J. Math. Soc. Japan
Vol. 71, No. 2 (2019) pp. 429–449
doi: 10.2969/jmsj/78307830

Completely positive isometries between matrix algebras

By Masamichi Hamana

(Received June 21, 2017)
(Revised Oct. 17, 2017)

Abstract. Let φ be a linear map between operator spaces. To measure the intensity of φ being isometric we associate with it a number, called the *isometric degree* of φ and written $\mathrm{id}(\varphi)$, as follows. Call φ a *strict m-isometry* with m a positive integer if it is an m-isometry, but is not an $(m+1)$-isometry. Define $\mathrm{id}(\varphi)$ to be 0, m, and ∞, respectively if φ is not an isometry, a strict m-isometry, and a complete isometry, respectively. We show that if $\varphi : M_n \to M_p$ is a unital completely positive map between matrix algebras, then $\mathrm{id}(\varphi) \in \{0, 1, 2, \ldots, [(n-1)/2], \infty\}$ and that when $n \geq 3$ is fixed and p is sufficiently large, the values $1, 2, \ldots, [(n-1)/2]$ are attained as $\mathrm{id}(\varphi)$ for some φ. The ranges of such maps φ with $1 \leq \mathrm{id}(\varphi) < \infty$ provide natural examples of operator systems that are isometric, but not completely isometric, to M_n. We introduce and classify, up to unital complete isometry, a certain family of such operator systems.

1. Introduction.

Since the publication of the pioneering paper of Choi [**1**] in 1972, an extensive literature has treated the difference between m-positivity and $(m+1)$-positivity on matrix algebras for a positive integer m (see, for example, the monograph of Paulsen [**5**] and the references cited there). However the difference between m-isometry and $(m+1)$-isometry seems to have been paid less attention. Here a linear map φ between operator spaces X and Y is called an *m-isometry* if $\mathrm{id}_m \otimes \varphi : M_m \otimes X \to M_m \otimes Y$, $(\mathrm{id}_m \otimes \varphi)(\sum_i a_i \otimes x_i) = \sum_i a_i \otimes \varphi(x_i)$, is an isometry, where M_m is the C^*-algebra of all complex $m \times m$ matrices, an operator space X is a linear subspace of some C^*-algebra A, and $M_n \otimes X$ is regarded as a normed linear subspace of the C^*-algebra $M_n \otimes A$. By a *complete isometry* we mean a map that is an m-isometry for all m. Clearly a complete isometry or an $(m+1)$-isometry is an m-isometry. We call an m-isometry *strict* if it is not an $(m+1)$-isometry. Hence, with any linear map φ between operator spaces we can associate a unique number, called the *isometric degree* of φ and written $\mathrm{id}(\varphi)$, defined as 0, m, and ∞, respectively if φ is not an isometry, a strict m-isometry, and a complete isometry, respectively.

We note that if φ is a *surjective* linear map between C^*-algebras, then $\mathrm{id}(\varphi) \in \{0, 1, \infty\}$, that is, $\mathrm{id}(\varphi)$ takes no integer value more than 1, or equivalently every surjective 2-isometry is a complete isometry. Indeed, more generally, for a surjective linear map between triple systems, the three notions of 2-isometry, triple isomorphism, and complete isometry coincide ([**3**], Proposition 2.1). Here a *triple system*, also called a *ternary ring of operators* (TRO), is a norm closed linear subspace of some C^*-algebra

2010 *Mathematics Subject Classification.* Primary 46L07; Secondary 46B04.
Key Words and Phrases. completely positive isometry, matrix algebra, operator system.

that is closed under the triple product $[x, y, z] := xy^*z$, and a triple isomorphism between triple systems is a linear bijection that preserves the triple products. A typical example of a surjective strict 1-isometry between C^*-algebras is the transpose $x \mapsto {}^t x$ of the matrix algebra M_n for $n \geq 2$ (see Tomiyama [**6**]).

The maps considered in this paper are unital completely positive maps $\varphi : M_n \to M_p$ between matrix algebras. In Section 3 we show that $\mathrm{id}(\varphi) \in \{0, 1, 2, \ldots, [(n-1)/2], \infty\}$ for such maps φ and that when $n \geq 3$ is fixed, the less trivial values $1, 2, \ldots, [(n-1)/2]$ are attained as $\mathrm{id}(\varphi)$ for some p and some $\varphi : M_n \to M_p$. The main ingredients for the study are a criterion for φ being an m-isometry (Lemma 3.3 (iii)) and a technique (Lemma 3.4(ii)) making the computation of $\mathrm{id}(\varphi)$ effective via the notion of *length* defined in Section 2.

In Section 4 we address the following problem. The ranges $\varphi(M_n)$ of the linear isometries $\varphi : M_n \to M_p$ with $1 \leq \mathrm{id}(\varphi) < \infty$ constructed in Section 3 are operator systems identical with M_n as normed spaces. But, how different are they from M_n as operator systems? Given a positive integer $n \geq 3$ we introduce a family $\{M_n^{q,\zeta}\}$ of operator systems $M_n^{q,\zeta}$ that are linearly isometric images of M_n, parametrized by positive integers q $(3 \leq q \leq n)$ and unit vectors ζ in certain Hilbert spaces, and classify them up to unital complete isometry. Moreover the group structure of all unital complete isometries of a fixed $M_n^{q,\zeta}$ onto itself is determined.

In Section 5 we state two questions that have remained unanswered in this paper and related remarks.

The author thanks the referee for his constructive critique and for suggesting many simplifications of proofs, which will be noted in appropriate places.

2. Preliminaries.

Let $\varphi : M_n \to M_p$ be a unital completely positive map between matrix algebras. Throughout the paper we always assume that it is written in the form $\varphi_L : B(H_1) \to B(L)$, which is the unital completely positive map defined as follows.

Let H_1 and H_2 be finite-dimensional Hilbert spaces, $\widetilde{H} := H_1 \otimes H_2$ their Hilbert space tensor product, and $L \subset \widetilde{H}$ a linear subspace. If $\dim H_1 = n$, $\dim L = p$ and we identify $B(H_1) = M_n$, $B(L) = M_p$, then we obtain a unital completely positive map $\varphi_L : M_n \to M_p$ defined by

$$\begin{aligned} \varphi_L : & B(H_1) \to B(H_1) \otimes B(H_2) = B(\widetilde{H}) \to P_L B(\widetilde{H}) P_L = B(L), \\ & x \quad \longmapsto \quad x \otimes 1_{H_2} \quad \longmapsto \quad P_L(x \otimes 1_{H_2})P_L =: \varphi_L(x). \end{aligned} \tag{2.1}$$

Here 1_{H_2} denotes the identity operator on H_2, P_L denotes the projection of $\widetilde{H}$ onto L, and we canonically identify $B(H_1) \otimes B(H_2)$ with $B(\widetilde{H})$ and $P_L B(\widetilde{H}) P_L$ with $B(L)$. Conversely, every unital completely positive map $\varphi : M_n \to M_p$ between matrix algebras is unitarily equivalent to the above map φ_L for some Hilbert spaces H_1, H_2 and some linear subspace L of $H_1 \otimes H_2$ such that $\dim H_1 = n$ and $\dim L = p$. Indeed, if we identify $M_p = B(H)$ for a Hilbert space H with $\dim H = p$, then by the Stinespring theorem (Paulsen [**5**], Theorem 4.1) there exist a finite-dimensional Hilbert space K, a unital *-homomorphism $\pi : M_n \to B(K)$, and a linear isometry $V : H \to K$ such that

$\varphi(x) = V^*\pi(x)V$ for all $x \in M_n$. Here, that $\dim K < \infty$ follows from the fact that K is obtained as the quotient space of the finite-dimensional tensor product $M_n \otimes H$. Since M_n is a simple C^*-algebra, we can identify the *-homomorphism π with the amplification $B(H_1) \to B(H_1) \otimes B(H_2)$, $x \mapsto x \otimes 1_{H_2}$, where $M_n = B(H_1)$ and $K = H_1 \otimes H_2$ for some Hilbert space H_2. Moreover, since φ is unital, V is an isometry of H onto $L := VH \subset K$, so that the map $V \cdot V^* : B(H) \to B(L)$, $x \mapsto VxV^*$, defines a unitary equivalence, and $VV^* = P_L \in B(H_1 \otimes H_2)$. Hence the map $\varphi : M_n \to M_p = B(H)$, $x \mapsto V^*\pi(x)V = V^*(x \otimes 1_{H_2})V$, is unitarily equivalent to the map $\varphi_L : B(H_1) \to B(L)$, $x \mapsto VV^*(x \otimes 1_{H_2})VV^* = P_L(x \otimes 1_{H_2})P_L$.

The uniqueness of $K = H_1 \otimes H_2$ and $L \subset H_1 \otimes H_2$, up to unitary equivalence, in the expression $\varphi = \varphi_L$ follows when we further require that $\pi(M_n)VH = K$, or equivalently that $(B(H_1) \otimes 1_{H_2})L = K = H_1 \otimes H_2$ (see [**5**], Proposition 4.2). But we will not assume this condition $(B(H_1) \otimes 1_{H_2})L = H_1 \otimes H_2$ to give flexibility in the choice of $L \subset H_1 \otimes H_2$.

As usual we write $B(H) = M_n$ when we need only specify $\dim H = n < \infty$.

In what follows we adopt the following notational convention. For H_1, H_2 and $H_1 \otimes H_2$ as above we denote by the letters ξ, η and ζ vectors in H_1, H_2 and $H_1 \otimes H_2$, respectively. Let $\overline{H_1} := \{\xi^* : \xi \in H_1\}$ be the complex conjugate of H_1, i.e., the Hilbert space with the linear space operation $\lambda_1\xi_1^* + \lambda_2\xi_2^* = (\overline{\lambda_1}\xi_1 + \overline{\lambda_2}\xi_2)^*$ and the inner product $\langle \xi_1^*, \xi_2^* \rangle_{\overline{H_1}} = \langle \xi_2, \xi_1 \rangle$ for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\xi_1, \xi_2 \in H_1$. Then the map $\xi^* \mapsto \langle \cdot, \xi \rangle$ gives a linear isomorphism of $\overline{H_1}$ onto the dual space of H_1, and it induces the canonical linear isomorphism $\rho : H_1 \otimes H_2 \to B(\overline{H_1}, H_2)$, $\zeta \mapsto \rho_\zeta$, defined by

$$\rho_{\xi_1 \otimes \eta_1}\xi^* = \langle \xi_1, \xi \rangle \eta_1, \quad \xi_1, \xi \in H_1, \ \ \eta_1 \in H_2. \tag{2.2}$$

The operator $\rho_\zeta \in B(\overline{H_1}, H_2)$, $\zeta \in H_1 \otimes H_2$, is reformulated by the following equality.

$$\langle \rho_\zeta \xi^*, \eta \rangle = \langle \zeta, \xi \otimes \eta \rangle, \quad \xi \in H_1, \eta \in H_2. \tag{2.3}$$

We use the following symbolic notation to denote inner products or operators:

$$\xi_2^*\xi_1 := \langle \xi_1, \xi_2 \rangle, \ \ \xi_1, \xi_2 \in H_1;$$
$$\xi_2\xi_1^* : H_1 \to H_1, \ \xi \mapsto (\xi_2\xi_1^*)\xi = \xi_2(\xi_1^*\xi) = \langle \xi, \xi_1 \rangle \xi_2, \ \ \xi_1, \xi_2 \in H_1;$$
$$\xi_1\eta_1 := \rho_{\xi_1 \otimes \eta_1} : \overline{H_1} \to H_2, \ \ \xi^* \mapsto \xi^*(\xi_1\eta_1) = (\xi^*\xi_1)\eta_1 = \langle \xi_1, \xi \rangle \eta_1, \ \ \xi_1 \in H_1, \ \eta_1 \in H_2,$$

etc. The meaning would be self-explanatory when we view vectors as column vectors with respect to some orthonormal basis and juxtapositions of them as matrix products. Then $\rho_{\xi_2 \otimes \eta_2}^* : H_2 \to \overline{H_1}$ and $\rho_{\xi_1 \otimes \eta_1}\rho_{\xi_2 \otimes \eta_2}^* : H_2 \to \overline{H_1} \to H_2$ are written formally as

$$\rho_{\xi_1 \otimes \eta_1}^* = \xi_1^*\eta_1^*, \quad \rho_{\xi_1 \otimes \eta_1}\rho_{\xi_2 \otimes \eta_2}^* = \langle \xi_1, \xi_2 \rangle \eta_1\eta_2^*, \tag{2.4}$$

meaning the maps $\eta \mapsto \xi_1^*\eta_1^*\eta = \langle \eta, \eta_1 \rangle \xi_1^*$ and $\eta \mapsto \langle \xi_1, \xi_2 \rangle \langle \eta, \eta_2 \rangle \eta_1$, respectively.

For any subsets $S \subset H_1 \otimes H_2$ and $T \subset H_1$ write

$$[S]_T := \mathrm{lin}\{\rho_\zeta \xi^* : \zeta \in S, \ \xi \in T\} = \mathrm{lin} \bigcup_{\zeta \in S} \rho_\zeta T^* \subset H_2. \tag{2.5}$$

Here and throughout, $\mathrm{lin}\{\dots\}$ denotes the linear span of $\{\dots\}$ in any linear space, and $T^* := \{\xi^* : \xi \in T\}$. In particular, if $T = \{\xi_1, \dots, \xi_k\}$, $\xi_i \in H_1$, write $[S]_{\xi_1,\dots,\xi_k} := [S]_T$, and if $T = H_1$, write $[[S]] := [S]_{H_1}$.

Definition 2.1. For a nonempty subset S of $H_1 \otimes H_2$ we call the following integer the *length* of S.

$$\mathrm{length}\, S := \min\{\dim T : T \subset H_1 \text{ linear}, [S]_T = [[S]]\}. \tag{2.6}$$

That is, $l = \mathrm{length}\, S$ if and only if $[S]_T \subsetneqq [[S]]$ for any linear subspace T of H_1 of $\dim T < l$ and $[S]_T = [[S]]$ for some linear subspace T of H_1 of $\dim T = l$.

