J. Math. Soc. Japan
Vol. 71, No. 2 (2019) pp. 451–469
doi: 10.2969/jmsj/78397839

The logarithmic derivative for point processes with equivalent Palm measures

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(Received July 6, 2017)
(Revised Oct. 17, 2017)

Abstract. The *logarithmic derivative* of a point process plays a key rôle in the general approach, due to the third author, to constructing diffusions preserving a given point process. In this paper we explicitly compute the logarithmic derivative for determinantal processes on $\mathbb{R}$ with integrable kernels, a large class that includes all the classical processes of random matrix theory as well as processes associated with de Branges spaces. The argument uses the quasi-invariance of our processes established by the first author.

1. Introduction.

Let $\mathbb{P}$ be a point process on $\mathbb{R}^d$, or, in other words, a Borel probability measure on the space of locally finite configurations $\mathrm{Conf}(\mathbb{R}^d)$. It is a natural question whether one can construct a diffusion $\xi(t) = (\xi^1(t), \xi^2(t), \dots, \xi^i(t), \dots)$ on the space $(\mathbb{R}^d)^{\mathbb{N}}$ such that the configuration $X(t) = \{\xi^1(t), \xi^2(t), \dots, \xi^i(t), \dots\}$ is almost surely locally finite for every $t \in \mathbb{R}_+$, and the process $X(t)$, considered as a process on the space $\mathrm{Conf}(\mathbb{R}^d)$, preserves the measure $\mathbb{P}$. For example, if $\mathbb{P}$ is the standard Poisson point process on $\mathbb{R}^d$, then $\xi^i(t)$ are independent Brownian motions. In the series of papers [**9**], [**12**]–[**18**] the third author with collaborators developed a general approach to constructing the process ξ. The key step is the computation of the *logarithmic derivative* $d^{\mathbb{P}}$ of the measure $\mathbb{P}$, $d^{\mathbb{P}} : \mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d) \mapsto \mathbb{R}^d$, introduced by the third author in [**13**]. The process ξ is then a solution of the infinite-dimensional stochastic differential equation

$$\xi^i(t) = \xi^i(0) + B^i(t) + \frac{1}{2}\int_0^t d^{\mathbb{P}}(\xi^i(u), X_i(\xi(u)))du, \quad i \in \mathbb{N},$$

where the configuration X_i is defined by the formula $X_i(\xi(u)) := \{\xi^j(u)\}_{j \neq i}$ and B^i are independent Brownian motions. In [**9**], [**13**], [**18**] logarithmic derivatives were calculated for determinantal processes arising in random matrix theory: sine_2, Airy_2, Bessel_2 and the Ginibre point processes. The computation was based on finite particle approximation and had to be adapted for each determinantal process separately.

Theorem 2.3, the main result of this paper, establishes existence and gives an explicit formula for the logarithmic derivative for determinantal point processes on $\mathbb{R}$ with integrable kernels studied in [**5**], a class that includes, in particular, determinantal processes

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2010 *Mathematics Subject Classification.* Primary 60G55; Secondary 60J60.

Key Words and Phrases. point processes, determinantal processes, logarithmic derivative, infinite-dimensional diffusion, Palm measure.

mentioned above and those corresponding to de Branges spaces [**7**] .

There are other methods to constructing infinite-dimensional diffusions. In particular, in [**10**], [**11**], using extended determinantal kernels, Katori and Tanemura constructed diffusions reversible with respect to the sine_2, Airy_2, and Bessel_2 point processes. A different approach to studying the diffusion preserving the sine_2 process is due to Tsai [**22**]. In [**1**], Borodin and Olshanski gave a construction of infinite-dimensional diffusions as scaling limits of random walks on partitions.

To explain our results in more details we first give an informal definition of the logarithmic derivative. Consider a point process $\mathbb{P}$ on $\mathbb{R}^d$ which admits a differentiable first correlation function $\rho_1 : \mathbb{R}^d \mapsto \mathbb{R}$. Denote by $\mathbb{P}^a$ the reduced Palm measure conditioned at the point $a \in \mathbb{R}^d$ and define the reduced Campbell measure $\mathcal{C}_{\mathbb{P}}$ as a Borel sigma-finite measure on the space $\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)$ given by

$$d\mathcal{C}_{\mathbb{P}}(a, X) = \rho_1(a) d\mathbb{P}^a(X) da.$$

Then, informally, the logarithmic derivative $d_{\mathbb{P}}$ is defined as a gradient of the logarithm of $\mathcal{C}_{\mathbb{P}}$,

$$d^{\mathbb{P}}(a, X) = \nabla_a \big(\ln \rho_1(a) + \ln \mathbb{P}^a(X) \big), \tag{1.1}$$

see Definition 2.1. The main problem when proving the existence of the logarithmic derivative is to give a sense to the term $\nabla_a \ln \mathbb{P}^a(X)$.

Our first result is Proposition 2.2, where we find the connection between equivalence of the Palm measures conditioned at different points and the existence of the logarithmic derivative. More specifically, we consider a point process $\mathbb{P}$ on $\mathbb{R}^d$ as above and assume that for any $a, b \in \mathbb{R}^d$ the reduced Palm measures $\mathbb{P}^a$ and $\mathbb{P}^b$ are equivalent,

$$d\mathbb{P}^b(X) = \mathcal{R}_{b,a}(X) d\mathbb{P}^a(X),$$

the Radon–Nikodym derivative $\mathcal{R}_{b,a}(X)$ is continuous with respect to b in $L^1(\mathbb{P}^a, \mathrm{Conf}(\mathbb{R}^d))$ and the derivative $\nabla_b \mathcal{R}_{b,a}$ exists in appropriate sense. We then prove that the logarithmic derivative $d^{\mathbb{P}}$ exists and the formula (1.1) is valid with

$$\nabla_a \ln \mathbb{P}^a := \nabla_b \big|_{b=a} \mathcal{R}_{b,a}.$$

Our second and main result is the mentioned above Theorem 2.3. To establish it, it suffices to check that assumptions of Proposition 2.2 are satisfied for the considered class of determinantal point processes. To show this, we use the results of the paper [**5**], where the first author proved that for this class of determinantal processes the reduced Palm measures are equivalent and the Radon–Nikodym derivative has the form

$$\mathcal{R}_{b,a} = \lim_{R\to\infty, \delta\to 0} \mathcal{R}_{b,a}^{R,\delta} \quad \text{where} \quad \mathcal{R}_{b,a}^{R,\delta} = C_{b,a}^{R,\delta} \prod_{\substack{X \in \mathrm{Conf}(\mathbb{R}) :\, |x|<R, \\ |x-a|, |x-b| > \delta}} \left(\frac{x-b}{x-a} \right)^2,$$

and $C_{b,a}^{R,\delta}$ are some normalizing constants. While the continuity in b of $\mathcal{R}_{b,a}$ follows immediately from the results of [**5**], the proof of its differentiability requires some efforts.

To get it, we approximate the Radon–Nikodym derivative $\mathcal{R}_{b,a}$ by the function $\mathcal{R}_{b,a}^{R,\delta}$, compute the derivative of the latter, and then pass to the limit $R \to \infty$, $\delta \to 0$ using the techniques of normalized additive and multiplicative functionals developed in [**5**], which we outline in the appendix. Finally, we find

$$\nabla_b\big|_{b=a} \mathcal{R}_{b,a} = \lim_{R\to\infty,\delta\to 0} (S_a^{R,\delta} - \mathbb{E}^a S_a^{R,\delta}),$$

where $S_a^{R,\delta} = \sum_{\substack{x\in X:\,|x|<R,\\ |x-a|>\delta}} 2/(a-x)$ and $\mathbb{E}^a$ stands for the expectation with respect to the reduced Palm measure $\mathbb{P}^a$.

The paper is organized as follows. In Section 2 we formulate our main results, Proposition 2.2 and Theorem 2.3. Section 3 is devoted to the proofs. In Appendix A we recall some results of [**5**] needed in the proof of Theorem 2.3.

