S'_p \neq \emptyset$ for some $p \in \{2, \dots, k\}$.

First we note that $[f_1 \circ f_2 \circ \cdots \circ f_k] = \Psi(b) \in \mathrm{SMod}(S)$ implies that

$$[f_1 \circ f_2 \circ \cdots \circ f_k] = [\iota \circ (f_1 \circ f_2 \circ \cdots \circ f_k) \circ \iota^{-1}] = [f_2 \circ f_3 \circ \cdots \circ f_k \circ f_1] = \cdots = [f_k \circ f_1 \circ \cdots \circ f_{k-1}].$$

In particular, we have

$$\Psi(b)[f_i] = [f_i]\Psi(b) \text{ for every } i \in \{1, \dots, k\}. \tag{5.2}$$

Let ν be the minimal subsurface of S with respect to inclusions such that

- ν contains $S'_1 \cup S'_2 \cup \cdots \cup S'_k$ and
- $\partial\nu$ is geodesic.

Since $\iota(S'_1 \cup S'_2 \cup \cdots \cup S'_k) = S'_1 \cup S'_2 \cup \cdots \cup S'_k$ we have $\iota(\nu) = \nu$. In particular, the boundary $\partial\nu$ is a multi-curve invariant under ι.

For simplicity, we put $\psi := f_1 \circ f_2 \circ \cdots \circ f_k \in \mathrm{Homeo}^+(S)$.

CLAIM 5.5. *$\psi(S'_i)$ is isotopic to S'_i.*

We will prove Claim 5.5 after the completion of the proof of Theorem 5.4.

By Claim 5.5 there is a homeomorphism $\phi \in \mathrm{Homeo}^+(S)$ which is isotopic to ψ and preserves S'_i setwise. Although ϕ may permute components of $\partial S'_i$, we may assume that there exists $d_0 > 0$ such that $\phi^{d_0} = id$ on $\partial S'_1 \cup \partial S'_2 \cup \cdots \cup \partial S'_k$. Let

$$\phi_i := \phi^{d_0}|_{S'_i} \in \mathrm{Homeo}^+(S'_i). \tag{5.3}$$

Claim 5.6. *For each $i = 1, \dots, k$ there is $d_i > 0$ such that $\phi_i^{d_i}$ is isotopic to a product of Dehn twists about the boundary components of S'_i. Namely,*

$$[\phi_i^{d_i}] = \prod_{C \subset \partial S'_i} T_C^{N(C)} \in \mathrm{Mod}(S'_i) \quad \text{for some } N(C) \in \mathbb{Z}. \tag{5.4}$$

Moreover, the exponent $N(C) = 0$ if C is essential in the minimal subsurface ν. Therefore, if S' is an annulus then $[\phi_i^{d_i}] = id$.

We will prove the claim after the completion of the proof of Theorem 5.4.

Let d' be the least common multiple of $d_1, \dots, d_k$ found in Claim 5.6. Put $d = d_0 d'$. As a consequence of Claim 5.6 the map $\phi^d|_{S_{i'}} = \phi_i^{d'} \in \mathrm{Homeo}^+(S'_i)$ is isotopic to a product of Dehn twists along common components of $\partial S'_i$ and $\partial\nu$. Let $C_1, \dots, C_m$ denote the boundary components of ν. We obtain

$$[\phi^d] = T_{C_1}^{N(1)} T_{C_2}^{N(2)} \cdots T_{C_m}^{N(m)}. \tag{5.5}$$

Since $[f] \in \mathrm{Dehn}^+(S')$ we see that $[\phi] = [f_1 \circ \cdots \circ f_k]$ is right-veering and $N(i) \geq 0$ for all $i = 1, \dots, m$.

We define an equivalence relation $C_i \sim C_{i'}$ if $\pi(C_i) = \pi(C_{i'}) \subset D_n$ and let $\mathcal{C} = \{C_1, \dots, C_m\}/\sim$. For $C \in \{C_1, C_2, \dots, C_m\}$ let $[C] \in \mathcal{C}$ be its equivalence class and $e(C) \in \mathbb{N}$ the smallest positive integer such that $\iota^{e(C)}(C) = C$. Since $\iota(\partial\nu) = \partial\nu$ we note that $C, \iota(C), \dots, \iota^{e(C)-1}(C) \subset \partial\nu = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m$. Put

$$T_{[C]} := T_C T_{\iota(C)} \cdots T_{\iota^{e(C)-1}(C)}.$$

The fact $[\phi] \in \mathrm{SMod}(S)$ implies that $N(i) = N(j)$ if $C_i \sim C_j$. We may define non-negative integers $N([C_i]) := N(i)$. The description (5.5) can be restated as

$$[\phi^d] = \prod_{[C] \in \mathcal{C}} T_{[C]}^{N([C])} \in \mathrm{Mod}(S).$$

Recall that the projection $\pi(C)$ is a simple closed curve or a simple proper arc in D_n joining distinct punctures. This is because $\pi^{-1}(\pi(C)) \subset \partial\nu$ is a simple closed multi-curve.