Note that replacing S and T in (2.5) and (2.6) by their linear spans does not affect the resulting sets and the value of $\mathrm{length}\, S$, i.e., $[S]_T = [\mathrm{lin}\, S]_T = [S]_{\mathrm{lin}\, T} = [\mathrm{lin}\, S]_{\mathrm{lin}\, T}$, $[[S]] = [[\mathrm{lin}\, S]]$ and $\mathrm{length}\, S = \mathrm{length}(\mathrm{lin}\, S)$. Note also that since the map $T \mapsto T^*$ gives a bijection between the set of all linear subspaces of H_1 and that of $\overline{H_1}$, the equality in (2.6) is written as $\sum_{\zeta \in S} \rho_\zeta T^* = \sum_{\zeta \in S} \rho_\zeta \overline{H_1}$, and (2.6) is reformulated as

$$\mathrm{length}\, S = \min\{\dim T : T \subset \overline{H_1} \text{ linear}, \sum_{\zeta \in S} \rho_\zeta T = \sum_{\zeta \in S} \rho_\zeta \overline{H_1}\}. \tag{2.7}$$

Definition 2.2. Let $\varphi : X \to Y$ be a linear map between operator spaces X and Y.

(i) For a positive integer m we call φ a *strict m-isometry* if $\varphi_m : M_m(X) \to M_m(Y)$ is an isometry, but $\varphi_{m+1} : M_{m+1}(X) \to M_{m+1}(Y)$ is not an isometry, where $M_m(X) = M_m \otimes X$, $M_m(Y) = M_m \otimes Y$, etc., and $\varphi_m = \mathrm{id}_m \otimes \varphi$ with id_m denoting the identity map on M_m.

(ii) We define the *isometric degree* of φ, written $\mathrm{id}(\varphi)$, to be 0, m, and ∞, respectively if φ is not an isometry, a strict m-isometry, and a complete isometry, respectively.

3. Isometric degrees of φ_L.

We describe the isometric degree $\mathrm{id}(\varphi_L)$ of the unital completely positive map φ_L defined in Section 2 in terms of the orthogonal complement $L^\perp$ of L as follows.

Theorem 3.1. *As in Section 2, let H_1, H_2 be finite-dimensional Hilbert spaces, L a linear subspace of $\widetilde{H} := H_1 \otimes H_2$, and $\varphi_L : B(H_1) \to B(L)$ the unital completely positive map associated with L. Let $n := \dim H_1$, $q := \dim H_2$, $L^\perp$ the orthogonal complement of L in $\widetilde{H}$, and $l := \mathrm{length}\, L^\perp$. Then*:

(i) *We have $l \le \min\{n, q\}$.*

(ii) *The following are equivalent*:

(ii1) *$\mathrm{id}(\varphi_L) = \infty$, i.e., φ_L is a complete isometry.*

(ii2) *$[[L^\perp]] \subsetneqq H_2$.*

(ii3) *There exists an $\eta_0 \in H_2 \setminus \{0\}$ such that $H_1 \otimes \eta_0 \subset L$.*

(iii) *Suppose that $\mathrm{id}(\varphi_L) < \infty$ and hence by* (ii) *that $[[L^\perp]] = H_2$. Then we have*

$$\mathrm{id}(\varphi_L) = \left[\frac{l-1}{2}\right], \tag{3.1}$$

where $[a]$ for a real number a is the largest integer $\leq a$. That is, if $l \leq 2$, then φ_L is not an isometry, and if $l \geq 3$, then φ_L is a strict $[(l-1)/2]$-isometry.

Since $l \leq n$, Theorem 3.1 means that if $1 \leq n \leq 2$, then $\mathrm{id}(\varphi_L) \in \{0, \infty\}$ and if $n \geq 3$, then $\mathrm{id}(\varphi_L) \in \{0, 1, 2, \ldots, [(n-1)/2], \infty\}$. In particular, if $1 \leq n \leq 2$, φ_L being an isometry implies its being a complete isometry. The following theorem shows that the values $1, 2, \ldots, [(n-1)/2]$ are indeed attained as $\mathrm{id}(\varphi_L)$ for some φ_L if $n \geq 3$ is fixed and p is sufficiently large.

THEOREM 3.2. *Let n and m be positive integers with $n \geq 3$ and $1 \leq m \leq [(n-1)/2]$. Then there exist a positive integer p and a map $\varphi_L : M_n \to M_p$ such that $\mathrm{id}(\varphi_L) = m$. Here we can take p to be $n(2m+1)-1$.*

We separate the proofs of Theorems 3.1 and 3.2 into several lemmas. In the following lemmas we retain the notation H_1, H_2, L, φ_L, $n = \dim H_1$, and $q = \dim H_2$ in Theorem 3.1.

LEMMA 3.3. (i) *For $\xi_1, \xi_2 \in H_1$ we have $\|P_L(\xi_2\xi_1^* \otimes 1_{H_2})P_L\| = \|\xi_2\xi_1^*\| = \|\xi_1\|\|\xi_2\|$ if and only if there exists an $\eta \in H_2 \setminus \{0\}$ such that $\xi_1 \otimes \eta, \xi_2 \otimes \eta \in L$, where $\xi_2\xi_1^* \in B(H_1)$ is the operator $\xi \mapsto (\xi_2\xi_1^*)\xi = \xi_2(\xi_1^*\xi) = \langle \xi, \xi_1\rangle \xi_2$ on H_1 of rank ≤ 1 as before.*

(ii) *The map $\varphi_L : B(H_1) \to B(L)$, $\varphi_L(x) = P_L(x \otimes 1_{H_2})P_L$, is an isometry if and only if*

$$\forall\, \xi_1, \xi_2 \in H_1, \exists\, \eta \in H_2 \setminus \{0\} :\ \xi_1 \otimes \eta,\ \xi_2 \otimes \eta \in L. \tag{3.2}$$

(iii) *For a positive integer m the map φ_L is an m-isometry if and only if*

$$\forall\, \xi_i \in H_1\ (1 \leq i \leq 2m),\ \exists\, \eta \in H_2 \setminus \{0\} :\ \xi_i \otimes \eta \in L\ (1 \leq i \leq 2m). \tag{3.3}$$

PROOF. (i) Clearly $\|\xi_2\xi_1^*\| = \|\xi_1\|\|\xi_2\|$, and for the proof we may assume that $\|\xi_1\| = \|\xi_2\| = \|\eta\| = 1$.

($\Leftarrow$): Suppose such an $\eta \in H_2$ exists. Then $\xi_i \otimes \eta \in L$, $\|\xi_i \otimes \eta\| = \|\xi_i\|\|\eta\| = 1$ $(i = 1, 2)$,

$$\begin{aligned}\|P_L(\xi_2\xi_1^* \otimes 1_{H_2})P_L\| &\geq |\langle P_L(\xi_2\xi_1^* \otimes 1_{H_2})P_L(\xi_1 \otimes \eta), \xi_2 \otimes \eta\rangle| \\ &= |\langle (\xi_2\xi_1^* \otimes 1_{H_2})(\xi_1 \otimes \eta), \xi_2 \otimes \eta\rangle| \\ &= \langle \xi_1, \xi_1\rangle\langle \xi_2, \xi_2\rangle\langle \eta, \eta\rangle = 1,\end{aligned}$$

and further, $\|P_L(\xi_2\xi_1^* \otimes 1_{H_2})P_L\| \leq \|\xi_2\xi_1^* \otimes 1_{H_2}\| = \|\xi_1\|\|\xi_2\| = 1$.

($\Rightarrow$): The following proof was suggested by the referee; the original proof was more lengthy. Let $v = \xi_2\xi_1^*$ and suppose that $\|P_L(v \otimes 1_{H_2})P_L\| = \|v\| = 1$. Then v is a partial isometry with $v^*v = \xi_1\xi_1^*$ and $vv^* = \xi_2\xi_2^*$. Since $H_1 \otimes H_2$ is finite-dimensional and its unit sphere is compact, there is a unit vector $\zeta \in H_1 \otimes H_2$ such that $\|P_L(v \otimes 1_{H_2})P_L\zeta\| = 1$. We show that ζ, $(v \otimes 1_{H_2})\zeta \in L$ and $(v^*v \otimes 1_{H_2})\zeta = \zeta$. Indeed,

$1 = \|P_L(v\otimes 1_{H_2})P_L\zeta\| \le \|P_L(v\otimes 1_{H_2})\|\|P_L\zeta\| \le \|P_L\zeta\| \le \|\zeta\| = 1$ implies that $\|P_L\zeta\| = \|\zeta\|$ and hence that $\zeta = P_L\zeta \in L$, since $\|\zeta\|^2 = \|P_L\zeta\|^2 + \|\zeta - P_L\zeta\|^2$. Similarly, $\|P_L(v\otimes 1_{H_2})\zeta\| = \|P_L(v\otimes 1_{H_2})P_L\zeta\| = 1 = \|(v\otimes 1_{H_2})\zeta\|$ implies $(v\otimes 1_{H_2})\zeta \in L$. Since v is a partial isometry, $\|(v^*v\otimes 1_{H_2})\zeta\| = \|(v\otimes 1_{H_2})\zeta\| = 1$, and $\|(v^*v\otimes 1_{H_2})\zeta\| = 1 = \|\zeta\|$. Then, since $v^*v\otimes 1_{H_2} = \xi_1\xi_1^*\otimes 1_{H_2}$ is the projection onto $\xi_1\otimes H_2$, it follows that $(v^*v\otimes 1_{H_2})\zeta = \zeta$ and hence that $\zeta = \xi_1\otimes\eta$ for some unit vector $\eta \in H_2$. Then $(v\otimes 1_{H_2})\zeta = (\xi_2\xi_1^*\otimes 1_{H_2})(\xi_1\otimes\eta) = \xi_2\otimes\eta$, and it follows that $\xi_1\otimes\eta$, $\xi_2\otimes\eta \in L$.

Note that the above argument shows that $\|P_L(\xi_2\xi_1^*\otimes 1_{H_2})P_L\zeta\| = \|\zeta\|$ for $\zeta \in H_1\otimes H_2$ if and only if $\zeta = \xi_1\otimes\eta$ for some $\eta \in H_2$ such that $\xi_1\otimes\eta$, $\xi_2\otimes\eta \in L$.

(ii) ($\Rightarrow$): If φ_L is an isometry, then $\|P_L(\xi_2\xi_1^*\otimes 1_{H_2})P_L\| = \|\varphi_L(\xi_2\xi_1^*)\| = \|\xi_2\xi_1^*\|$ for all ξ_1, $\xi_2 \in H_1$. Hence (3.2) follows from (i).

($\Leftarrow$): Let $x \in B(H_1)$ and take any unit vectors $\xi_i \in H_1$ $(i = 1, 2)$. Then there exists a unit vector $\eta \in H_2$ as in (3.2), and so

$$\begin{aligned}\|\varphi_L(x)\| &\ge |\langle P_L(x\otimes 1_{H_2})P_L(\xi_1\otimes\eta), \xi_2\otimes\eta\rangle| = |\langle (x\otimes 1_{H_2})(\xi_1\otimes\eta), \xi_2\otimes\eta\rangle| \\ &= |\langle x\xi_1, \xi_2\rangle|\langle\eta, \eta\rangle = |\langle x\xi_1, \xi_2\rangle|.\end{aligned}$$

Since ξ_1, ξ_2 are arbitrary, it follows that $\|\varphi_L(x)\| \ge \|x\|$, and the reverse inequality being obvious, $\|\varphi_L(x)\| = \|x\|$.

(iii) For $\varphi := \varphi_L : B(H_1) \to B(L)$ in (ii), $\varphi_m := \mathrm{id}_m\otimes\varphi : M_m\otimes B(H_1) \to M_m\otimes B(L)$ is given as follows. For $x = \sum_{1\le i,\,j\le m} e_{ij}\otimes x_{ij} \in M_m\otimes B(H_1)$, where $\{e_{ij}\}_{1\le i,\,j\le m}$ is a family of matrix units for M_m and $x_{ij} \in B(H_1)$,

$$\begin{aligned}\varphi_m(x) &= \sum_{1\le i,\,j\le m} e_{ij}\otimes\varphi(x_{ij}) = \sum_{1\le i,\,j\le m} e_{ij}\otimes P_L(x_{ij}\otimes 1_{H_2})P_L \\ &= (1_{\mathbb{C}^m}\otimes P_L)\Big(\sum_{1\le i,\,j\le m} e_{ij}\otimes x_{ij}\otimes 1_{H_2}\Big)(1_{\mathbb{C}^m}\otimes P_L) \\ &= P_{\mathbb{C}^m\otimes L}(x\otimes 1_{H_2})P_{\mathbb{C}^m\otimes L}.\end{aligned}$$

That is, φ_m is just the φ_L with H_1 replaced by $\mathbb{C}^m\otimes H_1$ and $L \subset H_1\otimes H_2$ replaced by $\mathbb{C}^m\otimes L \subset \mathbb{C}^m\otimes H_1\otimes H_2$. Hence, by (ii), φ_L is an m-isometry, i.e., φ_m is an isometry if and only if

$$\forall\,\xi_1', \xi_2' \in \mathbb{C}^m\otimes H_1, \exists\,\eta \in H_2\setminus\{0\} :\ \xi_1'\otimes\eta, \xi_2'\otimes\eta \in \mathbb{C}^m\otimes L. \tag{3.4}$$

For a fixed orthonormal basis $\{\varepsilon_j\}_{1\le j\le m}$ for $\mathbb{C}^m$, $\mathbb{C}^m\otimes H_1 = \varepsilon_1\otimes H_1\oplus\cdots\oplus\varepsilon_m\otimes H_1$, the orthogonal direct sum of right summands, and similarly $\mathbb{C}^m\otimes L = \varepsilon_1\otimes L\oplus\cdots\oplus\varepsilon_m\otimes L \subset \varepsilon_1\otimes(H_1\otimes H_2)\oplus\cdots\oplus\varepsilon_m\otimes(H_1\otimes H_2)$. Hence, taking two vectors ξ_1', ξ_2' in $\mathbb{C}^m\otimes H_1$ is equivalent to taking $2m$ vectors $\xi_1, \xi_2, \ldots, \xi_{2m}$ in H_1 so that $\xi_1' = \sum_{j=1}^m \varepsilon_j\otimes\xi_j$ and $\xi_2' = \sum_{j=1}^m \varepsilon_j\otimes\xi_{j+m}$, and for some $\eta \in H_2\setminus\{0\}$, $\xi_i'\otimes\eta \in \mathbb{C}^m\otimes L$ $(i = 1, 2)$ $\iff$ for some $\eta \in H_2\setminus\{0\}$, $\xi_1\otimes\eta$, $\xi_2\otimes\eta$, $\ldots$, $\xi_{2m}\otimes\eta \in L$. Thus the equivalence (3.4) $\iff$ (3.3) follows. □

Notation. For a linear subspace L of $H_1\otimes H_2$ and $\xi \in H_1$ we write

$$L^\xi := \{\eta \in H_2 :\ \xi\otimes\eta \in L\}. \tag{3.5}$$

Lemma 3.4. (i) *For* $\xi \in H_1$ *we have* $L^\xi = ([L^\perp]_\xi)^\perp$, *where* $[L^\perp]_\xi := \{\rho_\zeta \xi^* : \zeta \in L^\perp\}$ *as in* (2.5).