2. Formulation of the main results.

2.1. Configurations, point processes, Palm distributions.

Consider the space of locally finite configurations

$$\operatorname{Conf}(\mathbb{R}^d) := \big\{X \subset \mathbb{R}^d | \, X \text{ does not have limit points in } \mathbb{R}^d\big\}.$$

A Borel probability measure $\mathbb{P}$ on $\operatorname{Conf}(\mathbb{R}^d)$ is called a point process. Take a bounded Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ and consider a function $\#_B : \operatorname{Conf}(\mathbb{R}^d) \mapsto \mathbb{N}\cup\{0\}$ such that $\#_B(X)$ is equal to the cardinality of the set $B \cap X$. Assume that the process $\mathbb{P}$ admits the first correlation function ρ_1, that is for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$ the function $\#_B$ is integrable with respect to the measure $\mathbb{P}$ and there exists a function $\rho_1 \in L^1_{loc}(\mathbb{R}^d)$ satisfying

$$\int_B \rho_1(x)\, dx = \int_{\operatorname{Conf}(\mathbb{R}^d)} \#_B(X) d\mathbb{P}(X), \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \quad B \text{ is bounded.}$$

Define the first correlation measure $\hat\rho_1$ as $\hat\rho_1(B) = \int_B \rho_1(x)\, dx$.

The *Campbell measure* $\hat{\mathcal{C}}_{\mathbb{P}}$ is a sigma-finite Borel measure on $\mathbb{R}^d \times \operatorname{Conf}(\mathbb{R}^d)$ defined as

$$\hat{\mathcal{C}}_{\mathbb{P}}(B, \mathscr{X}) = \int_{\mathscr{X}} \#_B(X) d\mathbb{P}(X), \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \quad \mathscr{X} \in \mathcal{B}\big(\operatorname{Conf}(\mathbb{R}^d)\big),$$

where $\mathcal{B}\big(\operatorname{Conf}(\mathbb{R}^d)\big)$ stands for the Borel sigma-algebra on $\operatorname{Conf}(\mathbb{R}^d)$. Fix a Borel set $\mathscr{X} \subset \operatorname{Conf}(\mathbb{R}^d)$ and consider a sigma-finite measure $\mathcal{C}_{\mathbb{P}}^{\mathscr{X}}$ on $\mathbb{R}^d$ given by the formula

$$\mathcal{C}_{\mathbb{P}}^{\mathscr{X}}(B) = \hat{\mathcal{C}}_{\mathbb{P}}(B, \mathscr{X}), \quad \forall B \in \mathcal{B}(\mathbb{R}^d).$$

By definition, for any $\mathscr{X} \in \mathcal{B}\big(\operatorname{Conf}(\mathbb{R}^d)\big)$ the measure $\mathcal{C}_{\mathbb{P}}^{\mathscr{X}}$ is absolutely continuous with respect to $\hat\rho_1$. Then the *Palm measure* $\hat{\mathbb{P}}^a$, defined for $\hat\rho_1$-almost every $a \in \mathbb{R}^d$, is a measure on $\operatorname{Conf}(\mathbb{R}^d)$ given by the relation

$$\hat{\mathbb{P}}^a(\mathscr{X}) = \frac{d\mathcal{C}_{\mathbb{P}}^{\mathscr{X}}}{d\hat\rho_1}(a)\,.$$

Equivalently, the Palm measure $\hat{\mathbb{P}}^a$ is the canonical conditional measure of the Campbell measure $\hat{\mathcal{C}}_{\mathbb{P}}$ with respect to the measurable partition of the space $\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)$ into subsets of the form $\{a\} \times \mathrm{Conf}(\mathbb{R}^d)$, $a \in \mathbb{R}^d$. Thus, we can write

$$d\hat{\mathcal{C}}_{\mathbb{P}}(a, X) = d\hat{\mathbb{P}}^a(X)\rho_1(a)\,da.$$

By definition, the Palm measure $\hat{\mathbb{P}}^a$ is supported on the subset of configurations containing a particle at the position a. The *reduced Palm measure* $\mathbb{P}^a$ is defined as the push-forward of the Palm measure $\hat{\mathbb{P}}^a$ under the map $X \to X \setminus \{a\}$ erasing the particle a from the configuration X. We then define the *reduced Campbell measure* $\mathcal{C}_{\mathbb{P}}$ as

$$d\mathcal{C}_{\mathbb{P}}(a, X) = d\mathbb{P}^a(X)\rho_1(a)\,da. \tag{2.1}$$

Note that, in difference with the notions of the (reduced) Palm measure and the Campbell measure, this definition is not standard. Writing it in a more formal way, we obtain

$$\mathcal{C}_{\mathbb{P}}(B, \mathscr{X}) = \int_B \int_{\mathscr{X}} d\mathbb{P}^a(X)\rho_1(a)\,da, \quad \forall B \in \mathcal{B}(\mathbb{R}^d), \quad \mathscr{X} \in \mathcal{B}\big(\mathrm{Conf}(\mathbb{R}^d)\big).$$

For more details see e.g. [**5**] and [**8**].

2.2. Definition of the logarithmic derivative.

A function $\varphi : \mathrm{Conf}(\mathbb{R}^d) \mapsto \mathbb{R}$ is called *local* if there exists a compact set $K \subset \mathbb{R}^d$ such that $\varphi(X) \equiv \varphi(X \cap K)$. For a local function φ we define symmetric functions $\varphi_n : \mathbb{R}^{nd} \mapsto \mathbb{R}$, $n \geq 1$, by the relation

$$\varphi_n(x_1, \ldots, x_n) = \varphi\big(\{x_1, \ldots, x_n\}\big).$$

We say that a local function φ is smooth if the functions φ_n are smooth for all $n \in \mathbb{N}$. We denote by $\mathcal{D}_0$ the space of all bounded local smooth functions on $\mathrm{Conf}(\mathbb{R}^d)$.

Denote by B_R a ball in $\mathbb{R}^d$ of radius R. Let $L^1_{loc}(\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d), \mathcal{C}_{\mathbb{P}})$ be the space of vector-functions $f : \mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d) \mapsto \mathbb{R}^d$ satisfying $f \in L^1(B_R \times \mathrm{Conf}(\mathbb{R}^d), \mathcal{C}_{\mathbb{P}})$, for all $R > 0$.

Denote by C_0^∞ the space of smooth real-valued functions on $\mathbb{R}^d$ which have compact supports. We say that a function $\varphi : \mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d) \mapsto \mathbb{R}$ belongs to the space $C_0^\infty\mathcal{D}_0$ if $\varphi = \varphi_1\varphi_2$, where $\varphi_1 \in C_0^\infty(\mathbb{R}^d)$ and $\varphi_2 \in \mathcal{D}_0$.

DEFINITION 2.1. Let $\mathbb{P}$ be a point process on $\mathbb{R}^d$ that admits the first correlation function. A function $d^{\mathbb{P}} \in L^1_{loc}(\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d), \mathcal{C}_{\mathbb{P}})$ is called the *logarithmic derivative* of $\mathbb{P}$ if for any observable $\varphi \in C_0^\infty\mathcal{D}_0$ we have

$$\int_{\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)} \nabla_a\varphi(a, X)\,d\mathcal{C}_{\mathbb{P}}(a, X) = -\int_{\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)} d^{\mathbb{P}}(a, X)\varphi(a, X)\,d\mathcal{C}_{\mathbb{P}}(a, X).$$

2.3. Logarithmic derivative of a point process with equivalent Palm measures.

Consider a point process $\mathbb{P}$ on $\mathbb{R}^d$ that admits the first correlation function ρ_1; recall that we denote by $\hat{\rho}_1$ the first correlation measure,

$$\hat{\rho}_1(da) = \rho_1(a)\, da.$$

In this subsection we give a general scheme for the computation of the logarithmic derivative $d^{\mathbb{P}}$ under the following assumption.

Assumption 1.

1. The first correlation function ρ_1 is C^1-smooth.
2. For $\hat{\rho}_1$-almost all $a, b \in \mathbb{R}^d$ the reduced Palm measures $\mathbb{P}^a$ and $\mathbb{P}^b$ are equivalent.

Denote by $\mathcal{R}_{b,a}$ their Radon–Nikodym derivative, so that

$$d\mathbb{P}^b(X) = \mathcal{R}_{b,a}(X)\, d\mathbb{P}^a(X).$$

3. For $\hat{\rho}_1$-almost all $a \in \mathbb{R}^d$ we have $\mathcal{R}_{b,a} \to 1$ as $b \to a$ in $L^1(\mathbb{P}^a, \mathrm{Conf}(\mathbb{R}^d))$.

