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Vanishing theorems of L^2-cohomology groups on Hessian manifolds

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Abstract. We show vanishing theorems of L^2-cohomology groups of Kodaira–Nakano type on complete Hessian manifolds by introducing a new operator ∂'_F. We obtain further vanishing theorems of L^2-cohomology groups $L^2 H^{p,q}_{\bar\partial}(\Omega)$ on a regular convex cone Ω with the Cheng–Yau metric for $p > q$.

Introduction.

A *flat manifold* (M, D) is a manifold M with a flat affine connection D, where an affine connection is said to be *flat* if the torsion and the curvature vanish identically. A flat affine connection D gives an *affine local coordinate system* $\{x^1, \ldots, x^n\}$ satisfying

$$D_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = 0.$$

A Riemannian metric g on a flat manifold (M, D) is said to be a *Hessian metric* if g can be locally expressed in the Hessian form with respect to an affine coordinate system $\{x^1, \ldots, x^n\}$ and a *potential function* φ, that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

The triplet (M, D, g) is called a *Hessian manifold*. The Hessian structure (D, g) induces a holomorphic coordinate system $\{z^1, \ldots, z^n\}$ and a Kähler metric g^T on TM such that

$$z^i = x^i + \sqrt{-1} y^i,$$
$$g^T_{i\bar j}(z) = g_{ij}(x),$$

where $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ is a local coordinate system on TM induced by the affine coordinate system $\{x^1, \ldots, x^n\}$ and fiber coordinates $\{y^1, \ldots, y^n\}$. In this sense, Hessian geometry is a real analogue of Kähler geometry.

A (p, q)-form on a flat manifold (M, D) is a smooth section of $\wedge^{p,q} := \wedge^p T^*M \otimes \wedge^q T^*M$. On the space of (p, q)-forms, a flat connection D induces the Dolbeault-like operator $\bar\partial = \sum_i e(\overline{dx^i}) D_{\partial/\partial x^i}$, where $\overline{dx^i} = 1 \otimes dx^i$ and $e(\overline{dx^i}) = \overline{dx^i}\wedge$. For a flat line bundle (F, D^F) over M, the operator $\bar\partial$ can be extended on the space of F-valued

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(p,q)-forms and satisfies $\bar{\partial}^2 = 0$. Then the cohomology group $H^{p,q}_{\bar{\partial}}(M,F)$ is defined with respect to $\bar{\partial}$. On compact Hessian manifolds, Shima proved an analogue of the Kodaira–Nakano vanishing theorem for $H^{p,q}_{\bar{\partial}}(M,F)$ by using the theory of harmonic integrals when there exist a fiber metric h on F and a Riemannian metric g on M such that the *second Koszul forms* $B = -Dd\log h(s,s)$ and $\beta = (1/2)Dd\log\det[g_{ij}]$ satisfy $B+\beta > 0$, where s is a local frame field on F such that $D^F s = 0$.

Theorem 2.2.7 ([**2**]). *Let (M,D) be an oriented n-dimensional compact flat manifold and (F,D^F) be a flat line bundle over M. Assume there exist a fiber metric h on F and a Riemannian metric g on M such that $B+\beta$ is positive definite, where B and β are the second Koszul forms with respect to h and g, respectively. Then we have*

$$H^{p,q}_{\bar{\partial}}(M,F) = 0, \quad \textit{for } p+q>n.$$

However, many important examples of Hessian manifolds such as regular convex domains (cf. [**3**, Theorem 1.2.4]) are noncompact. In Section 3.2, we prove the following theorem which corresponds to Theorem 2.2.7 in the case of complete Hessian manifolds.

Main Theorem 2. *Let (M,D,g) be an oriented n-dimensional complete Hessian manifold and (F,D^F) a flat line bundle over M. We denote by h a fiber metric on F. Assume that there exists $\varepsilon>0$ such that $B+\beta = \varepsilon g$ where B and β are the second Koszul forms with respect to fiber metric h and Hessian metric g respectively. Then for $p+q>n$ and all $v \in L^2(M, F\otimes\wedge^{p,q})$ such that $\bar{\partial}v = 0$, there exists $u \in L^2(M, F\otimes\wedge^{p,q-1})$ such that*

$$\bar{\partial}u = v, \quad \|u\| \le \{\varepsilon(p+q-n)\}^{-1/2}\|v\|.$$

In particular, we have

$$L^2H^{p,q}_{\bar{\partial}}(M,F) = 0, \quad \textit{for } p+q>n.$$

The L^2-cohomology group $L^2H^{p,q}_{\bar{\partial}}(M,F)$ is often also written as $H^{p,q}_{(2)}(M,F)$.