(ii) (3.3) *holds if and only if* $[L^\perp]_T \subsetneqq H_2$ *for each linear subspace* T *of* H_1 *of* $\dim T \le 2m$.

Proof. (i) For $\eta \in H_2$, $\eta \in L^\xi \iff \xi \otimes \eta \in L \iff \langle \rho_\zeta \xi^*, \eta\rangle = \langle \zeta, \xi\otimes\eta\rangle = 0$ for all $\zeta \in L^\perp$ by (2.3) (since L is finite-dimensional and so $(L^\perp)^\perp = L$) $\iff \eta \in \{\rho_\zeta \xi^* : \zeta \in L^\perp\}^\perp = ([L^\perp]_\xi)^\perp$.

(ii) (3.3) holds $\iff \forall \xi_i \in H_1$ $(1 \le i \le 2m)$: $\bigcap_{1\le i\le 2m} L^{\xi_i} \ne \{0\} \iff \forall \xi_i \in H_1$ $(1 \le i \le 2m)$: $\sum_{1\le i\le 2m} (L^{\xi_i})^\perp \ne H_2$ (since $(\sum_i M_i)^\perp = \bigcap_i M_i^\perp$ for any linear subspaces M_i of H_2 and since H_2 is finite-dimensional). But, by (i) and (2.5), $\sum_{1\le i\le 2m}(L^{\xi_i})^\perp = \sum_{1\le i\le 2m}[L^\perp]_{\xi_i} = [L^\perp]_T$, where $T = \sum_{1\le i\le 2m} \mathbb{C}\xi_i$. When ξ_i $(1 \le i \le 2m)$ range over all $2m$ vectors in H_1, $T = \sum_{1\le i\le 2m}\mathbb{C}\xi_i$ ranges over all linear subspaces of H_1 of dimension $\le 2m$. Hence the assertion follows. □

Lemma 3.5. (i) *Let* K *be a finite-dimensional linear space,* $\{K_i\}_{i\in I}$ *a finite family of proper linear subspaces* K_i *of* K *with* $d_i := \dim K_i$, *and* $r := \dim K - \min_{i\in I} d_i > 0$. *Then there exists an* r*-dimensional linear subspace* T *of* K *such that* $K_i + T = K$ *for all* $i \in I$.

(ii) *Let* K *and* M *be finite-dimensional linear spaces,* $\{a_i\}_{i\in I}$ *a finite subset of* $B(K, M)$, *and* $r := \max_{i\in I} \operatorname{rank} a_i$. *Then there exists an* r*-dimensional linear subspace* T *of* K *such that* $a_i T = a_i K$ *for all* $i \in I$.

(iii) *For any subset* S *of* $H_1 \otimes H_2$ *we have* $\operatorname{length} S \le \min\{n, q\}$.

Proof. (i) We repeatedly use the following obvious fact: $(*)$ If $\{L_j\}$ is a finite family of proper linear subspaces of K, then $\bigcup_j L_j \ne K$. Indeed, each L_j is closed and has empty interior in K. So the same is true for their union $\bigcup_j L_j$, $K \setminus \bigcup_j L_j$ is open and dense in K, and it is non-empty.

By $(*)$ there exists $\xi_1 \in K \setminus \bigcup_{i\in I} K_i$. Let $K_i^{(1)} := K_i + \mathbb{C}\xi_1$ $(i \in I)$ and $I_1 := \{i \in I : K_i^{(1)} \subsetneqq K\}$. For $i \in I$ we have $i \in I \setminus I_1 \iff d_i + 1 = \dim K_i + 1 = \dim K_i^{(1)} = n$, i.e., $d_i = n-1$, and so $i \in I_1 \iff d_i \le n-2$. If $I_1 \ne \emptyset$, then again by $(*)$, there exists $\xi_2 \in K \setminus \bigcup_{i\in I_1} K_i^{(1)}$, and we can define $K_i^{(2)} := K_i^{(1)} + \mathbb{C}\xi_2$ $(i \in I_1)$, $I_2 := \{i \in I_1 : K_i^{(2)} \subsetneqq K\}$ so that for $i \in I$, $i \in I_2 \iff d_i \le n-3$ and $i \in I_1 \setminus I_2 \iff d_i = n-2$. As long as $I_j \ne \emptyset$ this procedure works, and since $d_i \ge n - r$ for all i with equality for some i, it terminates precisely at the rth step. Thus we obtain vectors $\xi_1, \xi_2, \dots, \xi_r \in K$ and sets $I_0 := I \supset I_1 \supset I_2 \supset \cdots \supset I_{r-1} \ne \emptyset$ so that $K_i \subsetneqq K_i^{(1)} \subsetneqq \cdots \subsetneqq K_i^{(j)} = K_i + \mathbb{C}\xi_1 + \cdots + \mathbb{C}\xi_j = K \iff i \in I_{j-1} \setminus I_j$. If we set $T := \mathbb{C}\xi_1 + \cdots + \mathbb{C}\xi_r$, it follows that $K_i + T = K$ for all $i \in I$.

(ii) We may assume $a_i \ne 0$ for all $i \in I$. Then $K_i := \operatorname{Ker} a_i \subsetneqq K$ $(i \in I)$, $\dim K_i = n - r_i$, and $n - \min_{i\in I}(n - r_i) = \max_{i\in I} r_i = r$, where $n = \dim K$ and $r_i := \operatorname{rank} a_i$. By (i) there exists an r-dimensional linear subspace T of K such that $K_i + T = K$ for all $i \in I$. Hence $a_i K = a_i(K_i + T) = a_i T$ for all $i \in I$.

(iii) Clearly $\operatorname{length} S \le n$ since $\dim T \le \dim \overline{H_1} = \dim H_1 = n$ for T in (2.7). Since $\dim \operatorname{lin} S \le \dim \widetilde{H} < \infty$, we have $\operatorname{lin} S = \operatorname{lin}\{\zeta_1, \dots, \zeta_k\}$ for some finite $\{\zeta_1, \dots, \zeta_k\} \subset S$. Then, by (2.7), $\operatorname{length} S = \min\{\dim T : T \subset \overline{H_1} \text{ linear}, \sum_{i=1}^k \rho_{\zeta_i} T = \sum_{i=1}^k \rho_{\zeta_i} \overline{H_1}\}$.

If $r := \max_{1\le i\le k} \operatorname{rank} \rho_{\zeta_i} = \max_{1\le i\le k} \dim(\rho_{\zeta_i}\overline{H_1}) \le \dim H_2 = q$, then by (ii) there exists an r-dimensional linear subspace T of $\overline{H_1}$ such that $\rho_{\zeta_i}T = \rho_{\zeta_i}\overline{H_1}$ for all i. Hence $\operatorname{length} S \le \dim T = r \le q$. □

Lemma 3.6. (i) *Let s be a positive integer with $1 \le s \le \min\{n, q\}$. Define $\zeta_0, \zeta_{ij} \in H_1 \otimes H_2$ by $\zeta_0 := \sum_{i=1}^s \xi_i \otimes \eta_i$, $\zeta_{ij} := \xi_i \otimes \eta_j$ $(1 \le i \le s,\ s+1 \le j \le q)$, where $\{\xi_i\}_{1\le i\le s} \subset H_1$ is linearly independent and $\{\eta_j\}_{1\le j\le q}$ is a basis for H_2. Then the linear span $M := \operatorname{lin}\{\zeta_0, \zeta_{ij} : 1 \le i \le s,\ s+1 \le j \le q\}$ satisfies that* $\operatorname{length} M = s$, $[[M]] = H_2$, *and* $\dim M = s(q-s)+1$.

(ii) *Suppose that $1 \le \dim H_2 = q \le \dim H_1 = n$. If $\zeta_0 = \sum_{i=1}^q \xi_i \otimes \eta_i \in H_1 \otimes H_2$ with both $\{\xi_i\}_{1\le i\le q} \subset H_1$ and $\{\eta_i\}_{1\le i\le q} \subset H_2$ linearly independent and $M := \mathbb{C}\zeta_0$, then* $\operatorname{length} M = q$ *and* $[[M]] = H_2$.

Proof. (i) There exist linearly independent vectors $\{\xi_i'\}_{1\le i\le s}$ in H_1 such that $\langle \xi_i, \xi_j'\rangle = \delta_{ij}$, the Kronecker symbol, for all i, j. Indeed, since $\{\xi_i\}_{1\le i\le s}$ is a basis for $H_1' := \operatorname{lin}\{\xi_i\}_{1\le i\le s}$, for each j $(1 \le j \le s)$ the linear functional $\sum_{i=1}^s \lambda_i\xi_i \mapsto \lambda_j$ $(\lambda_i \in \mathbb{C})$ on H_1' defines a unique element $\xi_j' \in H_1'$ such that $\langle \sum_{i=1}^s \lambda_i\xi_i, \xi_j'\rangle = \lambda_j$ for all $\lambda_i \in \mathbb{C}$ $(1 \le i \le s)$. Then it follows that for $1 \le k \le s$,

$$\begin{aligned}[M]_{\xi_k'} &= \{\rho_\zeta \xi_k'^* : \zeta \in M\} = \operatorname{lin}\{\rho_{\zeta_0}\xi_k'^*, \rho_{\zeta_{ij}}\xi_k'^* : 1 \le i \le s,\ s+1 \le j \le q\} \\ &= \operatorname{lin}\{\eta_k, \eta_{s+1}, \eta_{s+2}, \dots, \eta_q\},\end{aligned}$$

since by (2.2), $\rho_{\zeta_0}\xi_k'^* = \sum_{i=1}^s \langle \xi_i, \xi_k'\rangle\eta_i = \eta_k$ and $\rho_{\zeta_{ij}}\xi_k'^* = \langle \xi_i, \xi_k'\rangle\eta_j = \delta_{ki}\eta_j$. Hence, for the s-dimensional linear subspace $T_0 := \operatorname{lin}\{\xi_1', \dots, \xi_s'\}$ of H_1, $[M]_{T_0} = \sum_{k=1}^s [M]_{\xi_k'} = \operatorname{lin}\{\eta_1, \dots, \eta_s, \eta_{s+1}, \eta_{s+2}, \dots, \eta_q\} = H_2$. Since $[M]_{T_0} \subset [[M]] \subset H_2$, it also follows that $[[M]] = H_2$. On the other hand, if T is a k-dimensional linear subspace of H_1 with basis $\{\xi^{(r)} : 1 \le r \le k\}$ and if $k < s$, then, since $\rho_{\zeta_{ij}}(\xi^{(r)})^* \in \operatorname{lin}\{\eta_j : s+1 \le j \le q\}$,

$$\begin{aligned}[M]_T &= \operatorname{lin}\{\rho_{\zeta_0}(\xi^{(r)})^*, \rho_{\zeta_{ij}}(\xi^{(r)})^* : 1 \le r \le k,\ 1 \le i \le s,\ s+1 \le j \le q\} \\ &\subset \operatorname{lin}\{\rho_{\zeta_0}(\xi^{(r)})^* : 1 \le r \le k\} + \operatorname{lin}\{\eta_j : s+1 \le j \le q\}.\end{aligned}$$

The dimension of the right-hand side is at most $k + (q-s) < q = \dim H_2$, and so $[M]_T \subsetneqq H_2$. Thus it follows that $\operatorname{length} M = s$.

The set $\{\zeta_{ij}\}_{1\le i\le s,\, s+1\le j\le q}$ is linearly independent, and so its linear span N has dimension $s(q-s)$. Moreover $\zeta_0 = \sum_{i=1}^s \xi_i \otimes \eta_i \notin N$, since each element of N is uniquely written in the form $\sum_{i=1}^s \xi_i \otimes \sum_{j=s+1}^q \lambda_{ij}\eta_j$ $(\lambda_{ij} \in \mathbb{C})$. Hence $\dim M = \dim(N + \mathbb{C}\zeta_0) = s(q-s)+1$.

(ii) This is the special case of (i) where $s = q$ and the ζ_{ij}'s are missing. □

Proof of Theorem 3.1. (i) This follows from Lemma 3.5 (iii).

(ii) (ii1) $\iff$ (ii2): The map φ_L is a complete isometry $\iff$ φ_L is an m-isometry for all m $\iff$ by Lemma 3.3 (iii) and Lemma 3.4 (ii), $[L^\perp]_T \subsetneqq H_2$ for each linear subspace T of H_1 of $\dim T \le 2m$ and each m $\iff$ $[[L^\perp]] = [L^\perp]_{H_1} \subsetneqq H_2$.

(ii2) $\iff$ (ii3): For $\eta \in H_2$, $H_1 \otimes \eta \subset L$ $\iff$ $\eta \in \bigcap_{\xi\in H_1} L^\xi = \bigcap_{\xi\in H_1}([L^\perp]_\xi)^\perp = (\sum_{\xi\in H_1}[L^\perp]_\xi)^\perp = ([L^\perp]_{H_1})^\perp = ([[L^\perp]])^\perp$ by (3.5) and Lemma 3.4(i). Hence, $[[L^\perp]] \subsetneqq H_2$ $\iff$ $H_1 \otimes \eta_0 \subset L$ for some $\eta_0 \in H_1 \setminus \{0\}$.