For a function $\varphi \in C_0^\infty(\mathbb{R}^d)\mathcal{D}_0$ we define the function $f_\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ as

$$f_\varphi(\varepsilon) := \int_{\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)} \mathcal{R}_{a+\varepsilon,a}(X)\varphi(a, X)\, d\mathcal{C}_{\mathbb{P}}(a, X).$$

4. For any $\varphi \in C_0^\infty(\mathbb{R}^d)\mathcal{D}_0$ the function f_φ admits partial derivatives in ε at the point $\varepsilon = 0$. There exist functions $\partial_i \mathcal{R} : \mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d) \mapsto \mathbb{R}$ such that for any φ as above and any $1 \le i \le d$ we have

$$\partial_{\varepsilon_i} f_\varphi(0) = \int_{\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)} \partial_i \mathcal{R}(a, X)\varphi(a, X)\, d\mathcal{C}_{\mathbb{P}}(a, X).$$

Set

$$\nabla \mathcal{R} := (\partial_1 \mathcal{R}, \ldots, \partial_d \mathcal{R}).$$

Proposition 2.2. *Let $\mathbb{P}$ be a point process on $\mathbb{R}^d$ satisfying Assumption* 1. *Then for $\hat{\rho}_1$-almost all $a \in \mathbb{R}$ its logarithmic derivative $d^{\mathbb{P}}$ exists and has the form*

$$d^{\mathbb{P}}(a, X) = \nabla_a \ln \rho_1(a) + \nabla \mathcal{R}(a, X). \tag{2.2}$$

Note that there is no need to define the logarithmic derivative at the points $a \in \mathbb{R}^d$ where $\rho_1(a) = 0$ since the measure $\hat{\rho}_1$ of the set of such a is zero. Proof of Proposition 2.2 is given in Section 3.1. It is based on the differentiation by parts, that is why we crucially need the absolute continuity of the measure $\hat{\rho}_1$ and the differentiability of its density, which is the first correlation function ρ_1.

2.4. Logarithmic derivative of a determinantal process on $\mathbb{R}$ with an integrable kernel.

In this section we construct the logarithmic derivative for a class of determinantal processes on $\mathbb{R}$. A point process $\mathbb{P}$ on $\mathrm{Conf}(\mathbb{R})$ is called *determinantal* if there exists a locally trace class operator $\mathcal{P} : L^2(\mathbb{R}, dx) \mapsto L^2(\mathbb{R}, dx)$ such that for any bounded measurable function h, for which the support $\mathrm{supp}(h-1) =: D$ is a compact set, we have

$$\mathbb{E}\left(\prod_{x\in X} h(x)\right) = \det\big(1 + (h-1)\mathcal{P}\mathbb{I}_D\big).$$

Here the expectation is taken with respect to the measure $\mathbb{P}$, det stands for the Fredholm determinant and $\mathbb{I}_D$ denotes the indicator function of the set D. See for details [**19**], [**21**]. Since the operator $\mathcal{P}$ is locally trace class, it admits a kernel which we denote by Π. Note that for any $a \in \mathbb{R}$ the function $\Pi(a, \cdot)$ belongs to $L^2(\mathbb{R}, dx)$. Further on we will use the results of [**5**], so we impose on $\mathcal{P}$ and Π the following restrictions coming from [**5**].

Assumption 2.

1. The operator $\mathcal{P}$ is an orthogonal projection onto a closed subspace $L \subset L^2(\mathbb{R}, dx)$.
2. For $\hat\rho_1$-almost all $a \in \mathbb{R}$, given any function $\varphi \in L$ satisfying $\varphi(a) = 0$, we have $(x-a)^{-1}\varphi \in L$.
3. The kernel Π is C^2-smooth on $\mathbb{R}^2$.
4. We have $\displaystyle\int_{\mathbb{R}} \frac{\Pi(x,x)}{1+x^2}\,dx < \infty$.

Item 2 of Assumption 2 is called the *division property* (cf. [**6**]). Recall that the kernel Π is called integrable if there exists an open set U, such that $\hat\rho_1(\mathbb{R} \setminus U) = 0$, and linearly independent smooth functions A, B defined on U, satisfying

$$\Pi(x,y) = \frac{A(x)B(y) - A(y)B(x)}{x-y}, \tag{2.3}$$

if $x \neq y$. Proposition 1.2 from [**5**] proves that the division property is fulfilled for operators of orthogonal projection which admit integrable kernels. In particular, it follows that Assumption 2 is satisfied for the sine, Airy and Bessel kernels as well as for kernels corresponding to de Branges spaces [**7**]. We recall that the sine, Airy and Bessel kernels are given by the formula (2.3) with

$$A(x) = \begin{cases} \pi^{-1}\sin(\pi x) & \text{for the sine kernel,} \\ \mathrm{Ai}(x) & \text{for the Airy kernel,} \\ \mathrm{J}_\alpha(x) & \text{for the Bessel kernel,} \end{cases}$$

and $B(x) = (d/dx)A(x)$, where Ai is the Airy function and J_α is the Bessel function of order α, $\alpha > -1$.

Take $a \in \mathbb{R}$, $R \gg 1$ and $\delta \ll 1$, and consider the additive functional

$$S_a^{R,\delta} : \mathrm{Conf}(\mathbb{R}) \mapsto \mathbb{R}, \quad S_a^{R,\delta}(X) = \sum_{\substack{x\in X:\, |x|<R,\\ |x-a|>\delta}} \frac{2}{a-x}. \tag{2.4}$$

The additive functional $S_a^{R,\delta}$ may diverge as $R \to \infty$. To overcome this difficulty we define the normalized additive functional

$$\overline{S}_a^{R,\delta} := S_a^{R,\delta} - \mathbb{E}^a S_a^{R,\delta},$$

where $\mathbb{E}^a$ stands for the expectation with respect to the reduced Palm measure $\mathbb{P}^a$. Results obtained in [**5**] imply that, under Assumption 2, for $\hat{\rho}_1$-almost all $a \in \mathbb{R}$ there exists a function $\overline{S}_a : \mathrm{Conf}(\mathbb{R}) \mapsto \mathbb{R}$, such that

$$\overline{S}_a^{R,\delta} \to \overline{S}_a \quad \text{as} \quad R \to \infty,\, \delta \to 0 \quad \text{in} \quad L^2(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a). \tag{2.5}$$

Moreover, the convergence (2.5) holds uniformly in a as $a \in \mathbb{R}$ ranges in a compact set. The required theory from [**5**] is recalled in Appendix A.1 and the convergence (2.5) is established in Corollary A.2.

THEOREM 2.3. *Let $\mathbb{P}$ be a determinantal process on $\mathbb{R}$ satisfying Assumption* 2. *Then for $\hat{\rho}_1$-almost all $a \in \mathbb{R}$ the logarithmic derivative $d^{\mathbb{P}}$ exists and has the form*

$$d^{\mathbb{P}}(a, X) = \frac{d}{da} \ln \rho_1(a) + \overline{S}_a(X).$$

Theorem 2.3 is our main result and it is proven in Section 3.2. There, using results of [**5**], we show that Assumption 2 implies Assumption 1 with $\nabla R = \overline{S}_a$. Then Theorem 2.3 follows from Proposition 2.2.