If $\pi(C)$ is a simple closed curve, put $b_C := (T_{\pi(C)})^{k/e(C)} \in QP(n)$. Then by (5.1) we have $\Psi(b_C) = \Psi((T_{\pi(C)})^{k/e(C)}) = T_{[C]}$.

If $\pi(C)$ is a simple arc, i.e., $k = 2$ and $e(C) = 1$, put $b_C := h_{\pi(C)} \in QP(n)$ the positive half twist about $\pi(C)$. Then we have $\Psi(b_C) = \Psi(h_{\pi(C)}) = T_C = T_{[C]}$.

For either case, we have

$$\Psi(b^d) = [\phi^d] = \Psi\left(\prod_{[C] \in \mathcal{C}} (b_C)^{N([C])}\right).$$

For distinct $[C]$ and $[C']$ the projections $\pi(C)$ and $\pi(C')$ are disjoint because $C_1, \ldots, C_m$ are pairwise disjoint. Hence, by Proposition 2.2 the braid b^{2d} is a product of positive Dehn twists about pairwise disjoint simple closed curves. (We consider b^{2d} rather than b^d so that if $\pi(C)$ is an arc the half twist about $\pi(C)$ becomes a Dehn twist about a closed curve enclosing $\pi(C)$.) Corollary 4.7 shows that $b \in QP(n)$. $\square$

Proof of Claim 5.5. Suppose that S' is an annulus which is a neighborhood of a simple closed geodesic curve C and $[f] = (T_C)^m$. By (5.2) we have $(T_{\psi(C)})^m = ([\psi]T_C[\psi^{-1}])^m = [\psi](T_C)^m[\psi]^{-1} = (T_C)^m$, and $\psi(C)$ is isotopic to C. Hence, $\psi(S'_i)$ is isotopic to S'_i.

Next, suppose that $[f] \in \mathrm{Mod}(S')$ is pseudo-Anosov. The property (5.2) implies that $f_i \circ \psi(S'_i)$ is isotopic to $\psi \circ f_i(S'_i) = \psi(S'_i)$. Then either $\psi(S'_i)$ is isotopic to S'_i, or by isotopy one can make $\psi(S'_i)$ and S'_i disjoint. The latter possibility cannot be realized by the following reason. Take an essential simple closed curve $\alpha \subset S'_i$. If we can make $\psi(S'_i)$ disjoint from S'_i by isotopy, then $\psi(\alpha)$ can also be disjoint from S'_i. However by (5.2),

$$\alpha = f_i \circ f_i^{-1} \circ \psi^{-1} \circ \psi(\alpha) \sim f_i \circ \psi^{-1} \circ f_i^{-1}(\psi(\alpha)) = f_i \circ \psi^{-1}(\psi(\alpha)) = f_i(\alpha)$$

where "$\sim$" means isotopic. This contradicts the assumption that $[f]$ is pseudo-Anosov. $\square$

Proof of Claim 5.6. In the case where S' is an annulus, (5.4) is a direct consequence of Claim 5.5.

Assume that $[f] \in \mathrm{Mod}(S')$ is pseudo-Anosov and S' is not an annulus. By the symmetry, it is enough to prove the statement (5.4) for the case $i = 1$. Recall that $S'_1 \neq S'_p$ and $S'_1 \cap S'_p \neq \emptyset$. Let D be a connected component of $S'_1 \cap S'_p$. Since $\phi^{d_0} = id$ on $\partial S'_1 \cup \partial S'_2 \cup \cdots \cup \partial S'_k$ and $\partial D \subset (\partial S'_1 \cup \partial S'_2 \cup \cdots \cup \partial S'_k)$ we have $\phi^{d_0}(D) = D$ and $\phi^{d_0} = id$ on ∂D.

Since $\partial S'_1$ is geodesic D cannot be a bigon or an annulus. Let $\Gamma = \partial(S'_1 \setminus D)$. Then Γ is a simple closed multi-curve in S'_1. Since S'_1 is not an annulus Γ contains an arc γ which is essential in S'_1. Since $\phi^{d_0} = id$ on ∂D the curve $\phi_1(\gamma) = \phi^{d_0}(\gamma)$ is isotopic to γ; hence, the mapping class $[\phi_1] \in \mathrm{Mod}(S')$ cannot be pseudo-Anosov.

On the other hand, $[f] = [f_1|_{S'_1}] \in \mathrm{Mod}(S')$ is pseudo-Anosov, and by (5.2) $[\phi_1] \in Z([f]) \subset \mathrm{Mod}(S')$. Since $[\phi_1]$ is not pseudo-Anosov, Lemma 5.3 shows that $[\phi_1]$ is periodic. Namely, there is $d_1 > 0$ such that $[\phi_1^{d_1}]$ is a product of Dehn twists about the boundary components of S'_1 and we obtain (5.4).

Next we show the second statement of the claim. The surface S' is either annular or hyperbolic. Suppose that a boundary component C of S'_1 is an essential curve in the surface ν. Since C is not a boundary component of ν and ν is the minimal surface containing $S'_1 \cup \cdots \cup S'_k$ with respect to inclusions, there must exist $p \neq 1$ such that $C \cap S'_p \neq \emptyset$.