(iii) As noted above, Lemma 3.3 (iii) and Lemma 3.4 (ii) show that $(*)$ φ_L is an m-isometry for $m \geq 1$ if and only if $[L^\perp]_T \subsetneqq H_2$ for each linear subspace T of H_1 of $\dim T \leq 2m$. Since we are assuming that $[[L^\perp]] = H_2$, the definition of length (Definition 2.1) implies that $l = \dim T$ for some linear subspace T of H_1 with $[L^\perp]_T = H_2$ and that $[L^\perp]_T \subsetneqq H_2$ for each linear subspace T of H_1 of $\dim T < l$.

If $l = \operatorname{length} L^\perp \leq 2$, then $[L^\perp]_T = H_2$ for some linear subspace T of H_1 of $\dim T \leq 2$. Hence, by $(*)$, φ_L is not an isometry.

If $l \geq 3$ and $m := [(l-1)/2] \geq 1$, then $m \leq (l-1)/2 < m+1$. Hence $2m \leq l-1$, $2(m+1) > l-1$, and so $2m < l$, $2(m+1) \geq l$. The inequality $2m < l$ shows that $[L^\perp]_T \subsetneqq H_2$ for each linear subspace T of H_1 of $\dim T \leq 2m$ and hence by $(*)$ that φ_L is an m-isometry. Since $[L^\perp]_T = H_2$ for some linear subspace T of H_1 of $\dim T = l$ and since $2(m+1) \geq l$, the condition in $(*)$ with m replaced by $m+1$ does not hold. Hence φ_L is not an $(m+1)$-isometry. Thus φ_L is a strict m-isometry. $\square$

Proof of Theorem 3.2. Set $q := 2m+1$ so that $3 \leq q \leq n$ since $1 \leq m \leq [(n-1)/2] \leq (n-1)/2$, and take Hilbert spaces H_1 and H_2 with $\dim H_1 = n$ and $\dim H_2 = q$. Lemma 3.6 (ii) shows that for $\zeta_0 \in H_1 \otimes H_2$ as in the statement there, $\operatorname{length} \mathbb{C}\zeta_0 = q$ and $[[\mathbb{C}\zeta_0]] = H_2$. Then Theorem 3.1 (iii) shows that φ_L for $L := \{\zeta_0\}^\perp$ is a strict m-isometry since $[(q-1)/2] = m$. Since $\dim L = \dim(H_1 \otimes H_2) - 1 = nq - 1 = n(2m+1) - 1$, $\varphi_L : B(H_1) \to B(L)$ may be regarded as a unital completely positive map of M_n into $M_{n(2m+1)-1}$. $\square$

Remark 3.7. Part (ii) of Theorem 3.1 may be well-known although we cannot provide suitable references, and the implication (ii3) $\Rightarrow$ (ii1) is obvious without any consideration used above, since $M := H_1 \otimes \eta_0 \subset L$ with $\eta_0 \in H_2 \setminus \{0\}$ implies that the map $B(H_1) \to B(M)$, $x \mapsto \varphi_L(x)|M = P_L(x \otimes 1_{H_2})P_L|M$, is an injective *-homomorphism, so a complete isometry and that φ_L itself is a complete isometry.

4. Classification of a family $\{M_n^{q,\zeta}\}$.

The notation H_1, H_2, $n = \dim H_1 < \infty$, $q = \dim H_2 < \infty$, $\widetilde{H} = H_1 \otimes H_2$, φ_L : $B(H_1) \to B(L)$ for $L \subset \widetilde{H}$, etc. will be as before.

In this section we assume $n \geq q \geq 3$, and introduce operator systems $M_n^{q,\zeta}$, linearly isometric to M_n, as follows. Consider the following condition for a vector ζ in $\widetilde{H}$:

$$\zeta = \sum_{i=1}^{q} \xi_i \otimes \eta_i, \quad \{\xi_i\}_{1 \leq i \leq q} \subset H_1,\ \{\eta_i\}_{1 \leq i \leq q} \subset H_2 \text{ linearly independent,} \tag{4.1}$$

and set

$$Z_{n,q} := \{\zeta \in \widetilde{H} : \|\zeta\| = 1,\ \zeta \text{ satisfies (4.1)}\}. \tag{4.2}$$

For $\zeta \in Z_{n,q}$ denote by φ_ζ the map φ_L defined for $L := \{\zeta\}^\perp$. Then $\operatorname{id}(\varphi_\zeta) = [(q-1)/2]$, since $\operatorname{length} \mathbb{C}\zeta = q$ and $[[\mathbb{C}\zeta]] = H_2$ by Lemma 3.6(ii) and so Theorem 3.1(iii) applies. We have $\dim L = \dim\{\zeta\}^\perp = \dim \widetilde{H} - 1 = nq - 1$, and $[(q-1)/2] \geq 1$ since $q \geq 3$. Hence we may regard φ_ζ as a unital completely positive isometry of M_n into M_{nq-1}, and we obtain an operator system $M_n^{q,\zeta} := \varphi_\zeta(M_n) \subset M_{nq-1}$ as its range.

We will classify the family $\{M_n^{q,\zeta}\}$, where $n \geq q \geq 3$ and $\zeta \in Z_{n,q}$, up to unital complete isometry. That is, we will show when

$$M_n^{q,\zeta} \cong M_{n'}^{q',\zeta'} \tag{4.3}$$

holds for $n \geq q \geq 3$, $\zeta \in Z_{n,q}$, $n' \geq q' \geq 3$, and $\zeta' \in Z_{n',q'}$. Here, for operator systems X and Y we write $X \cong Y$ if there exists a unital complete isometry of X onto Y.

We first deduce that $M_n^{q,\zeta} \ncong M_n$ from the following:

Proposition 4.1. *Let X be an operator system and suppose that there is a unital completely positive isometry of M_n onto X that is not a complete isometry. Then X is not unitally completely isometric to M_n.*

Proof. Let $\varphi : M_n \to X$ be a surjective unital completely positive isometry that is not a complete isometry. Suppose that there exists a surjective unital complete isometry $\kappa : M_n \to X$. Note in general that any surjective unital isometry ι between operator systems V and W is positive. Indeed, for $a \in V$ we have $a \geq 0$ if and only if $f(a) \geq 0$ for all $f \in S(V) := \{f \in V^* : \|f\| = f(1) = 1\}$, and similarly for W. Hence, the condition on ι implies $\iota^*(S(W)) = S(V)$, and the assertion follows. Then κ^{-1}, being also a surjective unital complete isometry, is completely positive, and $\psi := \kappa^{-1}\circ\varphi : M_n \to M_n$ is a surjective unital completely positive isometry. By Kadison's structure theorem of surjective linear isometries between unital C^*-algebras [**4**], there exists a unitary $u \in M_n$ such that (i) $\psi(x) = uxu^*$ for all $x \in M_n$ or (ii) $\psi(x) = u^txu^*$ for all $x \in M_n$. Indeed, since M_n is a factor, ψ is a *-automorphism or an anti-*-automorphism. In the former case, (i) is true. In the latter case, ψ composed with the transpose map, $x \mapsto {}^t\psi(x)$, is a *-automorphism, and so ψ is of the form (ii). The map in case (ii) is not 2-positive (Tomiyama [**6**], Corollary 2.3), and so the case (i) occurs. Hence $\varphi = \kappa \circ \psi$ is also a complete isometry. This is a contradiction. □

Clearly (4.3) implies $n = n'$ since $\dim M_n^{q,\zeta} = \dim M_n = n^2$ and $\dim M_{n'}^{q',\zeta'} = n'^2$. The following result shows that it also implies $q = q'$.

Theorem 4.2. *The C^*-envelope $C_e^*(M_n^{q,\zeta})$ of $M_n^{q,\zeta}$ equals M_{nq-1}.*

Here we recall the notion of the *C^*-envelope*, written $C_e^*(X)$, of an operator system X [**2**]. (We follow the usage of the notation $C_e^*(X)$ to denote the C^*-envelope of X in the recent literature.) An operator system X is a norm closed linear subspace of some unital C^*-algebra such that $1 \in X$ and $x \in X$ implies $x^* \in X$. The C^*-envelope of X is the C^*-algebra $C_e^*(X)$ uniquely determined by the following properties:

(i) $X \subset C_e^*(X)$ and X generates $C_e^*(X)$ as a C^*-algebra;

(ii) if $Y \subset B$ with B a unital C^*-algebra is an operator system, there is a unital complete isometry κ of Y onto X, and $C^*(Y)$ is the C^*-subalgebra of B generated by Y, then there exists a *-homomorphism π of $C^*(Y)$ onto $C_e^*(X)$ extending κ so that $C^*(Y)/\mathrm{Ker}\,\pi \cong C_e^*(X)$ (*-isomorphic as C^*-algebras).

If Theorem 4.2 were true, then (4.3) would imply by the uniqueness of the C^*-envelope that $M_{nq-1} = C_e^*(M_n^{q,\zeta}) \cong C_e^*(M_n^{q',\zeta'}) = M_{nq'-1}$ and hence that $nq-1 = nq'-1$ and $q = q'$ as stated above. To show Theorem 4.2 it suffices to show that $M_n^{q,\zeta} = \varphi_\zeta(M_n) \subset M_{nq-1}$ generates M_{nq-1} as a C^*-algebra. Indeed, the C^*-envelope $C_e^*(M_n^{q,\zeta})$ is realized as the quotient C^*-algebra B/I, where B is the C^*-subalgebra of M_{nq-1} generated by $M_n^{q,\zeta}$ and I is its ideal. But, since M_{nq-1} is simple, $B = M_{nq-1}$ implies $I = \{0\}$, and $C_e^*(M_n^{q,\zeta}) = B = M_{nq-1}$. Moreover, since M_{nq-1} is finite-dimensional, $B = M_{nq-1}$ if and only if $(M_n^{q,\zeta})' := \{x \in M_{nq-1} : xy = yx,\ \forall y \in M_n^{q,\zeta}\} = \mathbb{C}1_{nq-1}$.

Hence Lemma 4.3(iii) below completes the proof of Theorem 4.2 if we take $B(H_1) = M_n$, $P_L B(\widetilde{H}) P_L = B(L) = M_{nq-1}$ and $P_L(B(H_1) \otimes 1_{H_2})P_L = \varphi_L(B(H_1)) = \varphi_\zeta(M_n)$ there.

Lemma 4.3. (i) *For any subset S of $\widetilde{H}$, $[[S]] = [S]_{H_1} := \operatorname{lin}\{\rho_\zeta \overline{H_1} : \zeta \in S\} \subset H_2$ is the smallest linear subspace M of H_2 such that $S \subset H_1 \otimes M$, and*

$$\operatorname{lin}(B(H_1) \otimes 1_{H_2})S := \operatorname{lin}\{(x \otimes 1_{H_1})\zeta : x \in B(H_1),\ \zeta \in S\} = H_1 \otimes [[S]]. \tag{4.4}$$

(ii) *We have*

$$(P_L(B(H_1) \otimes 1_{H_2})P_L)' \cap P_L B(\widetilde{H}) P_L = \{xP_L : x \in 1_{H_1} \otimes B(H_2),\ xP_L = P_L x\}, \tag{4.5}$$

where $T' := \{x \in B(\widetilde{H}) : xy = yx,\ \forall y \in T\}$ for any $T \subset B(\widetilde{H})$.

(iii) *If $L = \{\zeta\}^\perp$ for $\zeta \in Z_{n,q}$, then*

$$(P_L(B(H_1) \otimes 1_{H_2})P_L)' \cap P_L B(\widetilde{H}) P_L = \mathbb{C}P_L. \tag{4.6}$$

Proof. (i) For $\eta \in H_2$, $[[S]] \subset \{\eta\}^\perp \iff \eta \in [[S]]^\perp \iff \langle \rho_\zeta \xi^*, \eta\rangle = 0$, $\forall \xi \in H_1, \forall \zeta \in S \iff \langle \zeta, \xi \otimes \eta\rangle = 0, \forall \xi \in H_1, \forall \zeta \in S$ by (2.3) $\iff H_1 \otimes \{\eta\} \subset S^\perp \iff S \subset S^{\perp\perp} \subset (H_1 \otimes \{\eta\})^\perp = H_1 \otimes \{\eta\}^\perp$. Since $[[S]] = \bigcap\{\{\eta\}^\perp : \eta \in H_2,\ [[S]] \subset \{\eta\}^\perp\}$, the first assertion follows. Hence $S \subset H_1 \otimes [[S]]$ implies $N := \operatorname{lin}(B(H_1) \otimes 1_{H_2})S \subset (B(H_1) \otimes 1_{H_2})(H_1 \otimes [[S]]) = H_1 \otimes [[S]]$. Moreover, since $(B(H_1) \otimes 1_{H_2})N \subset N$, $P_N \in (B(H_1) \otimes 1_{H_2})' = 1_{H_1} \otimes B(H_2)$, and $P_N = 1_{H_1} \otimes P_M$ for some linear subspace M of H_2. It follows that $S \subset N = H_1 \otimes M$, $[[S]] \subset M$, and $H_1 \otimes [[S]] \subset H_1 \otimes M = N$.

(ii) To elucidate the point we start from a slightly general setting. Let M be a von Neumann algebra, $N \subset M$ a von Neumann subalgebra, $P := N' \cap M$, and $p \in M$ a projection. Then $(*)$ $p(P \cap \{p\}') \subset (pNp)' \cap pMp$, since $p \in (pNp)'$, $P \cap \{p\}' \subset N' \cap \{p\}' \subset (pNp)'$, and so $p(P \cap \{p\}') \subset (pNp)' \cap pMp$. Under certain conditions on M, N and p we show the reverse inclusion. Then (4.5) follows if we take $M = B(\widetilde{H}) = B(H_1) \otimes B(H_2)$, $N = B(H_1) \otimes 1_{H_2}$ and $p = P_L$, and show that the conditions hold for such M, N and p.