3. Proofs of the main results.

3.1. Proof of Proposition 2.2.

Take a function $\varphi \in C_0^\infty \mathcal{D}_0$. We have

$$I := -\int_{\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)} \partial_{a_i} \varphi(a, X)\, d\mathcal{C}_{\mathbb{P}}(a, X) = -\int_{\mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)} \lim_{\varepsilon \to 0} \frac{\varphi(a{+}\varepsilon_i, X){-}\varphi(a, X)}{\varepsilon}\, d\mathcal{C}_{\mathbb{P}}(a, X),$$

where $\varepsilon_i := \varepsilon e_i$ and e_i is the i-th basis vector of $\mathbb{R}^d$. Using the dominated convergence theorem, we exchange the limit with the integral. The latter can be applied since

$$\left| \frac{\varphi(a + \varepsilon_i, X) - \varphi(a, X)}{\varepsilon} \right| \le \sup_{X \in \mathrm{Conf}(\mathbb{R}^d),\, x \in \mathbb{R}^d} \left| \partial_{x_i} \varphi(x, X) \right|$$

and $\varphi \in C_0^\infty \mathcal{D}_0$. We get

$$
\begin{aligned}
I &= -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a+\varepsilon_i,X)\,d\mathcal{C}_{\mathbb{P}}(a,X)-\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X)\Big)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a+\varepsilon_i,X)-\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X)\Big),
\end{aligned}
\tag{3.1}
$$

where in the last line of (3.1) we put $\varepsilon := -\varepsilon$. Using the definition of the reduced Campbell measure (2.1), we find

$$
\begin{aligned}
I = &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\rho_1(a+\varepsilon_i)\,d\mathbb{P}^{a+\varepsilon_i}(X)da\\
&-\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\rho_1(a)\,d\mathbb{P}^{a}(X)da\Big)\\
= &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\big(\rho_1(a+\varepsilon_i)-\rho_1(a)\big)\,d\mathbb{P}^{a+\varepsilon_i}(X)da\\
&+\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\rho_1(a)\,\big(d\mathbb{P}^{a+\varepsilon_i}(X)-d\mathbb{P}^{a}(X)\big)da\Big)\\
= &\lim_{\varepsilon\to 0}(I_1^\varepsilon+I_2^\varepsilon).
\end{aligned}
$$

Using Assumption 1(3), we obtain

$$
\lim_{\varepsilon\to 0} I_1^\varepsilon = \int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\partial_{a_i}\rho_1(a)\,d\mathbb{P}^{a}(X)da = \int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\partial_{a_i}\big(\ln\rho_1(a)\big)\,d\mathcal{C}_{\mathbb{P}}(a,X).
\tag{3.2}
$$

In view of Assumption 1(2), we have

$$
\lim_{\varepsilon\to 0} I_2^\varepsilon = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\mathcal{R}_{a+\varepsilon_i,a}(X)\,d\mathcal{C}_{\mathbb{P}}(a,X)-\int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X)\Big).
$$

Then, because of the identity $\mathcal{R}_{a,a}(X)\equiv 1$, Assumption 1(4) implies

$$
\lim_{\varepsilon\to 0} I_2^\varepsilon = \int_{\mathbb{R}^d\times\mathrm{Conf}(\mathbb{R}^d)}\varphi(a,X)\partial_i\mathcal{R}(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X).
\tag{3.3}
$$

Combining (3.2) with (3.3) we obtain the desired relation (2.2). □

3.2. Proof of Theorem 2.3.

In view of Proposition 2.2, it suffices to check that Assumption 1 is satisfied with $\nabla\mathcal{R}(a,X) = \overline{S}_a(X)$. Item 1 of Assumption 1 immediately follows from item 3 of Assumption 2. The proof of the other items relies on the results obtained in the paper [**5**]; see also [**4**]. One of the main tools we use borrowed from the works above is the following lemma. Take $a,b\in\mathbb{R}$ and consider the normalized multiplicative functional $\overline{\Psi}^{R,\delta}_{b,a}:\mathrm{Conf}(\mathbb{R})\mapsto\mathbb{R}$ given by

$$\overline{\Psi}^{R,\delta}_{b,a}(X) := C^{R,\delta}_{b,a} \prod_{\substack{x\in X:\,|x|<R,\\ |x-a|,|x-b|>\delta}} \left(\frac{x-b}{x-a}\right)^2, \tag{3.4}$$

where the constant $C^{R,\delta}_{a,b}$ is specified by the normalization requirement $\mathbb{E}^a\overline{\Psi}^{R,\delta}_{b,a} = 1$. Here and further on we set $\prod_{x\in\emptyset} f(x) = 1$, for any function f.

Lemma 3.1. 1. *Under Assumption 2, there exists $\alpha > 0$ and a function $\overline{\Psi}_{b,a}$: $\mathrm{Conf}(\mathbb{R}) \mapsto \mathbb{R}$ satisfying*

$$\overline{\Psi}^{R,\delta}_{b,a} \to \overline{\Psi}_{b,a} \quad as \quad R\to\infty,\, \delta\to 0 \quad in \quad L^{1+\alpha}(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a), \tag{3.5}$$

for $\hat\rho_1$-almost all $a\in\mathbb{R}$, uniformly in a, b which range in compact subsets of $\mathbb{R}$.

2. *For $\hat\rho_1$-almost all $a, b\in\mathbb{R}^d$, the function $\overline{\Psi}_{b,a}$ is the Radon–Nikodym derivative of the Palm measure $\mathbb{P}^b$ with respect to the Palm measure $\mathbb{P}^a$, i.e.*

$$d\mathbb{P}^b(X) = \overline{\Psi}_{b,a}(X)\, d\mathbb{P}^a(X). \tag{3.6}$$

Proof. Item 1 is established in Corollary A.5(1) and follows from results obtained in [**5**], which we explain in Appendix A.2. Item 2 is a particular case of Theorem 1.4 (1) from [**5**] which is one of the main results of [**5**]. The key point of the proof of (3.6) is to show that Palm subspaces $L(p)$ and $L(q)$ corresponding to conditioning at different points p and q satisfy the relation $L(p) = ((x-p)/(x-q))L(q)$; this relation uses Assumption 2(2). Then the proof is concluded by using a general result of [**2**], [**3**] stating the following. If $\mathcal{P}_0$ is an orthogonal projection operator onto a subspace $L_0 \subset L^2(\mathbb{R})$ that induces a determinantal measure $\mathbb{P}_0$, then, under certain assumptions, multiplication of the space L_0 by a function g corresponds to taking the product of the determinantal measure $\mathbb{P}_0$ with the normalized multiplicative functional $const\prod_{x\in X}(g(x))^2$. In Appendix A.3 we give a more detailed outline of the proof of the identity (3.6).

Note that in [**5**] the multiplicative functional $\overline{\Psi}^{R,\delta}_{b,a} = 1$ is defined as the product over the set $\{x\in X : |x| < R, |x-a| > \delta\}$, so that in difference with the definition (3.4) the point b is not isolated. It can be checked directly that this does not affect the proof at all. □

Lemma 3.1(2) implies item 2 of Assumption 1 with $\mathcal{R}_{b,a} = \overline{\Psi}_{b,a}$. Because of the bounds $|x| < R$ and $|x-a| > \delta$, the functions $\overline{\Psi}^{R,\delta}_{b,a}(X)$ are $\mathbb{P}^a$-almost surely continuous with respect to b. Then, using the dominated convergence theorem, we see that they are continuous in $L^1(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a)$. Then the uniformity in b of convergence from Lemma 3.1(1) implies that the functions $\overline{\Psi}_{b,a}$ also are continuous with respect to b in $L^1(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a)$. So that item 3 of Assumption 1 is fulfilled as well.

It remains to check that item 4 of Assumption 1 holds with $\nabla\mathcal{R}(a, X) = \overline{S}_a(X)$. Due to Lemma 3.1(2), we have

$$f_\varphi(\varepsilon) = \int_{\mathbb{R}\times\mathrm{Conf}(\mathbb{R})} \overline{\Psi}_{a+\varepsilon,a}(X)\varphi(a, X)\, d\mathcal{C}_{\mathbb{P}}(a, X).$$

We need to show that the function f_φ is differentiable at zero and

$$\frac{d}{d\varepsilon} f_\varphi(0) = \int_{\mathbb{R}\times\mathrm{Conf}(\mathbb{R})} \overline{S}_a(X)\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X). \tag{3.7}$$

Due to Lemma 3.1(1), we have

$$f_\varphi(\varepsilon) = \lim_{R\to\infty,\delta\to 0} f_\varphi^{R,\delta}(\varepsilon), \tag{3.8}$$

where

$$f_\varphi^{R,\delta}(\varepsilon) = \int_{\mathbb{R}\times\mathrm{Conf}(\mathbb{R})} \overline{\Psi}^{R,\delta}_{a+\varepsilon,a}(X)\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X).$$