Note that C cannot be a boundary component of S'_p (that is, one side of C is S'_1 and the other side of C is S'_p) because in such a case the quotient space S/ι cannot be a topological disk.

If C is not isotopic to a boundary component of S'_p then the descriptions of $[\phi_1^{d_1}]$ and $[\phi_p^{d_p}]$ in (5.4) and the definition of ϕ_i in (5.3) show that $N(C) = 0$.

If C transversely intersects a boundary component, say C', of S'_p, we also get $N(C) = 0$, because otherwise (5.3) and (5.4) show that $\phi(C') \not\subset S'_p$ which contradicts Claim 5.5. $\square$

We close the paper with an example which shows that Theorem 5.4 can be viewed as a generalization of the *daisy relation* found in [**9**].

Example 5.7. Let F be a sphere with $k+1$ holes ($k \geq 3$) that is obtained as the k-fold cyclic branched covering $\pi_F : F \to A$ of an annulus A branched at one point. Let $a_0, \ldots, a_k$ be the boundary components of F. Let $\iota_F : F \to F$ be a deck transformation defined by a $2\pi/k$ rotation of F about the unique branch point such that $\iota_F(a_0) = a_0$ and $\iota_F(a_i) = a_{i+1}$ for $i = 1, \ldots, k-1$, and $\iota_F(a_k) = a_1$. See the left hand side picture of Figure 4. Let x_i ($i = 1, \ldots, k$) be simple closed curves on F enclosing a_0 and a_i such that $\iota_F(x_i) = x_{i+1}$. According to the *daisy relation* as stated in [**9**] we have

$$T_{x_1} \cdots T_{x_k} = T_{a_0}^{k-2} T_{a_1} \cdots T_{a_k} \in \mathrm{Mod}(F). \tag{5.6}$$

(The case of $k = 3$ yields the famous lantern relation.)

Recall the k-fold cyclic branched covering $\pi : S \to D$ with n branch points and the deck transformation $\iota : S \to S$. Take an embedding $i : F \hookrightarrow S$ such that $i \circ \iota_F(x) = \iota \circ i(x)$ for all $x \in F$. Let $C = i(x_1), A_0 = i(a_0)$, and $A = i(a_1)$. See the right hand side picture in Figure 4 for the simplest case $(n,k) = (3,3)$.

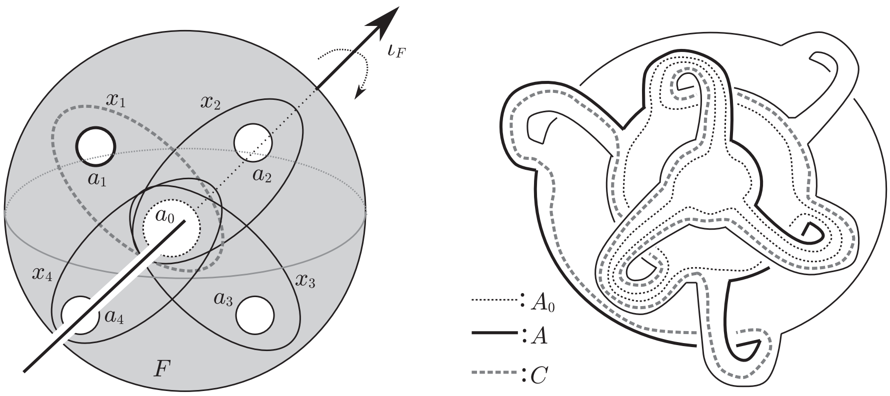

Figure 4. Left: The $(k+1)$ holed sphere F for $k = 4$. Right: The surface S for $(n,k) = (3,3)$. The curves C, A, A_0 satisfy the daisy (lantern) relation $T_C T_{\iota(C)} T_{\iota^2(C)} = T_{A_0} T_A T_{\iota(A)} T_{\iota^2(A)}$. The surface $S \setminus i(F)$ is the Bennequin surface of a $(2,3)$ torus knot, which is A_0.

The daisy relation (5.6) gives

$$T_C T_{\iota(C)} \cdots T_{\iota^{k-1}(C)} = \left(T_{A_0}^{k-2}\right)\left(T_A T_{\iota(A)} \cdots T_{\iota^{k-1}(A)}\right) \in \mathrm{Mod}(S). \tag{5.7}$$

The left hand side of (5.7) is of the form of $[f_1 f_2 \cdots f_k]$ which is studied in Theorem 5.4. One can show, using [**18**, Lemma 3.1] and the chain relation [**11**, Proposition 4.12] in mapping class group theory, the term $T_{A_0}^{k-2}$ in the right hand side of (5.7) is the image of a quasipositive braid under the homomorphism $\Psi : B_n \to \mathrm{Mod}(S)$. We can also see that the link $A \cup \iota(A) \cup \cdots \cup \iota^{k-1}(A)$ is a (k,k) torus link and the term $T_A T_{\iota(A)} \cdots T_{\iota^{k-1}(A)}$ is the image under Ψ of a positive full twist of k-stranded trivial braid, which is clearly a quasipositive element in B_n. In this sense, Theorem 5.4 can be seen as a generalization of the daisy relation.

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