The argument in this and the next paragraphs is due to the referee. Suppose there is a faithful conditional expectation ψ of M onto P such that

$x\psi(p) = p\psi(x)$, $\forall x \in (pNp)' \cap pMp$, and (a)

if q is the support projection of $\psi(p)$ in P, then $\psi(p)$ is invertible in qPq. (b)

Then q is the smallest projection in P such that $p \leq q$, since ψ is faithful, so $\psi((1-q)p(1-$

$q)) = (1-q)\psi(p)(1-q) = 0$ implies $(1-q)p(1-q) = 0$ and $p \le q$, and since $p \le q'$ for a projection q' in P implies $\psi(p) \le \psi(q') = q'$ and $q \le q'$. Replacing x by x^* in (a) shows $\psi(p)x = \psi(x)p$, and (a) implies that $x = xq = x\psi(p)\psi(p)^{-1} = p\psi(x)\psi(p)^{-1}$ and similarly $x = \psi(p)^{-1}\psi(x)p$ for $x \in (pNp)' \cap pMp$. Here $\psi(x)\psi(p)^{-1} = \psi(p)^{-1}\psi(x) =: y \in P$, so $x = py = yp$ holds, and it follows that $y \in P \cap \{p\}'$ and $x = py \in p(P \cap \{p\}')$, showing the reverse inclusion in $(*)$. Indeed, by (a), $\psi(x)\psi(p) = \psi(x\psi(p)) = \psi(\psi(p)x) = \psi(p)\psi(x)$, so $\psi(p)^{-1}\psi(x)q = q\psi(x)\psi(p)^{-1}$, and $\psi(p)^{-1}\psi(x) = \psi(x)\psi(p)^{-1}$, since $p \le q \in P$ and $x \in pMp$ imply that $\psi(x)q = \psi(xq) = \psi(x)$ and $q\psi(x) = \psi(x)$.

It remains only to show the existence of ψ as above for $M = B(\widetilde{H})$, $N = B(H_1) \otimes 1_{H_2}$, and $p = P_L$. The unitary group $\mathcal{U}$ of $B(H_1) \otimes 1_{H_2}$ is a compact group with the unique, normalized, left and right invariant Haar measure du. Then the left invariance of du shows that the map $\psi : B(\widetilde{H}) \to B(\widetilde{H})$ defined by $\psi(x) = \int_{\mathcal{U}} uxu^*\,du$, $x \in B(\widetilde{H})$, is a conditional expectation of $B(\widetilde{H})$ onto $(B(H_1) \otimes 1_{H_2})' = 1_{H_1} \otimes B(H_2)$. Moreover, $\psi(B(H_1) \otimes 1_{H_2}) \subset (B(H_1) \otimes 1_{H_2}) \cap (1_{H_1} \otimes B(H_2)) = \mathbb{C}1_{\widetilde{H}}$ and the right invariance of du show that $\psi(a \otimes 1_{H_2}) = \mathrm{tr}(a)1_{\widetilde{H}} = 1_{H_1} \otimes \mathrm{tr}(a)1_{H_2}$ and so $\psi(a \otimes b) = 1_{H_1} \otimes \mathrm{tr}(a)b$ for $a \in B(H_1)$ and $b \in B(H_2)$, where tr is the unique normalized trace of $B(H_1)$. Hence, if we denote by $\mathrm{tr} \otimes \mathrm{id}_{B(H_2)} : B(\widetilde{H}) = B(H_1) \otimes B(H_2) \to B(H_2)$ the right slice map $\sum_i a_i \otimes b_i \mapsto \sum_i \mathrm{tr}(a_i)b_i$, $a_i \in B(H_1)$, $b_i \in B(H_2)$, then $\psi(x) = 1_{H_1} \otimes (\mathrm{tr} \otimes \mathrm{id}_{B(H_2)})(x)$, $x \in B(\widetilde{H})$. Since tr is faithful, ψ is also faithful. If $x \in (P_L(B(H_1) \otimes 1_{H_2})P_L)' \cap P_L B(\widetilde{H}) P_L$, then for all $u \in \mathcal{U}$, $xP_LuP_Lu^* = P_LuP_Lxu^*$, and $xuP_Lu^* = P_Luxu^*$ since $xP_L = P_Lx = x$. Hence integration over $\mathcal{U}$ shows $x\psi(P_L) = P_L\psi(x)$, and (a) above is true. By (i), $1_{H_1} \otimes P_{[[L]]}$ is the smallest projection in $1_{H_1} \otimes B(H_2)$ majorizing P_L, and by the previous paragraph it is the support projection of $\psi(P_L)$. Finally, since $1_{H_1} \otimes B(H_2)$ is finite-dimensional, $\psi(P_L)$ is invertible in $1_{H_1} \otimes P_{[[L]]}B(H_2)P_{[[L]]}$, showing (b).

(iii) It suffices to show that if $Q \in (P_L(B(H_1) \otimes 1_{H_2})P_L)' \cap P_L B(\widetilde{H}) P_L$ is a projection, then $Q = 0$ or P_L. By (ii), $Q = (1_{H_1} \otimes q)P_L$ for some projection $q \in B(H_2)$ such that $1_{H_1} \otimes q \in \{P_L\}'$. Since $L = \{\zeta\}^\perp$ and $1_{\widetilde{H}} - P_L = P_{\mathbb{C}\zeta}$, $(1_{H_1} \otimes q)P_{\mathbb{C}\zeta} = P_{\mathbb{C}\zeta}(1_{H_1} \otimes q)$ equals 0 or $P_{\mathbb{C}\zeta}$. Hence $P_{\mathbb{C}\zeta} \le 1_{H_1} \otimes (1_{H_2} - q)$ or $P_{\mathbb{C}\zeta} \le 1_{H_1} \otimes q$. Since $[[\mathbb{C}\zeta]] = H_2$ as noted before, (i) implies $1_{H_1} \otimes 1_{H_2} \le 1_{H_1} \otimes (1_{H_2} - q)$ or $1_{H_1} \otimes 1_{H_2} \le 1_{H_1} \otimes q$. Therefore $q = 0$ or 1_{H_2}, $Q = 0$ or P_L, as desired. □

The following is a key to the classification of $\{M_n^{q,\zeta}\}$.

THEOREM 4.4. *For $i = 1, 2$ let $\zeta_i \in Z_{n,q}$, $L_i := \{\zeta_i\}^\perp$, and regard $M_n^{q,\zeta_i} = \varphi_{\zeta_i}(B(H_1)) = P_{L_i}(B(H_1) \otimes 1_{H_2})P_{L_i} \subset B(H_1 \otimes H_2)$.*

(i) *A linear map $\kappa : M_n^{q,\zeta_1} \to M_n^{q,\zeta_2}$ is a surjective unital complete isometry if and only if $\kappa(P_{L_1}(x \otimes 1_{H_2})P_{L_1}) = P_{L_2}(uxu^* \otimes 1_{H_2})P_{L_2}$ for all $x \in B(H_1)$, where $u \in B(H_1)$ is a unitary such that $(u \otimes v)\zeta_1 = \zeta_2$ for some unitary $v \in B(H_2)$.*

(ii) *We have $M_n^{q,\zeta_1} \cong M_n^{q,\zeta_2}$ if and only if there exist unitaries $u \in B(H_1)$ and $v \in B(H_2)$ such that $(u \otimes v)\zeta_1 = \zeta_2$.*

For the proof we need the following two lemmas, which take care of u and v as in the above statement, respectively.

LEMMA 4.5. *For $i = 1, 2$ let $\zeta_i \in Z_{n,q}$, $L_i := \{\zeta_i\}^\perp$ and let $U \in B(H_1 \otimes H_2)$ be a unitary such that $U\zeta_1 = \zeta_2$. If*

$$UP_{L_1}(B(H_1) \otimes 1_{H_2})P_{L_1}U^* = P_{L_2}(B(H_1) \otimes 1_{H_2})P_{L_2}, \tag{4.7}$$

then there exists a unitary $u \in B(H_1)$ such that

$$UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^* = P_{L_2}(uxu^* \otimes 1_{H_2})P_{L_2}, \ \ \forall x \in B(H_1). \tag{4.8}$$

PROOF. The following map $\psi : B(H_1) \to B(H_1)$ is a surjective unital linear isometry:

$$\begin{aligned} x \mapsto \varphi_{\zeta_1}(x) = P_{L_1}(x \otimes 1_{H_2})P_{L_1} &\mapsto UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^* \\ &\mapsto \varphi_{\zeta_2}^{-1}(UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^*) =: \psi(x). \end{aligned}$$

Indeed, $\varphi_{\zeta_i} : B(H_1) \to \varphi_{\zeta_i}(B(H_1)) = P_{L_i}(B(H_1) \otimes 1_{H_2})P_{L_i}$ $(i = 1, 2)$ are linear isometries, and by (4.7), $UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^* \in UP_{L_1}(B(H_1) \otimes 1_{H_2})P_{L_1}U^* = P_{L_2}(B(H_1) \otimes 1_{H_2})P_{L_2} = \varphi_{\zeta_2}(B(H_1))$. Then

$$UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^* = \varphi_{\zeta_2}(\psi(x)) = P_{L_2}(\psi(x) \otimes 1_{H_2})P_{L_2}, \ \ \forall x \in B(H_1). \tag{4.9}$$

As used in the proof of Proposition 4.1, Kadison's result [**4**] shows that the unital linear isometry ψ is of the following form: for some unitary u in $B(H_1)$, (i) $\psi(x) = uxu^*$ for all $x \in B(H_1)$ or (ii) $\psi(x) = u\,{}^t x u^*$ for all $x \in B(H_1)$.

We show that the case (ii) does not occur. Indeed, if (ii) holds, then (4.9) implies

$$\begin{aligned} &(u^* \otimes 1_{H_2})UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^*(u \otimes 1_{H_2}) \\ =&(u^* \otimes 1_{H_2})P_{L_2}(u \otimes 1_{H_2})({}^t x \otimes 1_{H_2})(u^* \otimes 1_{H_2})P_{L_2}(u \otimes 1_{H_2}) \\ =&P_{(u^*\otimes 1_{H_2})L_2}({}^t x \otimes 1_{H_2})P_{(u^*\otimes 1_{H_2})L_2} = P_0({}^t x \otimes 1_{H_2})P_0 \end{aligned}$$

for all $x \in B(H_1)$, where $P_0 := P_{(u^*\otimes 1_{H_2})L_2}$. Since the map $x \mapsto (u^* \otimes 1_{H_2})UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^*(u \otimes 1_{H_2})$ on $B(H_1)$ is completely positive, so is the map $\tau : x \mapsto P_0({}^t x \otimes 1_{H_2})P_0$ on $B(H_1)$. But the latter is not 2-positive. To see this we use a well-known argument showing that the transpose is not 2-positive (see [**1**]). Let $\zeta_0 := (u^* \otimes 1_{H_2})\zeta_2 = \sum_{i=1}^n \varepsilon_i \otimes \eta_i^{(0)} \in \widetilde{H}$, where $\eta_i^{(0)} \in H_2$ and $\{\varepsilon_i\}_{1\le i\le n}$ is an orthonormal basis for H_1. Since $\|\zeta_0\| = \|\zeta_2\| = 1$, by renumbering if necessary we may assume that $\eta_1^{(0)} \neq 0$. Let $\varepsilon_1' := \|\eta_1^{(0)}\|^{-1}\eta_1^{(0)} \in H_2$ so that $\eta_1^{(0)} = \|\eta_1^{(0)}\|\varepsilon_1'$ and $\|\varepsilon_1'\| = 1$, and let

$$\zeta_1' := \lambda_1(\varepsilon_1 \otimes \varepsilon_1') + \varepsilon_3 \otimes \varepsilon_1', \quad \zeta_2' := \lambda_2(\varepsilon_1 \otimes \varepsilon_1') - \varepsilon_2 \otimes \varepsilon_1',$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ are specified later (note that $n \geq 3$). Since $\langle \zeta_1', \zeta_0 \rangle = \lambda_1\|\eta_1^{(0)}\| + \langle \varepsilon_1', \eta_3^{(0)} \rangle$, $\langle \zeta_2', \zeta_0 \rangle = \lambda_2\|\eta_1^{(0)}\| - \langle \varepsilon_1', \eta_2^{(0)} \rangle$, we may take λ_1, λ_2 so that $\langle \zeta_1', \zeta_0 \rangle = \langle \zeta_2', \zeta_0 \rangle = 0$ and hence so that $\zeta_1', \zeta_2' \in \{\zeta_0\}^\perp = (u^* \otimes 1_{H_2})\{\zeta_2\}^\perp = (u^* \otimes 1_{H_2})L_2 = P_0\widetilde{H}$. If $x_{11} := e_{22}$, $x_{12} := e_{23}$, $x_{21} := e_{32}$, $x_{22} := e_{33} \in B(H_1)$, where $e_{ij} := \varepsilon_i\varepsilon_j^*$, then

$[x_{ij}]_{1\le i,\,j\le 2} \in B(H_1)\otimes M_2$ is positive, since $x/2$ is a projection, but $\tau_2\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}\right) = \begin{bmatrix} P_0 & 0 \\ 0 & P_0\end{bmatrix}\begin{bmatrix} {}^t x_{11}\otimes 1_{H_2} & {}^t x_{12}\otimes 1_{H_2} \\ {}^t x_{21}\otimes 1_{H_2} & {}^t x_{22}\otimes 1_{H_2}\end{bmatrix}\begin{bmatrix} P_0 & 0 \\ 0 & P_0\end{bmatrix}$ is not positive, since $P_0\zeta_1' = \zeta_1'$, $P_0\zeta_2' = \zeta_2'$,

$$\begin{aligned}
&\left\langle \begin{bmatrix} P_0 & 0 \\ 0 & P_0\end{bmatrix}\begin{bmatrix} {}^t x_{11}\otimes 1_{H_2} & {}^t x_{12}\otimes 1_{H_2} \\ {}^t x_{21}\otimes 1_{H_2} & {}^t x_{22}\otimes 1_{H_2}\end{bmatrix}\begin{bmatrix} P_0 & 0 \\ 0 & P_0\end{bmatrix}\begin{bmatrix}\zeta_1' \\ \zeta_2'\end{bmatrix}, \begin{bmatrix}\zeta_1' \\ \zeta_2'\end{bmatrix}\right\rangle \\
&= \left\langle \begin{bmatrix} e_{22}\otimes 1_{H_2} & e_{32}\otimes 1_{H_2} \\ e_{23}\otimes 1_{H_2} & e_{33}\otimes 1_{H_2}\end{bmatrix}\begin{bmatrix}\zeta_1' \\ \zeta_2'\end{bmatrix}, \begin{bmatrix}\zeta_1' \\ \zeta_2'\end{bmatrix}\right\rangle \\
&= \left\langle \begin{bmatrix} -\varepsilon_3\otimes\varepsilon_1' \\ \varepsilon_2\otimes\varepsilon_1'\end{bmatrix}, \begin{bmatrix}\lambda_1(\varepsilon_1\otimes\varepsilon_1') + \varepsilon_3\otimes\varepsilon_1' \\ \lambda_2(\varepsilon_1\otimes\varepsilon_1') - \varepsilon_2\otimes\varepsilon_1'\end{bmatrix}\right\rangle = -2.
\end{aligned}$$