Because of the truncation $|x| < R$, $|x-a| > \delta$ in the definition (3.4) of the function $\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}$, for $\mathcal{C}_{\mathbb{P}}$-almost all (a,X) the function $\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}(X)$ is smooth with respect to ε and, together with its derivatives in ε, $\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}$ is integrable over $(a,X)\in\mathbb{R}\times\mathrm{Conf}(\mathbb{R})$ with respect to the measure $\mathcal{C}_{\mathbb{P}}$, uniformly in ε from a compact set. Then the function $f_\varphi^{R,\delta}$ is smooth in ε and we have

$$\frac{d}{d\varepsilon} f_\varphi^{R,\delta}(\varepsilon) = \int_{\mathbb{R}\times\mathrm{Conf}(\mathbb{R})} \frac{d}{d\varepsilon}\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}(X)\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X). \tag{3.9}$$

Proposition 3.2. *Under Assumption* 2, *the derivative* $(d/d\varepsilon)f_\varphi^{R,\delta}(\varepsilon)$ *converges as* $R\to\infty, \delta\to 0$, *uniformly in* ε *from a small neighbourhood of zero.*

Proof of Proposition 3.2 is given in the next subsection. Together with (3.8), Proposition 3.2 implies that the function f_φ is differentiable in a small neighbourhood of zero and

$$\frac{d}{d\varepsilon} f_\varphi(\varepsilon) = \lim_{R\to\infty,\delta\to 0} \frac{d}{d\varepsilon} f_\varphi^{R,\delta}(\varepsilon). \tag{3.10}$$

Let us compute the derivative (3.9). By definition (3.4) of the multiplicative functional $\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}$, we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\overline{\Psi}^{R,\delta}_{a+\varepsilon,a} &= \frac{d}{d\varepsilon}\exp\Bigg(\ln C^{R,\delta}_{a+\varepsilon,a} + 2\sum_{\substack{x\in X:\,|x|<R,\\ |x-a|,|x-(a+\varepsilon)|>\delta}} \ln\left|\frac{x-(a+\varepsilon)}{x-a}\right|\Bigg)\\
&= \overline{\Psi}^{R,\delta}_{a+\varepsilon,a}\Bigg(\frac{d}{d\varepsilon}\ln C^{R,\delta}_{a+\varepsilon,a} + S^{R,\delta}_{a,a+\varepsilon}\Bigg),
\end{aligned} \tag{3.11}$$

where

$$S^{R,\delta}_{a,b} := \sum_{\substack{x\in X:\,|x|<R,\\ |x-a|,|x-b|>\delta}} \frac{2}{b-x}.$$

Since, by definition, $\mathbb{E}^a\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}\equiv 1$, we have

$$\mathbb{E}^a\frac{d}{d\varepsilon}\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}=\frac{d}{d\varepsilon}\mathbb{E}^a\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}=0.$$

Then, taking the expectation $\mathbb{E}^a$ of the both sides of (3.11), we get

$$\frac{d}{d\varepsilon}\ln C^{R,\delta}_{a+\varepsilon,a}=-\mathbb{E}^a\big(\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}S^{R,\delta}_{a,a+\varepsilon}\big). \tag{3.12}$$

Now (3.11) together with (3.12) implies

$$\frac{d}{d\varepsilon}f^{R,\delta}_{\varphi}(\varepsilon)=\int_{\mathbb{R}\times\mathrm{Conf}(\mathbb{R})}\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}\big(S^{R,\delta}_{a,a+\varepsilon}-\mathbb{E}^a\overline{\Psi}^{R,\delta}_{a+\varepsilon,a}S^{R,\delta}_{a,a+\varepsilon}\big)\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X). \tag{3.13}$$

Since $\overline{\Psi}^{R,\delta}_{a,a}=1$ and $S^{R,\delta}_{a,a}=S^{R,\delta}_a$, where the additive functional $S^{R,\delta}_a$ is defined in (2.4), we obtain

$$\frac{d}{d\varepsilon}f^{R,\delta}_{\varphi}(0)=\int_{\mathbb{R}\times\mathrm{Conf}(\mathbb{R})}\big(S^{R,\delta}_a-\mathbb{E}^aS^{R,\delta}_a\big)\varphi(a,X)\,d\mathcal{C}_{\mathbb{P}}(a,X). \tag{3.14}$$

Due to (2.5), the function $S^{R,\delta}_a-\mathbb{E}^aS^{R,\delta}_a=\overline{S}^{R,\delta}_a$ converges to $\overline{S}_a$ in $L^2(\mathrm{Conf}(\mathbb{R}),\mathbb{P}^a)$ as $R\to\infty$, $\delta\to 0$, uniformly in $\hat\rho_1$-almost all $a\in\bigcup_{X\in\mathrm{Conf}(\mathbb{R})}\operatorname{supp}\varphi(\cdot,X)$, since the latter set is compact. Then the right-hand side of (3.14) converges to that of (3.7). In view of (3.10), this concludes the proof of the theorem. □

3.3. Proof of Proposition 3.2.

In view of the formula (3.13), to establish the desired convergence it suffices to show that the function

$$J^{R,\delta}(a,b):=\overline{\Psi}^{R,\delta}_{b,a}\big(S^{R,\delta}_{a,b}-\mathbb{E}^a\overline{\Psi}^{R,\delta}_{b,a}S^{R,\delta}_{a,b}\big)\quad\text{converges as } R\to\infty,\ \delta\to 0, \tag{3.15}$$

in $L^1(\mathrm{Conf}(\mathbb{R}),\mathbb{P}^a)$ uniformly in b and $\hat\rho_1$-almost all a, where a ranges in a compact set and b satisfies $|a-b|<\vartheta$, for some fixed $\vartheta\ll 1$. All convergences below will be uniform in a,b satisfying these restrictions, and we do not mention it any more. Further on we assume R to be sufficiently large and δ to be sufficiently small where it is needed.

For real Borel functions f,g where g is non-negative we define the additive and multiplicative functionals $S_f,\overline{S}_f,\Psi_g,\widetilde{\Psi}_g,\overline{\Psi}_g:\mathrm{Conf}(\mathbb{R})\mapsto\mathbb{R}$ by the formulas (A.1), (A.2), (A.5) and (A.6). Clearly, they are well-defined if the functions f and g are bounded and the supports $\operatorname{supp} f$, $\operatorname{supp}(g-1)$ are compact. However, the normalized functionals $\overline{S}_f,\widetilde{\Psi}_g$ and $\overline{\Psi}_g$ can be defined for larger classes of functions, see Appendices A.1 and A.2.

Let us define

$$h^R_>(x):=\frac{2}{b-x}\mathbb{I}_{\{|x|<R,\,|x-a|>\delta,\,|x-b|\ge 1\}}(x)=\frac{2}{b-x}\mathbb{I}_{\{|x|<R,\,|x-b|\ge 1\}}(x),$$

where we have used that $|a-b|\ll 1$, so that the constraint $|x-a|>\delta$ holds automatically. Set also

$$h^\delta_<(x) := \frac{2}{b-x}\mathbb{I}_{\{|x|>R,\,|x-a|>\delta,\,\delta<|x-b|<1\}}(x) = \frac{2}{b-x}\mathbb{I}_{\{|x-a|>\delta,\,\delta<|x-b|<1\}}(x).$$

Then we have

$$S^{R,\delta}_{a,b} = S_{h^R_>} + S_{h^\delta_<}.$$

Recall that $\mathbb{E}^a\overline{\Psi}^{R,\delta}_{b,a} = 1$. Then, subtracting in the parenthesis of (3.15) the term $\mathbb{E}^a S_{h^R_>}$ and adding the term $\mathbb{E}^a\overline{\Psi}^{R,\delta}_{b,a}\mathbb{E}^a S_{h^R_>}$, we obtain

$$J^{R,\delta}(a,b) = \overline{\Psi}^{R,\delta}_{b,a}(\overline{S}_{h^R_>} + S_{h^\delta_<}) - \overline{\Psi}^{R,\delta}_{b,a}\mathbb{E}^a\big(\overline{\Psi}^{R,\delta}_{b,a}(\overline{S}_{h^R_>} + S_{h^\delta_<})\big). \tag{3.16}$$

Now, to establish the convergence (3.15) it suffices to show that the functions $\overline{\Psi}^{R,\delta}_{b,a}\overline{S}_{h^R_>}$ and $\overline{\Psi}^{R,\delta}_{b,a}S_{h^\delta_<}$ converge as $R\to\infty$, $\delta\to 0$ in $L^1(\mathrm{Conf}(\mathbb{R}),\mathbb{P}^a)$. Indeed, then the convergence of the first summand from (3.16) will be obvious while the convergence of the second summand will follow from Lemma 3.1(1), which states that the function $\overline{\Psi}^{R,\delta}_{b,a}$ converges in $L^1(\mathrm{Conf}(\mathbb{R}),\mathbb{P}^a)$ as well.

Term $\overline{\Psi}^{R,\delta}_{b,a}\overline{S}_{h^R_>}$. Let

$$h_>(x) := \frac{2}{b-x}\mathbb{I}_{\{|x-b|\geq 1\}}(x).$$

Due to Corollary A.2(2), the additive functional $\overline{S}_{h_>}$ is well-defined and we have the convergence

$$\overline{S}_{h^R_>} \to \overline{S}_{h_>} \quad \text{as} \quad R\to\infty \quad \text{in} \quad L^p(\mathrm{Conf}(\mathbb{R}),\mathbb{P}^a) \tag{3.17}$$

with $p=2$. We claim that it takes place for any $p>2$ as well. This concludes consideration of the term $\overline{\Psi}^{R,\delta}_{b,a}\overline{S}_{h^R_>}$ since, using the Hölder inequality, from (3.17) joined with Lemma 3.1(1) we obtain

$$\overline{\Psi}^{R,\delta}_{b,a}\overline{S}_{h^R_>} \to \overline{\Psi}_{b,a}\overline{S}_{h_>} \quad \text{as} \quad \delta\to 0,\, R\to\infty \quad \text{in} \quad L^1(\mathrm{Conf}(\mathbb{R}),\mathbb{P}^a).$$

Denote

$$\Delta^R := h^R_> - h_>.$$

Due to the Cauchy–Bunyakovsky–Schwarz inequality, we have

$$\mathbb{E}^a|\overline{S}_{h^R_>} - \overline{S}_{h_>}|^p = \mathbb{E}^a|\overline{S}_{\Delta^R}|^p \leq \sqrt{\mathbb{E}^a(\overline{S}_{\Delta^R})^{2p-2}}\sqrt{\mathbb{E}^a(\overline{S}_{\Delta^R})^2}.$$

Due to the convergence (3.17) with $p=2$, we have $\mathbb{E}^a(\overline{S}_{\Delta^R})^2\to 0$ as $R\to\infty$. Thus, it suffices to prove that the expectation $\mathbb{E}^a|\overline{S}_{\Delta^R}|^q$ is bounded uniformly in R, for any $q>0$. We have

$$|\overline{S}_{\Delta^R}|^q \leq C_q\big(e^{\overline{S}_{\Delta^R}} + e^{-\overline{S}_{\Delta^R}}\big). \tag{3.18}$$

Let us write

$$e^{\overline{S}_{\Delta^R}} = \widetilde{\Psi}_{\exp(\Delta^R)}.$$

Due to Corollary A.5(2), we have

$$\widetilde{\Psi}_{\exp(\Delta^R)} \to \widetilde{\Psi}_1 = 1 \quad \text{as} \quad R \to \infty \quad \text{in} \quad L^1(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a), \tag{3.19}$$

where $\widetilde{\Psi}_1$ is the multiplicative functional $\widetilde{\Psi}_g$ corresponding to the function $g = 1$. In particular, the L^1-norm $\mathbb{E}^a e^{\overline{S}_{\Delta^R}} = \mathbb{E}^a \widetilde{\Psi}_{\exp(\Delta^R)}$ is bounded uniformly in R. Replacing Δ^R by $-\Delta^R$, the same argument implies that the expectation $\mathbb{E}^a e^{-\overline{S}_{\Delta^R}}$ is also bounded uniformly in R. Then, due to (3.18), we see that the expectation $\mathbb{E}^a |\overline{S}_{\Delta^R}|^q$ is bounded uniformly in R as well. So that, we obtain the desired convergence (3.17).

Term $\overline{\Psi}^{R,\delta}_{b,a} S_{h^\delta_<}$. Let us factorize

$$\overline{\Psi}^{R,\delta}_{b,a} S_{h^\delta_<} = \frac{\widetilde{\Psi}_{g_1^R} \widetilde{\Psi}_{g_2^\delta} \Psi_{g_3^\delta} S_{h^\delta_<}}{\mathbb{E}^a \big(\widetilde{\Psi}_{g_1^R} \widetilde{\Psi}_{g_2^\delta} \Psi_{g_3^\delta}\big)} \tag{3.20}$$

where

$$g_1^R := \left(\left(\frac{x-b}{x-a} \right)^2 - 1 \right) \mathbb{I}_{\{x\colon |x|<R,\, |x-b| \ge 1\}} + 1$$

and

$$g_2^\delta := \left(\frac{1}{(x-a)^2} - 1 \right) \mathbb{I}_{\{x\colon |x-a|>\delta,\, \delta<|x-b|<1\}} + 1, \quad g_3^\delta := \big((x-b)^2 - 1\big) \mathbb{I}_{\{x\colon |x-a|>\delta,\, \delta<|x-b|<1\}} + 1.$$

Set

$$g_1 := \left(\left(\frac{x-b}{x-a} \right)^2 - 1 \right) \mathbb{I}_{\{x\colon |x-b| \ge 1\}} + 1$$

and

$$g_2 := \left(\frac{1}{(x-a)^2} - 1 \right) \mathbb{I}_{\{x\colon |x-b|<1\}} + 1, \quad g_3 := \big((x-b)^2 - 1\big) \mathbb{I}_{\{x\colon |x-b|<1\}} + 1.$$

Corollary A.5(2) states that

$$\widetilde{\Psi}_{g_1^R} \to \widetilde{\Psi}_{g_1} \quad \text{as} \quad R \to \infty \qquad \text{in} \quad L^p(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a), \tag{3.21}$$

for any $p > 0$, and

$$\widetilde{\Psi}_{g_2^\delta} \to \widetilde{\Psi}_{g_2} \quad \text{as} \quad \delta \to 0 \quad \text{in} \quad L^{1+\alpha}(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a), \tag{3.22}$$

for some $\alpha > 0$. Since the functions g_3^δ, g_3 are bounded uniformly in δ and $(g_3^\delta - 1), (g_3 - 1)$ have compact supports, we obviously have

$$\Psi_{g_3^\delta} \to \Psi_{g_3} \quad \text{as} \quad \delta \to 0 \quad \text{in} \quad L^p(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a),$$

for any $p > 0$. Then the Hölder inequality implies

$$\mathbb{E}^a\big(\widetilde{\Psi}_{g_1^R}\widetilde{\Psi}_{g_2^\delta}\Psi_{g_3^\delta}\big) \to \mathbb{E}^a\big(\widetilde{\Psi}_{g_1}\widetilde{\Psi}_{g_2}\Psi_{g_3}\big) \quad \text{as} \quad R \to \infty, \quad \delta \to 0.$$

To prove that the numerator of (3.20) converges, we note that

$$\Psi_{g_3^\delta} S_{h_<^\delta} = 2 \sum_{\substack{x \in X: \\ |x-a|>\delta,\, \delta<|x-b|<1}} (b-x) \prod_{\substack{y \in X:\, y \neq x \\ |y-a|>\delta,\, \delta<|y-b|<1}} (y-b)^2. \tag{3.23}$$

Clearly, the right-hand side of (3.23) converges as $\delta \to 0$ in $L^p(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a)$, for any $p > 0$. Together with (3.21)–(3.22), by the Hölder inequality this implies that the numerator of (3.20) converges in $L^1(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a)$ as $R \to \infty$, $\delta \to 0$, so that the function $\overline{\Psi}_{b,a}^{R,\delta} S_{h_>^\delta}$ also does. □

Acknowledgements. A. Bufetov's research has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 647133 (ICHAOS)). It has also been funded by the Russian Academic Excellence Project '5-100', the grant MD 5991.2016.1 of the President of the Russian Federation, by RFBR according to the research project 18-31-20031 and by the Gabriel Lamé Chair at the Chebyshev Laboratory of the SPbSU, a joint initiative of the French Embassy in the Russian Federation and the Saint-Petersburg State University.