Hence (i) holds, and substitution of (i) for (4.9) shows (4.8). □

Lemma 4.6. *Let $\zeta_1 \in Z_{n,q}$ and $L_1 := \{\zeta_1\}^\perp$. If there exists a unitary $U_1 \in B(H_1\otimes H_2)$ such that $\zeta_2 = U_1\zeta_1 \in Z_{n,q}$ and*

$$P_{L_1}(x\otimes 1_{H_2})P_{L_1} = P_{L_1}U_1^*(x\otimes 1_{H_2})U_1P_{L_1}, \ \ \forall x\in B(H_1), \tag{4.10}$$

then there exist a unitary $v\in B(H_2)$ and $\lambda_0\in\mathbb{C}$ such that

$$U_1 = 1_{H_1}\otimes v + \lambda_0\zeta_2\zeta_1^*, \quad |1-\lambda_0| = 1. \tag{4.11}$$

Proof. We use the technique in the proof of Lemma 4.3 (ii) suggested by the referee. We have (4.10) $\iff$

$$U_1P_{L_1}(x\otimes 1_{H_2})P_{L_1} = P_{L_2}(x\otimes 1_{H_2})U_1P_{L_1}, \quad \forall x\in B(H_1) \tag{4.12}$$

(since $U_1P_{L_1}U_1^* = P_{U_1L_1} = P_{L_2}$) $\iff$ $U_1P_{L_1}uP_{L_1}u^* = P_{L_2}uU_1P_{L_1}u^*$, $\forall u\in\mathcal{U}$, the unitary group of $B(H_1)\otimes 1_{H_2}$, which implies as in the proof of Lemma 4.3 (ii) that $U_1P_{L_1}(1_{H_1}\otimes(\mathrm{tr}\otimes\mathrm{id}_{B(H_2)})(P_{L_1})) = P_{L_2}(1_{H_1}\otimes(\mathrm{tr}\otimes\mathrm{id}_{B(H_2)})(U_1P_{L_1}))$ and the support projection of $(\mathrm{tr}\otimes\mathrm{id}_{B(H_2)})(P_{L_1})$ equals $P_{[[L_1]]}$. Here $P_{[[L_1]]} = 1_{H_2}$, since $P_{L_1}\le 1_{H_1}\otimes P_{[[L_1]]}$ by Lemma 4.3 (i) and so $nq-1 = \dim\widetilde{H}-1 = \mathrm{rank}\,P_{L_1}\le n\cdot\mathrm{rank}\,P_{[[L_1]]}\le nq$ and $n\ge q\ge 3$ imply $\mathrm{rank}\,P_{[[L_1]]} = q = \dim H_2$. Hence $(\mathrm{tr}\otimes\mathrm{id}_{B(H_2)})(P_{L_1})$ is invertible in $B(H_2)$, and if we set $v := (\mathrm{tr}\otimes\mathrm{id}_{B(H_2)})(U_1P_{L_1})(\mathrm{tr}\otimes\mathrm{id}_{B(H_2)})(P_{L_1})^{-1}\in B(H_2)$, then

$$U_1P_{L_1} = P_{L_2}(1_{H_1}\otimes v). \tag{4.13}$$

By substituting (4.13) for (4.12) it follows that $P_{L_2}(B(H_1)\otimes 1_{H_2})P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v)P_{L_1} = \{0\}$. Then we have $P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v)P_{L_1} = 0$, so $(1_{H_1}\otimes v)P_{L_1} = P_{L_2}(1_{H_1}\otimes v)P_{L_1}$, and since (4.13) implies $P_{L_2}(1_{H_1}\otimes v)P_{L_1} = P_{L_2}(1_{H_1}\otimes v)$, it follows that

$$(1_{H_1}\otimes v)P_{L_1} = P_{L_2}(1_{H_1}\otimes v). \tag{4.14}$$

Indeed, otherwise $P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v)P_{L_1}\widetilde{H} = \mathbb{C}\zeta_2$, and

$$\begin{aligned}\{0\} &= P_{L_2}(B(H_1)\otimes 1_{H_2})P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v)P_{L_1}\widetilde{H} = P_{L_2}(B(H_1)\otimes 1_{H_2})(\mathbb{C}\zeta_2)\\ &= P_{L_2}(H_1\otimes[[\mathbb{C}\zeta_2]]) = P_{L_2}(H_1\otimes H_2) = L_2\end{aligned}$$

by (4.4) and the fact that $\zeta_2\in Z_{n,\,q}$, a contradiction.

Now we show that v is a unitary in $B(H_2)$. Indeed, by (4.13) and (4.14), $U_1P_{L_1} = (1_{H_1}\otimes v)P_{L_1}$, and by substituting this for (4.10) it follows that

$$\{0\} = P_{L_1}(1_{H_1}\otimes(1_{H_2}-v^*v))(B(H_1)\otimes 1_{H_2})P_{L_1},$$

and by (4.4) and the fact that $[[L_1]] = H_2$ shown above,

$$\begin{aligned}\{0\} &= P_{L_1}(1_{H_1}\otimes(1_{H_2}-v^*v))(H_1\otimes[[L_1]])\\ &= P_{L_1}(H_1\otimes(1_{H_2}-v^*v)H_2).\end{aligned}$$

Hence $H_1\otimes(1_{H_2}-v^*v)H_2\subset L_1^\perp = \mathbb{C}\zeta_1$. But, since $\dim H_1 = n\geq 3$, $(1_{H_2}-v^*v)H_2 = \{0\}$, $v^*v = 1_{H_2}$. Since $\dim H_2<\infty$, it follows that v is a unitary.

We have $U_1P_{\mathbb{C}\zeta_1} = \zeta_2\zeta_1^*$ and $P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v) = \zeta_2\zeta_3^*$ for some $\zeta_3\in\widetilde{H}$, since $U_1\zeta_1 = \zeta_2$ and $P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v)\widetilde{H}\subset\mathbb{C}\zeta_2$, and

$$\begin{aligned}U_1 &= U_1P_{L_1} + U_1P_{\mathbb{C}\zeta_1} = P_{L_2}(1_{H_1}\otimes v) + U_1P_{\mathbb{C}\zeta_1}\\ &= 1_{H_1}\otimes v - P_{\mathbb{C}\zeta_2}(1_{H_1}\otimes v) + U_1P_{\mathbb{C}\zeta_1} = 1_{H_1}\otimes v + \zeta_2\zeta_4^*,\end{aligned}\tag{4.15}$$

where $\zeta_4 := \zeta_1-\zeta_3\in\widetilde{H}$. Then $\zeta_4 = \overline{\lambda_0}\zeta_1$ for some $\lambda_0\in\mathbb{C}$, since $P_{L_2}U_1 = U_1P_{L_1}$ and $P_{L_2}(1_{H_1}\otimes v) = (1_{H_1}\otimes v)P_{L_1}$ imply that by (4.15), $\zeta_2\zeta_4^* = P_{\mathbb{C}\zeta_2}\zeta_2\zeta_4^* = P_{\mathbb{C}\zeta_2}(U_1-1_{H_1}\otimes v) = (U_1-1_{H_1}\otimes v)P_{\mathbb{C}\zeta_1}$ and $\zeta_2\zeta_4^* = \zeta_2\zeta_4^*P_{\mathbb{C}\zeta_1}$. Hence the first equality in (4.11) follows. Finally, since $(1_{H_1}\otimes v)\zeta_1 = U_1\zeta_1-\lambda_0\zeta_2\zeta_1^*\zeta_1 = (1-\lambda_0)\zeta_2$, $|1-\lambda_0| = \|(1-\lambda_0)\zeta_2\| = \|(1_{H_1}\otimes v)\zeta_1\| = \|\zeta_1\| = 1$. □

Proof of Theorem 4.4. (i) ($\Leftarrow$): Suppose that there exist unitaries $u\in B(H_1)$ and $v\in B(H_2)$ such that $(u\otimes v)\zeta_1 = \zeta_2$ and let $U := u\otimes v\in B(H_1\otimes H_2)$. Then U is a unitary and $UP_{L_1} = P_{L_2}U$, since $U\zeta_1 = \zeta_2$ implies that $UL_1 = U\{\zeta_1\}^\perp = \{U\zeta_1\}^\perp = \{\zeta_2\}^\perp = L_2$ and $UP_{L_1}U^* = P_{UL_1} = P_{L_2}$. Hence, for all $x\in B(H_1)$,

$$UP_{L_1}(x\otimes 1_{H_2})P_{L_1}U^* = P_{L_2}U(x\otimes 1_{H_2})U^*P_{L_2} = P_{L_2}(uxu^*\otimes 1_{H_2})P_{L_2},$$

and

$$UM_n^{q,\,\zeta_1}U^* = UP_{L_1}(B(H_1)\otimes 1_{H_2})P_{L_1}U^* = P_{L_2}(B(H_1)\otimes 1_{H_2})P_{L_2} = M_n^{q,\,\zeta_2}.$$

So the map $P_{L_1}(x\otimes 1_{H_2})P_{L_1}\mapsto P_{L_2}(uxu^*\otimes 1_{H_2})P_{L_2}$, $x\in B(H_1)$, is a unital complete isometry of $M_n^{q,\,\zeta_1}$ onto $M_n^{q,\,\zeta_2}$.

($\Rightarrow$): If there exists a surjective unital complete isometry $\kappa : M_n^{q,\,\zeta_1}\to M_n^{q,\,\zeta_2}$, then κ extends to a surjective unital complete isometry $\hat{\kappa} : P_{L_1}B(H_1\otimes H_2)P_{L_1} = B(L_1)\to P_{L_2}B(H_1\otimes H_2)P_{L_2} = B(L_2)$, since $C_e^*(M_n^{q,\,\zeta_i}) = P_{L_i}B(H_1\otimes H_2)P_{L_i}$ by Theorem 4.2 and the C^*-envelopes are unique. Then there exists a surjective linear isometry $U_0 : L_1\to L_2$ such that $\hat{\kappa}(x) = U_0xU_0^*$ for all $x\in P_{L_1}B(H_1\otimes H_2)P_{L_1}$. Since $H = L_i\oplus L_i^\perp = L_i\oplus\mathbb{C}\zeta_i$

($i = 1, 2$), we obtain a unitary $U \in B(H_1 \otimes H_2)$ such that $U|L_1 = U_0$ and $U\zeta_1 = \zeta_2$. Then, since $\hat{\kappa}(M_n^{q,\zeta_1}) = \kappa(M_n^{q,\zeta_1}) = M_n^{q,\zeta_2}$ and $U_0 = U|L_1$, it follows that

$$UP_{L_1}(B(H_1) \otimes 1_{H_2})P_{L_1}U^* = P_{L_2}(B(H_1) \otimes 1_{H_2})P_{L_2}.$$

Now Lemma 4.5 together with $U\zeta_1 = \zeta_2$ shows that there exists a unitary $u \in B(H_1)$ such that

$$UP_{L_1}(x \otimes 1_{H_2})P_{L_1}U^* = P_{L_2}(uxu^* \otimes 1_{H_2})P_{L_2}, \ \ \forall x \in B(H_1).$$

If we set $U_1 := (u^* \otimes 1_{H_2})U$, then $P_{L_2}(u \otimes 1_{H_2}) = UP_{L_1}U^*(u \otimes 1_{H_2}) = UP_{L_1}U_1^*$, since $U\zeta_1 = \zeta_2$ implies that $P_{L_2} = UP_{L_1}U^*$ as seen above. Substituting this for the above equality we have the following:

$$P_{L_1}(x \otimes 1_{H_2})P_{L_1} = P_{L_1}U_1^*(x \otimes 1_{H_2})U_1P_{L_1}, \ \ \forall x \in B(H_1).$$

Since $\zeta_2 = U\zeta_1 \in Z_{n,q}$, we have, in view of (4.1), $\zeta_3 := U_1\zeta_1 = (u^* \otimes 1_{H_2})U\zeta_1 = (u^* \otimes 1_{H_2})\zeta_2 \in Z_{n,q}$. Hence Lemma 4.6 applies, and it follows that there exist a unitary $v \in B(H_2)$ and $\lambda_0 \in \mathbb{C}$ such that

$$U_1 = 1_{H_1} \otimes v + \lambda_0\zeta_3\zeta_1^*, \quad |1 - \lambda_0| = 1.$$

Thus

$$U = (u \otimes 1_{H_2})U_1 = u \otimes v + \lambda_0(u \otimes 1_{H_2})\zeta_3\zeta_1^* = u \otimes v + \lambda_0\zeta_2\zeta_1^*.$$

Since $U\zeta_1 = \zeta_2$ and $|1 - \lambda_0| = 1$, we have $(u \otimes v)\zeta_1 = U\zeta_1 - \lambda_0\zeta_2\zeta_1^*\zeta_1 = (1 - \lambda_0)\zeta_2$, $u_1 := (1 - \lambda_0)^{-1}u \in B(H_1)$ is a unitary, and $(u_1 \otimes v)\zeta_1 = \zeta_2$. Moreover, $UP_{L_1} = (u \otimes v)P_{L_1}$, since $\zeta_2\zeta_1^*P_{L_1} = \zeta_2\zeta_1^*(1_{\widetilde{H}} - \zeta_1\zeta_1^*) = 0$; $(u_1 \otimes v)P_{L_1} = P_{L_2}(u_1 \otimes v)$, since $(u_1 \otimes v)\zeta_1 = \zeta_2$; and for all $x \in B(H_1)$,

$$\begin{aligned}\kappa(P_{L_1}(x \otimes 1_{H_1})P_{L_1}) &= \hat{\kappa}(P_{L_1}(x \otimes 1_{H_1})P_{L_1}) = UP_{L_1}(x \otimes 1_{H_1})P_{L_1}U^* \\ &= (u \otimes v)P_{L_1}(x \otimes 1_{H_1})P_{L_1}(u \otimes v)^* \\ &= (u_1 \otimes v)P_{L_1}(x \otimes 1_{H_1})P_{L_1}(u_1 \otimes v)^* \\ &= P_{L_2}(u_1 \otimes v)(x \otimes 1_{H_1})(u_1^* \otimes v^*)P_{L_2} \\ &= P_{L_2}(u_1xu_1^* \otimes 1_{H_2})P_{L_2}.\end{aligned}$$

(ii) This is obvious from the above argument in (i). □

To state the following theorem we need some notation and a lemma. Write

$$\mathcal{M}_{n,q} := \{M_n^{q,\zeta} : \ \zeta \in Z_{n,q}\};$$

define an equivalence relation $\sim$ on $\mathcal{M}_{n,q}$ by writing $M_n^{q,\zeta_1} \sim M_n^{q,\zeta_2}$ if and only if $M_n^{q,\zeta_1} \cong M_n^{q,\zeta_2}$; and denote by $\mathcal{M}_{n,q}/\!\sim$ the set of all equivalence classes. Consider the following set:

$$\Lambda_q := \{\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^q : \lambda_1 \geq \cdots \geq \lambda_q > 0, \ \sum_{i=1}^{q} \lambda_i^2 = 1\}. \tag{4.16}$$

Since $q = \dim H_2 \leq \dim H_1 = n$, we may assume $H_2 \subset H_1$, and we identify $B(H_2) = P_{H_2} B(H_1) P_{H_2} \subset B(H_1, H_2) = P_{H_2} B(H_1) \subset B(H_1)$. Take a fixed orthonormal basis $\{\varepsilon_i^0\}_{1 \leq i \leq n}$ for H_1 so that $H_2 = \sum_{i=1}^{q} \mathbb{C}\varepsilon_i^0$ and $\{\varepsilon_i^0\}_{1 \leq i \leq q}$ is an orthonormal basis for H_2. For each $\lambda = (\lambda_i) \in \Lambda_q$ write

$$\zeta_\lambda := \sum_{i=1}^{q} \lambda_i \varepsilon_i^0 \otimes \varepsilon_i^0 \in Z_{n,\,q}, \quad L_\lambda := \{\zeta_\lambda\}^\perp \subset H_1 \otimes H_2,$$
$$M_n^{q,\,\lambda} := M_n^{q,\,\zeta_\lambda} = P_{L_\lambda}(B(H_1) \otimes 1_{H_2}) P_{L_\lambda} \subset P_{L_\lambda} B(H_1 \otimes H_2) P_{L_\lambda}.$$

Hence we obtain the following subsets of $Z_{n,\,q}$ and $\mathcal{M}_{n,\,q}$ parametrized by Λ_q:

$$Z_{n,\,q}^0 := \{\zeta_\lambda : \ \lambda \in \Lambda_q\},$$
$$\mathcal{M}_{n,\,q}^0 := \{M_n^{q,\,\lambda} : \ \lambda \in \Lambda_q\}.$$

Denote by $\mathcal{U}_1 = U(H_1)$, $\mathcal{U}_2 = U(H_2)$ the unitary groups of $B(H_1)$, $B(H_2)$, respectively, and define an action of the product group $\mathcal{U}_1 \times \mathcal{U}_2$ on $H_1 \otimes H_2$ by

$$(u,\, v)\zeta := (u \otimes v)\zeta, \ \ (u,\, v) \in \mathcal{U}_1 \times \mathcal{U}_2, \ \zeta \in H_1 \otimes H_2.$$

LEMMA 4.7. (i) *Each ζ in $H_1 \otimes H_2$ is written in the form*

$$\zeta = \sum_{i=1}^{q} \lambda_i \varepsilon_i' \otimes \varepsilon_i, \tag{4.17}$$

where $\lambda_i \in \mathbb{R}$ $(1 \leq i \leq q)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q \geq 0$, and $\{\varepsilon_i'\}_{1 \leq i \leq q} \subset H_1$ and $\{\varepsilon_i\}_{1 \leq i \leq q} \subset H_2$ are orthonormal.

(ii) *The vector ζ in* (i) *has another expression $\zeta = \sum_{i=1}^{q} \mu_i \delta_i' \otimes \delta_i$ for $\{\mu_i\}$, $\{\delta_i'\}$ and $\{\delta_i\}$ as above if and only if $\lambda_i = \mu_i$ $(1 \leq i \leq q)$ and there exist unitary matrices $[\alpha_{ij}^{(k)}]_{i,\,j \in I_k}$ $(1 \leq k \leq s)$ such that*

$$\delta_i' = \sum_{j \in I_k} \overline{\alpha_{ij}^{(k)}} \varepsilon_j', \quad \delta_i = \sum_{j \in I_k} \alpha_{ij}^{(k)} \varepsilon_j \ \ (i \in I_k, \ 1 \leq k \leq s), \tag{4.18}$$

where I_k $(1 \leq k \leq s)$ are the partition of $\{1,\, 2,\, \ldots,\, q'\}$ that we define by taking $q' \leq q$ as the largest i with $\lambda_i > 0$ and by setting $\{\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_{q'}\} = \{\lambda_1',\, \ldots,\, \lambda_s'\}$ $(\lambda_1' > \cdots > \lambda_s' > 0)$ and $I_k = \{i \in \{1,\, 2,\, \ldots,\, q'\} : \lambda_i = \lambda_k'\}$ $(1 \leq k \leq s)$.

PROOF. (i) For the linear isomorphism $\rho : H_1 \otimes H_2 \to B(\overline{H_1},\, H_2)$ defined in Section 2 consider the polar decomposition $\rho_\zeta^* = u_0|\rho_\zeta^*|$ of $\rho_\zeta^* \in B(H_2,\, \overline{H_1})$, where $|\rho_\zeta^*| \in B(H_2)$ and $u_0 \in B(H_2,\, \overline{H_1})$ is the unique partial isometry such that $u_0^* u_0 H_2 = |\rho_\zeta^*| H_2$. The spectral decomposition of $|\rho_\zeta^*|$ is of the form $|\rho_\zeta^*| = \sum_{i=1}^{q} \lambda_i \varepsilon_i \varepsilon_i^*$, where $\lambda_1 \geq \cdots \geq \lambda_q \geq 0$ and $\{\varepsilon_i\}_{1 \leq i \leq q}$ is an orthonormal basis for H_2. Let $q' \leq q$ be such that $\lambda_{q'} > 0$

and $\lambda_i = 0$ for $i > q'$. Then $u_0^* u_0 H_2 = \sum_{i=1}^{q'} \mathbb{C}\varepsilon_i$, $\{u_0\varepsilon_i\}_{1\le i\le q'}$ is an orthonormal set in $\overline{H_1}$, and we may take an orthonormal set $\{\varepsilon_i'\}_{1\le i\le q}$ in H_1 so that $u_0\varepsilon_i = (\varepsilon_i')^*$ $(1 \le i \le q')$, $= 0$ $(i > q')$. It follows that $\zeta = \sum_{i=1}^q \lambda_i \varepsilon_i' \otimes \varepsilon_i$. Indeed, let $\zeta' = \sum_{i=1}^q \lambda_i \varepsilon_i' \otimes \varepsilon_i$. Then $\rho_\zeta^* \varepsilon_j = u_0|\rho_\zeta^*|\varepsilon_j = u_0(\lambda_j\varepsilon_j) = \lambda_j(\varepsilon_j')^*$ $(1 \le j \le q)$; by (2.4), $\rho_{\zeta'}^*\varepsilon_j = (\sum_{i=1}^q \lambda_i(\varepsilon_i')^*\varepsilon_i{}^*)\varepsilon_j = \lambda_j(\varepsilon_j')^*$ $(1 \le j \le q)$; and since ρ is injective, $\zeta = \zeta'$.

(ii) For simplicity we assume that $\lambda_q > 0$ and hence that $q' = q$. The case $\lambda_q = 0$ is treated similarly.

($\Rightarrow$): Suppose $\zeta = \sum_{i=1}^q \lambda_i\varepsilon_i' \otimes \varepsilon_i = \sum_{i=1}^q \mu_i \delta_i' \otimes \delta_i$. The argument in (i) shows that $\sum_{i=1}^q \lambda_i\varepsilon_i' \otimes \varepsilon_i = \sum_{i=1}^q \mu_i\delta_i' \otimes \delta_i \iff$ (a) $|\rho_\zeta^*| = \sum_{i=1}^q \lambda_i \varepsilon_i\varepsilon_i{}^* = \sum_{i=1}^q \mu_i\delta_i\delta_i{}^*$ (by (2.4)) and (b) $u_0\varepsilon_i = \varepsilon_i'{}^*$, $u_0\delta_i = \delta_i'{}^*$ $(1 \le i \le q)$. Then (a) holds $\iff \lambda_i = \mu_i$ $(1 \le i \le q)$ and $\sum_{i\in I_k} \varepsilon_i' \otimes \varepsilon_i = \sum_{i\in I_k} \delta_i' \otimes \delta_i$ $(1 \le k \le s)$. The latter condition implies that $\delta_i = \sum_{j\in I_k} \alpha_{ij}^{(k)}\varepsilon_j$ for some $\alpha_{ij}^{(k)} \in \mathbb{C}$ $(i \in I_k,\ 1 \le k \le s)$. By (b), $\delta_i'{}^* = u_0\delta_i = \sum_{j\in I_k} \alpha_{ij}^{(k)} u_0\varepsilon_j = \sum_{j\in I_k} \alpha_{ij}^{(k)} \varepsilon_j'{}^* = (\sum_{j\in I_k} \overline{\alpha_{ij}^{(k)}}\varepsilon_j')^*$, and $\delta_i' = \sum_{j\in I_k} \overline{\alpha_{ij}^{(k)}}\varepsilon_j'$ $(i \in I_k,\ 1 \le k \le s)$. Finally, since $\{\delta_i\}_{i\in I_k}$ and $\{\varepsilon_i\}_{i\in I_k}$ are both orthonormal, the matrices $[\alpha_{ij}^{(k)}]_{i,\,j\in I_k}$ are unitary.

The implication ($\Leftarrow$) follows from a direct computation. □

Theorem 4.8. *We have* $\mathcal{M}_{n,\,q}^0 = \{M_n^{q,\,\lambda} : \lambda \in \Lambda_q\} \subset \mathcal{M}_{n,\,q} = \{M_n^{q,\,\zeta} : \zeta \in Z_{n,\,q}\}$; *for each* $\zeta \in Z_{n,\,q}$ *there exists a unique* $\lambda \in \Lambda_q$ *so that* $M_n^{q,\,\zeta} \cong M_n^{q,\,\lambda}$; *and if* $\lambda_1, \lambda_2 \in \Lambda_q$ *and* $\lambda_1 \ne \lambda_2$, *then* $M_n^{q,\,\lambda_1} \not\cong M_n^{q,\,\lambda_2}$. *Hence we can identify the set* $\mathcal{M}_{n,\,q}/\sim$ *of all equivalence classes with* Λ_q.

Proof. In view of (4.1), the set $Z_{n,\,q}$ is stable under the action of $\mathcal{U}_1 \times \mathcal{U}_2$ defined above, and so we can consider the set $Z_{n,\,q}/\sim$ consisting of all orbits $[\zeta] := \{(u, v)\zeta : (u, v) \in \mathcal{U}_1 \times \mathcal{U}_2\}$ of elements ζ of $Z_{n,\,q}$. Then Theorem 4.4(ii) shows that $M_n^{q,\,\zeta_1} \cong M_n^{q,\,\zeta_2}$ if and only if $[\zeta_1] = [\zeta_2]$ and hence that the map $\mathcal{M}_{n,\,q} \to Z_{n,\,q}/\sim$, $M_n^{q,\,\zeta} \mapsto [\zeta]$, induces a bijection between $\mathcal{M}_{n,\,q}/\sim$ and $Z_{n,\,q}/\sim$.

Now we define a map $\sigma : Z_{n,\,q}/\sim \to \Lambda_q$ by using (4.17) in Lemma 4.7. Let $\zeta \in Z_{n,\,q}$. Then $\lambda := (\lambda_1, \dots, \lambda_q) \in \Lambda_q$ for $\lambda_1 \ge \cdots \ge \lambda_q \ge 0$ in (4.17), since $\operatorname{rank}|\rho_\zeta^*| = \operatorname{rank}\rho_\zeta^* = \operatorname{rank}\rho_\zeta = q$, so $\lambda_q > 0$, and $\|\zeta\| = 1$. Then define $\sigma([\zeta]) := \lambda$. That σ is a well-defined bijection is almost obvious. Indeed, for $\zeta, \zeta' \in Z_{n,q}$, $\zeta = \sum_{i=1}^q \lambda_i\varepsilon_i' \otimes \varepsilon_i$ and $\zeta' = \sum_{i=1}^q \lambda_i\delta_i' \otimes \delta_i$ for some $\lambda = (\lambda_i) \in \Lambda_q$ and orthonormal $\{\varepsilon_i'\}, \{\delta_i'\} \subset H_1$ and $\{\varepsilon_i\}, \{\delta_i\} \subset H_2$ if and only if there exists $(u, v) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that $\zeta' = (u \otimes v)\zeta$, i.e., $[\zeta] = [\zeta']$. This shows that σ is a well-defined injection. Further, $\sigma([\zeta_\lambda]) = \lambda$ for each $\lambda \in \Lambda_q$, and σ is a surjection. □

Let X be an operator system. We call a unital complete isometry of X onto itself an *automorphim* of X, and denote by $\operatorname{Aut} X$ the group of all automorphisms of X. We determine the automorphism group $\operatorname{Aut} M_n^{q,\,\lambda}$ of the operator system $M_n^{q,\,\lambda}$. It turns out that $\operatorname{Aut} M_n^{q,\,\lambda}$ is rather different from $\operatorname{Aut} M_n$, which is isomorphic to the quotient group $U(n)/\mathbb{T}1_n$, where $U(n) := \{u \in M_n : u^*u = uu^* = 1_n\}$ is the unitary group of M_n and $\mathbb{T} := \{\mu \in \mathbb{C} : |\mu| = 1\}$.