The work of A. Dymov was supported by the Russian Science Foundation under grant 14-21-00162 and performed in Steklov Mathematical Institute of RAS. In conformity with the reglementation of the Russian Science Foundation supporting the research of A. Dymov, we must indicate that A. Dymov prepared Section 3, while A. Bufetov and H. Osada prepared Sections 1, 2 and Appendix A.

The research of H. Osada was supported by JSPS KAKENHI Grant Number JP16H06338.

A. Regularization of additive and multiplicative functionals.

In this appendix we consider a determinantal point process $\mathbb{P}$ on $\mathbb{R}$ with the kernel Π and assume that Π satisfies Assumption 2. We explain results from [**5**] which we use in this paper and prove some auxiliary convergence results.

A.1. Additive functionals.

Let $f : \mathbb{R} \to \mathbb{C}$ be a Borel function. Define the corresponding *additive functional*

$$S_f : \mathrm{Conf}(\mathbb{R}) \mapsto \mathbb{R}, \quad S_f(X) = \sum_{x \in X} f(x), \tag{A.1}$$

where the series may diverge. If $S_f \in L_1(\mathrm{Conf}(\mathbb{R}), \mathbb{P})$, then we introduce the *normalized additive functional*

$$\overline{S}_f = S_f - \mathbb{E}S_f. \tag{A.2}$$

Now we will show that the normalized additive functional can be defined even when the additive functional itself is not well-defined. Introduce the Hilbert space $\mathcal{V}(\Pi)$ of real functions with the norm

$$\|f\|^2_{\mathcal{V}(\Pi)} = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)-f(y)|^2|\Pi(x,y)|^2 dxdy.$$

Here we identify functions which differ by a constant. If a function f is such that $S_f \in L^2(\mathrm{Conf}(\mathbb{R}),\mathbb{P})$, we have

$$\mathbb{E}|\overline{S}_f|^2 = \mathrm{Var}\, S_f = \|f\|^2_{\mathcal{V}(\Pi)}.$$

In particular, this is the case if the function f is bounded and has compact support. Thus, the correspondence $f \to \overline{S}_f$ is an isometric embedding of a dense subset of $\mathcal{V}(\Pi)$ into $L^2(\mathrm{Conf}(\mathbb{R}),\mathbb{P})$. It therefore admits a unique isometric extension onto the whole space $\mathcal{H}$, and we get

PROPOSITION A.1. *There exists a unique linear isometric embedding*

$$\overline{S} : \mathcal{V}(\Pi) \hookrightarrow L^2(\mathrm{Conf}(\mathbb{R}),\mathbb{P}), \quad \overline{S} : f \to \overline{S}_f$$

such that

1. $\mathbb{E}\overline{S}_f = 0$ *for all* $f \in \mathcal{V}(\Pi)$;
2. *if* $S_f \in L^1(\mathrm{Conf}(\mathbb{R}),\mathbb{P})$, *then* $\overline{S}_f$ *is given by* (A.2).

For more details see Proposition 4.1 in [**5**].

Let $\mathbb{P}^a$ be the reduced Palm measure of the measure $\mathbb{P}$, conditioned at the point a.

COROLLARY A.2. 1. *For $\hat{\rho}_1$-almost every $a \in \mathbb{R}$ there exists a function $\overline{S}_a$: $\mathrm{Conf}(\mathbb{R}) \mapsto \mathbb{R}$ such that the convergence* (2.5) *takes place, uniformly in $a \in \mathbb{R}$ ranging in a compact set.*

2. *The convergence* (3.17) *takes place for $p = 2$, uniformly in $a, b \in \mathbb{R}$ ranging in a compact set.*

To establish the corollary we need the following result. For $a, x, y \in \mathbb{R}$ set

$$\Pi^a(x,y) := \Pi(x,y) - \frac{\Pi(x,a)\Pi(a,y)}{\Pi(a,a)} \quad \text{if} \quad \Pi(a,a) \neq 0 \tag{A.3}$$

and $\Pi^a(x,y) := \Pi(x,y)$ if $\Pi(a,a) = 0$. Let also $L(a) \subset L^2(\mathbb{R},dx)$ be a subspace defined as

$$L(a) := \{\varphi \in L : \varphi(a) = 0\}, \tag{A.4}$$

where we recall that L is a range of the orthogonal projection operator $\mathcal{P}$ corresponding to the kernel Π.

Theorem A.3 (Shirai–Takahashi [**20**]). *For $\hat{\rho}_1$-almost every $a \in \mathbb{R}$, the reduced Palm measure $\mathbb{P}^a$ coincides with the determinantal measure associated to the kernel Π^a. Moreover, Π^a is an orthogonal projection kernel onto the space $L(a)$.*

Proof of Corollary A.2. In view of Theorem A.3, item 1 follows from Proposition A.1 applied to the kernel Π^a. Indeed, we have

$$\overline{S}_a^{R,\delta} = \overline{S}_{f_a^{R,\delta}} \quad \text{with} \quad f_a^{R,\delta}(x) = \frac{2}{a-x} \mathbb{I}_{\{|x|<R,\, |x-a|>\delta\}}.$$

Clearly, $f_a^{R,\delta} \to f_a$ in $\mathcal{V}(\Pi^a)$ uniformly in a as $R \to \infty$, $\delta \to 0$, where

$$f_a(x) := \frac{2}{a-x}.$$

Then Proposition A.1 implies the desired convergence with $\overline{S}_a := \overline{S}_{f_a}$. Item 2 can be proven by the same argument. □

A.2. Multiplicative functionals.

For a bounded nonnegative function g with compact support we define the *multiplicative functionals* $\Psi_g, \widetilde{\Psi}_g : \mathrm{Conf}(\mathbb{R}) \mapsto \mathbb{R}$ as

$$\Psi_g = \prod_{x \in X} g(x) = e^{S_{\log g}} \quad \text{and} \quad \widetilde{\Psi}_g = e^{\overline{S}_{\log g}}. \tag{A.5}$$

Here we set $\Psi_g(X) = \widetilde{\Psi}_g(X) = 0$ if there is $x \in X$ such that $g(x) = 0$. In view of Proposition A.1, we can extend the multiplicative functional $\widetilde{\Psi}_g$ to functions g satisfying $\|\log g\|_{\mathcal{V}(\Pi)} < \infty$. If $\widetilde{\Psi}_g \in L^1(\mathrm{Conf}(\mathbb{R}), \mathbb{P})$ we define the *normalized multiplicative functional* as

$$\overline{\Psi}_g = \frac{\widetilde{\Psi}_g}{\mathbb{E}\widetilde{\Psi}_g}. \tag{A.6}$$

Fix positive numbers $\alpha > 0$, $\varepsilon > 0$, $M > \varepsilon$ and two bounded Borel subsets $B^1, B^2 \in \mathcal{B}(\mathbb{R})$ satisfying

$$||\mathbb{I}_{B^1 \cup B^2} \mathcal{P}|| < 1.$$

Denote by $\mathscr{G}$ the set of nonnegative Borel functions $g : \mathbb{R} \mapsto \mathbb{R}$ satisfying

1. $\{x : g(x) < \varepsilon\} \subset B^1$;
2. $\{x : g(x) > M\} \subset B^2$;
3. $\displaystyle\int_{B^2} |g(x)|^{1+\alpha} \Pi(x,x) dx + \int_{\mathbb{R} \setminus B^2} |g(x) - 1|^2 \Pi(x,x) dx < \infty.$

We metrize $\mathscr{G}$ by equipping it with the distance

$$d_{\mathscr{G}}(g_1, g_2) = \int_{B^2} |g_1(x) - g_2(x)|^{1+\alpha} \Pi(x,x) dx + \int_{\mathbb{R}\setminus B^2} |g_1(x) - g_2(x)|^2 \Pi(x,x) dx.$$

Then $\mathscr{G}$ becomes a complete separable metric space. Below we formulate Proposition 4.3 from [**5**].

Proposition A.4. *For any $\alpha' : 0 < \alpha' < \alpha$, the correspondences $g \to \widetilde{\Psi}_g$, $g \to \overline{\Psi}_g$ induce continuous mappings from $\mathscr{G}$ to $L^{1+\alpha'}(\mathrm{Conf}(\mathbb{R}), \mathbb{P})$.*

Corollary A.5. 1. *Assertion of Lemma 3.1(1) is satisfied.*

2. *For $\hat{\rho}_1$-almost all $a \in \mathbb{R}$, convergences (3.19), (3.21) and (3.22) take place and are uniform in $a, b \in \mathbb{R}$ as a range in a compact set and b satisfies $|b - a| < \theta \ll 1$.*

Proof.

Item 1: Due to Theorem A.3, the reduced Palm measure $\mathbb{P}^a$ coincides for $\hat{\rho}_1$-almost all $a \in \mathbb{R}$ with the determinantal measure associated with the kernel Π^a. Moreover, it is immediate to check that the kernel Π^a satisfies Assumption 2. Let

$$g_{a,b}(x) := \left(\frac{x-b}{x-a}\right)^2 \quad \text{and} \quad g_{a,b}^{R,\delta}(x) := (g_{a,b}(x) - 1)\mathbb{I}_{\{|x|>R,\, |x-b|>\delta,\, |x-a|>\delta\}} + 1.$$

Using that the function $\Pi^a_{diag}(x) := \Pi^a(x,x)$ has zero of second order at the point $x = a$, we find that $g_{a,b}, g_{a,b}^{R,\delta} \in \mathscr{G}^a$, for an appropriate choice of the sets B^1, B^2 and numbers α, M (independent from R, δ and a, b), where the space $\mathscr{G}^a$ is defined as the space $\mathscr{G}$ above, but with respect to the kernel Π^a. Moreover,

$$d_{\mathscr{G}^a}(g_{a,b}^{R,\delta}, g_{a,b}) \to 0 \quad \text{as} \quad R \to \infty, \delta \to 0, \tag{A.7}$$

uniformly in a, b as they range in compact sets. Then Proposition A.4 implies that

$$\overline{\Psi}_{g_{a,b}^{R,\delta}} \to \overline{\Psi}_{g_{a,b}} \quad \text{as} \quad R \to \infty, \delta \to 0 \quad \text{in} \quad L^{1+\alpha'}(\mathrm{Conf}(\mathbb{R}), \mathbb{P}^a),$$

for any $0 < \alpha' < \alpha$. Since $\overline{\Psi}_{a,b}^{R,\delta} = \overline{\Psi}_{g_{a,b}^{R,\delta}}$, we get the desired convergence with $\overline{\Psi}_{a,b} = \overline{\Psi}_{g_{a,b}}$. Its uniformity in a, b follows from the uniformity of convergence (A.7) by a direct analysis of the proof of Proposition A.4 (i.e. of Proposition 4.3 from [**5**]).

Item 2: Similarly with the last item, the desired convergences follow by applying Proposition A.4 with the kernel Π^a. The convergence (3.21) takes place for arbitrary $p > 0$ since we assume $|a - b| \ll 1$, so that the functions g_1^R and g_1 are bounded, and then α can be chosen arbitrarily large. □

A.3. Quasi-invariance.

In this section we briefly recall the proof of (3.6), as presented in [**5**]. Let the kernels Π^a and Π^b be defined as in (A.3) and the spaces $L(a)$ and $L(b)$—as in (A.4). Recall that the kernels Π^a, Π^b and the spaces $L(a), L(b)$ satisfy Theorem A.3. The key point of the proof is the identity

$$L(b) = \frac{x-b}{x-a} L(a), \tag{A.8}$$

which is obtained with help of the division property (see Assumption 2(2)). The remaining argument deduces the relation (3.6) from (A.8). It relies on an abstract result obtained in [**3**] which states that a determinantal measure multiplied by a multiplicative functional and appropriately normalized is again a determinantal measure. More specifically, let $\mathcal{P}_0$ be an operator of orthogonal projection in $L^2(\mathbb{R})$ onto a closed subspace L_0 and $\mathbb{P}_0$ be the corresponding determinantal measure. Let also $g : \mathbb{R} \mapsto \mathbb{R}$ be a function which is bounded away from 0 and ∞, $\mathcal{P}_g$ be an orthogonal projection operator onto the subspace gL_0 and $\mathbb{P}_g$ be the corresponding determinantal measure. Then, under certain additional assumptions, we have

$$\mathbb{P}_g = \frac{\Psi_{g^2}\mathbb{P}_0}{\mathbb{E}_{\mathbb{P}_0}\Psi_{g^2}}, \tag{A.9}$$

where Ψ_{g^2} is the multiplicative functional corresponding to the function g^2 by (A.5) and $\mathbb{E}_{\mathbb{P}_0}$ denotes the expectation with respect to the measure $\mathbb{P}_0$. Together with (A.8), the relation (A.9) with $g = (x-b)/(x-a)$ implies the desired equality (3.6). Equality (A.9) can not be applied directly to our situation since the function g is not bounded away neither from zero nor from infinity and the regularization procedure described in Section A.2 must be performed firstly.

References

[1] A. Borodin and G. Olshanski, Infinite-dimensional diffusions as limits of random walks on partitions, Probability Theory and Related Fields, **144** (2009), 281–318.

[2] A. Bufetov, Infinite determinantal measures, Electron. Res. Announc. Math. Sci., **20** (2013), 12–30.

[3] A. I. Bufetov, Multiplicative functionals of determinantal processes, Uspekhi Mat. Nauk, **67** (2012), 177–178.

[4] A. I. Bufetov, On the action of the diffeomorphisms group on determinantal measures, Russian Math. Surveys, **70** (2015), 953–954.

[5] A. I. Bufetov, Quasi-symmetries of determinantal point processes, Ann. Probab., **46** (2018), 956–1003.

[6] A. I. Bufetov and R. V. Romanov, Division subspaces and integrable kernels, Bulletin of the London Math. Soc., 2018, online first.

[7] A. I. Bufetov and T. Shirai, Quasi-symmetries and rigidity for determinantal point processes associated with de Branges spaces, Proc. Japan Acad. Ser. A Math. Sci., **93** (2017), 1–5.

[8] D. J. Daley and D. Vere-Jones, An Introduction to the Theory of Point Processes, Springer, New York, 1988.

[9] R. Honda and H. Osada, Infinite-dimensional stochastic differential equations related to Bessel random point fields, Stochastic Processes and their Applications, **125** (2015), 3801–3822.

[10] M. Katori and H. Tanemura, Noncolliding Brownian motion and determinantal processes, J. Stat. Phys., **129** (2007), 1233–1277.

[11] M. Katori and H. Tanemura, Markov property of determinantal processes with extended sine, Airy, and Bessel kernels, Markov Process. Relat., **17** (2011), 541–580.

[12] Y. Kawamoto and H. Osada, Finite-particle approximations for interacting Brownian particles with logarithmic potentials, J. Math. Soc. Japan, **70** (2018), 921–952.

[13] H. Osada, Infinite-dimensional stochastic differential equations related to random matrices, Probability Theory and Related Fields, **153** (2012), 471–509.

[14] H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials, Ann. Probab., **41** (2013), 1–49.

[15] H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II : Airy random point field, Stochastic Processes and their Applications, **123** (2013), 813–838.

[16] H. Osada and H. Tanemura, Strong Markov property of determinantal processes with extended kernels, Stochastic Processes and their Applications, **126** (2016), 186–208.

[17] H. Osada and H. Tanemura, Infinite-dimensional stochastic differential equations and tail σ-fields, arXiv:1412.8674.

[18] H. Osada and H. Tanemura, Infinite-dimensional stochastic differential equations arising from Airy random point fields, arXiv:1408.0632.

[19] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and Boson processes, J. Funct. Anal., **205** (2003), 414–463.

[20] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants II: Fermion shifts and their ergodic and Gibbs properties, Ann. Probab., **31** (2003), 1533–1564.

[21] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys, **55** (2000), 923–975.

[22] L.-C. Tsai, Infinite dimensional stochastic differential equations for Dyson's model, Probability Theory and Related Fields, **166** (2016), 801–850.

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