In order to describe $\operatorname{Aut} M_n^{q,\,\lambda}$ we introduce some notation. For $\lambda = (\lambda_1, \dots, \lambda_q) \in \Lambda_q$ define a subgroup U_λ of $U(n)$ as follows. As in the statement of Lemma 4.7 (ii), let $\{\lambda_1, \ldots, \lambda_q\} = \{\lambda'_1, \ldots, \lambda'_s\}$ $(\lambda'_1 > \cdots > \lambda'_s)$ and $I_k = \{i \in \{1, \ldots, q\} : \lambda_i = \lambda'_k\}$ $(1 \leq k \leq s)$. Further, let $I_0 = \{q+1, \ldots, n\}(= \emptyset$ if $n = q)$,

$$K_1 := \sum_{i \in I_1} \mathbb{C}\varepsilon_i^0, \ \ldots, \ K_s := \sum_{i \in I_s} \mathbb{C}\varepsilon_i^0, \quad K_0 := \sum_{i \in I_0} \mathbb{C}\varepsilon_i^0,$$

so that $K_1 \oplus \cdots \oplus K_s = \sum_{i=1}^q \mathbb{C}\varepsilon_i^0 = H_2 \subset K_1 \oplus \cdots \oplus K_s \oplus K_0 = \sum_{i=1}^n \mathbb{C}\varepsilon_i^0 = H_1$. Define a subgroup U_λ of $U(n) = U(B(H_1))$ by

$$U_\lambda := U(K_1) \oplus \cdots \oplus U(K_s) \oplus U(K_0),$$

where $U(K_k) := U(B(K_k))$ is the unitary group of $B(K_k)$ $(0 \leq k \leq s)$ and when $n = q$ we regard the last summand $U(K_0)$ as missing.

Proposition 4.9. *For $\lambda \in \Lambda_q$ and U_λ as above, every automorphism of $M_n^{q,\lambda} = P_{L_\lambda}(B(H_1) \otimes 1_{H_2})P_{L_\lambda}$ is of the form $P_{L_\lambda}(x \otimes 1_{H_2})P_{L_\lambda} \mapsto P_{L_\lambda}(uxu^* \otimes 1_{H_2})P_{L_\lambda}$, $x \in B(H_1)$, for some $u \in U_\lambda$; two such automorphisms corresponding to u, $u' \in U_\lambda$ coincide if and only if $u^*u' \in \mathbb{T}1_n$; and the automorphism group* Aut $M_n^{q,\lambda}$ *of $M_n^{q,\lambda}$ is isomorphic to $U_\lambda/\mathbb{T}1_n$.*

Proof. By Theorem 4.4 (i) an automorphism of $M_n^{q,\lambda}$ is characterized as the map

$$P_{L_\lambda}(x \otimes 1_{H_2})P_{L_\lambda} \mapsto P_{L_\lambda}(uxu^* \otimes 1_{H_2})P_{L_\lambda}, \ \ x \in B(H_1),$$

for some $u \in U(H_1)$ for which $(*)$ there exists $v \in U(H_2)$ such that $(u \otimes v)\zeta_\lambda = \zeta_\lambda$. Since $\varphi_{\zeta_\lambda} : B(H_1) \to P_{L_\lambda}(B(H_1) \otimes 1_{H_2})P_{L_\lambda}$, $x \mapsto P_{L_\lambda}(x \otimes 1_{H_2})P_{L_\lambda}$, is a linear isometry, for u, $u' \in U(H_1)$ we have $P_{L_\lambda}(uxu^* \otimes 1_{H_2})P_{L_\lambda} = P_{L_\lambda}(u'xu'^* \otimes 1_{H_2})P_{L_\lambda}$ for all $x \in B(H_1)$ if and only if $uxu^* = u'xu'^*$ for all $x \in B(H_1)$, i.e., $u^*u' \in \mathbb{T}1_n$.

Hence it remains only to show that for $u \in U(H_1)$ we have $(*)$ if and only if $u \in U_\lambda$. In the notation λ'_k, I_k, etc. as above we have $\zeta_\lambda = \sum_{i=1}^q \lambda_i(\varepsilon_i^0 \otimes \varepsilon_i^0) = \sum_{k=1}^s \lambda'_k \sum_{i \in I_k}(\varepsilon_i^0 \otimes \varepsilon_i^0)$ and $(u \otimes v)\zeta_\lambda = \sum_{k=1}^s \lambda'_k \sum_{i \in I_k}(u\varepsilon_i^0 \otimes v\varepsilon_i^0)$. If $(u \otimes v)\zeta_\lambda = \zeta_\lambda$, then, by Lemma 4.7 (ii), $u\varepsilon_i^0 = \sum_{j \in I_k} \overline{\alpha_{ij}^{(k)}}\varepsilon_j^0$, $v\varepsilon_i^0 = \sum_{j \in I_k} \alpha_{ij}^{(k)}\varepsilon_j^0$ $(i \in I_k, 1 \leq k \leq s)$ for some unitary matrices $[\alpha_{ij}^{(k)}]_{i,j \in I_k}$ $(1 \leq k \leq s)$. Hence $uK_k = K_k$ $(1 \leq k \leq s)$, so $uK_0 = K_0$, too, and $u \in U_\lambda$. Conversely, let $u \in U_\lambda$ and so $u = u_1 \oplus \cdots \oplus u_s \oplus u_0$ for $u_k \in U(K_k)$ $(k = 1, \ldots, s, 0)$. Define unitary matrices $[\beta_{ij}^{(k)}]_{i,j \in I_k}$ $(1 \leq k \leq s)$ by $u_k\varepsilon_i^0 = \sum_{j \in I_k} \beta_{ij}^{(k)}\varepsilon_j^0$ $(i \in I_k, 1 \leq k \leq s)$. Then $[\overline{\beta_{ij}^{(k)}}]_{i,j \in I_k}$ $(1 \leq k \leq s)$ are also unitary, and a unitary $v \in U(H_2)$ is defined by $v = v_1 \oplus \cdots \oplus v_s$, where $v_k \in U(K_k)$ and $v_k\varepsilon_i^0 = \sum_{j \in I_k} \overline{\beta_{ij}^{(k)}}\varepsilon_j^0$ $(i \in I_k, 1 \leq k \leq s)$. It follows again from Lemma 4.7 (ii) that $(u \otimes v)\zeta_\lambda = \zeta_\lambda$. □

5. Two questions.

Theorem 3.1 describes the isometric degree $\mathrm{id}(\varphi_L)$ of φ_L in terms of $[[L^\perp]] \subset H_2$ and $l :=$ length $L^\perp$. That is, $\mathrm{id}(\varphi_L) = \infty$ if and only if $[[L^\perp]] \subsetneqq H_2$, and if $\mathrm{id}(\varphi_L) < \infty$ and so $[[L^\perp]] = H_2$, then $\mathrm{id}(\varphi_L) = [(l-1)/2]$. But our satisfactory computation of length $L^\perp$ is essentially confined to the case $\dim L^\perp = 1$ (Lemma 3.6). So it would be interesting to answer the following:

QUESTION 1. Can we compute $\operatorname{length} M$ for any linear subspace M of $H_1 \otimes H_2$ effectively?

The following remark may be useful in treating the case $\dim M \geq 2$. If we set $N := \rho_M = \{\rho_\zeta : \zeta \in M\} \subset B(\overline{H_1}, H_2)$, then

$$\operatorname{length} M = \min\{\dim T : T \subset \overline{H_1} \text{ linear}, \operatorname{lin} NT = \operatorname{lin} N\overline{H_1}\},$$

and by the proof of Lemma 3.5 (iii) we have the estimate:

$$\operatorname{length} M \leq \min\{\max_{1\leq i\leq k} \operatorname{rank} a_i : a_1, \ldots, a_k \in N, \operatorname{lin}\{a_1, \ldots, a_k\} = N, k = 1, 2, \ldots\}.$$

Indeed, if $N = \operatorname{lin}\{a_1, \ldots, a_k\}$ for some finite $\{a_1, \ldots, a_k\} \subset N$, then, by Lemma 3.5 (ii) there exists a linear subspace T_0 of $\overline{H_1}$ with $\dim T_0 = \max_{1\leq i\leq k} \operatorname{rank} a_i =: r$ such that $a_i T_0 = a_i \overline{H_1}$ for all i. Hence $\operatorname{lin} NT_0 = a_1 T_0 + \cdots + a_k T_0 = a_1\overline{H_1} + \cdots + a_k\overline{H_1} = \operatorname{lin} N\overline{H_1}$, and $(*)$ $\operatorname{length} M \leq r$. By varying the a_i's the inequality follows.

Equality in $(*)$ holds provided that the a_i's $(1 \leq i \leq k)$ satisfy further the condition that the sum $a_1\overline{H_1} + \cdots + a_k\overline{H_1}$ is a direct sum. For, we have $\operatorname{rank} a_{i_0} = r$ for some i_0, and $\dim a_{i_0}\overline{H_1} = r$. If T is a linear subspace of $\overline{H_1}$ with $\dim T \leq r - 1$, then $\dim a_{i_0} T \leq \dim T \leq r - 1$, and $a_{i_0} T \subsetneqq a_{i_0}\overline{H_1}$. By the assumption on the a_i's it follows that $\operatorname{lin} NT = a_1 T + \cdots + a_k T \subsetneqq a_1\overline{H_1} + \cdots + a_k\overline{H_1} = \operatorname{lin} N\overline{H_1}$. Thus this and the argument in the preceding paragraph show that $\operatorname{length} M = r$.

QUESTION 2. Given positive integers n, m with $n \geq 3$ and $1 \leq m \leq [(n-1)/2]$, what is the least number p for which there exists $\varphi_L : M_n \to M_p$ with $\operatorname{id}(\varphi_L) = m$?

Theorem 3.2 shows that such a least number, p_0, exists and $p_0 \leq n(2m+1) - 1$. Note also that if we can find one $\varphi_{L_0} : M_n \to M_{p_0}$ with $\operatorname{id}(\varphi_{L_0}) = m$, then, for each $p > p_0$ there exists $\varphi_L : M_n \to M_p$ such that $\operatorname{id}(\varphi_L) = m$. Indeed, take Hilbert spaces K_1, K_2 so that $\dim K_1 = p_0$, $\dim K_2 =: q < \infty$, and $p_0 < p \leq p_0 q$. Then there is a linear subspace L of $K_1 \otimes K_2$ so that $\dim L = p$ and $K_1 \otimes \eta_0 \subset L \subset K_1 \otimes K_2$ for some unit vector $\eta_0 \in K_2$. By Theorem 3.1(ii), the map $\kappa : M_{p_0} = B(K_1) \to B(K_1) \otimes B(K_2) = B(K_1 \otimes K_2) \to P_L B(K_1 \otimes K_2) P_L = B(L) = M_p$, $x \mapsto x \otimes 1_{K_2} \mapsto P_L(x \otimes 1_{K_2})P_L$, is a unital complete isometry. So it follows that $\kappa \circ \varphi_{L_0} : M_n \to M_{p_0} \to M_p$ is a unital completely positive map with $\operatorname{id}(\kappa \circ \varphi_{L_0}) = \operatorname{id}(\varphi_{L_0}) = m$.

The map $\varphi_L : M_n \to M_p$ is determined by Hilbert spaces H_1, H_2 and a linear subspace L of $H_1 \otimes H_2$ such that $\dim H_1 = n$ and $\dim L = p$. As noted above, $\operatorname{id}(\varphi_L) < \infty$ if and only if $[[L^\perp]] = H_2$, and in this case, $\operatorname{id}(\varphi_L) = [(l-1)/2]$ with $l = \operatorname{length} L^\perp$. Hence Question 2 is equivalent to the problem of minimizing $\dim L$ when we vary H_2 and $L \subset H_1 \otimes H_2$ under the following condition:

$$m = \left[\frac{l-1}{2}\right], \quad [[L^\perp]] = H_2, \quad \text{and} \quad l = \operatorname{length} L^\perp. \qquad (**)$$

In the proof of Theorem 3.2 we obtained the value $n(2m+1) - 1$ for $p = \dim L$

by taking $M = \mathbb{C}\zeta_0$ in Lemma 3.6(ii) as $L^{\perp}$. But, even if we take M in Lemma 3.6(i) as $L^{\perp}$, we cannot reduce this number $n(2m+1)-1$. Indeed, in the notation there, we have $1 \leq s \leq \min\{n, q\}$, length $M = s$, $[[M]] = H_2$, and $\dim M = s(q-s)+1$. If $(**)$ holds for $L^{\perp} = M$, then $m = [(s-1)/2]$ implies $s = 2m+1$ or $2m+2$, and $\dim L = \dim(H_1 \otimes H_2) - \dim M = nq - (s(q-s)+1) = (n-s)q + s^2 - 1$. Since $n - s \geq 0$ and $s \leq q$, the minimum value of $\dim L$ when q varies is $(n-s)s + s^2 - 1 = ns - 1 \geq n(2m+1) - 1$.

References

[1] M.-D. Choi, Positive linear maps on C^*-algebras, Canad. J. Math., **24** (1972), 520–529.

[2] M. Hamana, Injective envelopes of operator systems, Publ. Res. Inst. Math. Sci., Kyoto Univ., **15** (1979), 773–785.

[3] M. Hamana, Triple envelopes and Šilov boundaries of operator spaces, Math. J. Toyama Univ., **22** (1999), 77–93.

[4] R. V. Kadison, Isometries of operator algebras, Ann. Math., **54** (1951), 325–338.

[5] V. Paulsen, Completely bounded maps and operator algebras, Cambridge Stud. Adv. Math. **78**, Cambridge Univ. Press, 2002.

[6] J. Tomiyama, On the transpose map of matrix algebras, Proc. Amer. Math. Soc., **88** (1983), 635–638.

Masamichi Hamana
Ooizumi 1551-1
Toyama, 939-8058, Japan
E-mail: m.hamana@amber.plala.or.jp