Note that we cannot use the harmonic theory for the proof and we need the method of functional analysis as in the case of complete Kähler manifolds. To prove Main Theorem 2, we introduce a operator ∂'_F (cf. Definition 2.3.1) which is not defined in [**2**] and we obtain the following as an analogue of Kodaira–Nakano identity.

Theorem 2.3.8. *Let (D,g) be a Hessian structure. Then we have*

$$\bar{\Box}_F = \Box'_F + [e(\beta+B), \Lambda],$$

where $\bar{\Box}_F$ and $\Box'_F$ are the Laplacians with respect to $\bar{\partial}$ and ∂'_F, respectively, and Λ is the adjoint operator with respect to $e(g)$.

An open convex cone Ω in $\mathbb{R}^n$ is said to be *regular* if Ω contains no complete straight lines. We can apply Main Theorem 2 to regular convex cones with the *Cheng–Yau metric* (cf. [**3**, Theorem 1.2.4]). Further, we have stronger vanishing theorems as follows in

Section 3.3.

Main Theorem 3. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng–Yau metric. Then for $p > q \geq 1$ and all $v \in L^2(\Omega, \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(\Omega, \wedge^{p,q-1})$ such that*

$$\bar{\partial} u = v, \quad \|u\| \leq (p-q)^{-1/2}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\Omega, \wedge^{p,0})$ satisfies $\bar{\partial} v = 0$, then $v = 0$. In particular, we have

$$L^2 H^{p,q}_{\bar{\partial}}(\Omega) = 0, \quad \textit{for } p > q.$$

In the case of a regular convex cone $(\mathbb{R}^n, D, g = -Dd\log(x^1 \dots x^n))$, we have sharp vanishing theorem in Section 3.4.

Main Theorem 4. *For $p \geq 1$, $q \geq 1$ and $v \in L^2(\mathbb{R}^n_+, \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(\mathbb{R}^n_+, \wedge^{p,q-1})$ such that*

$$\bar{\partial} u = v, \quad \|u\| \leq p^{-1/2}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\mathbb{R}^n_+, \wedge^{p,0})$ satisfies $\bar{\partial} v = 0$, then $v = 0$. In particular, we have

$$L^2 H^{p,q}_{\bar{\partial}}(\mathbb{R}^n_+) = 0, \quad \textit{for } p \geq 1 \textit{ and } q \geq 0.$$

1. Hessian manifolds.

In this chapter we give a brief review of Hessian manifolds.

1.1. Hessian manifolds.

An affine connection D on a manifold M is said to be *flat* if the torsion tensor T^D and the curvature tensor R^D vanish identically. A manifold M endowed with a flat connection D is called a *flat manifold*, which is denoted by (M, D). On a flat manifold (M, D), there exists a local coordinate system $\{x^1, \dots, x^n\}$ such that $D_{\partial/\partial x^i}(\partial/\partial x^j) = 0$, which is called an *affine coordinate system* with respect to D. The changes between such a local coordinate system are affine transformations.

In this paper, every local coordinate system on flat manifolds is given as an affine coordinate system.

Definition 1.1.1. A Riemannian metric g on a flat manifold (M, D) is said to be a *Hessian metric* if g is locally expressed by

$$g = Dd\varphi,$$

that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

Then the pair (D,g) is called a *Hessian structure* on M, and φ is said to be a *potential* of (D,g). A manifold M with a Hessian structure (D,g) is called a *Hessian manifold*, which is denoted by (M,D,g).

Let (M,D) be a flat manifold and TM the tangent bundle over M. We denote by $\{x^1,\dots,x^n,y^1,\dots,y^n\}$ a local coordinate system on TM induced by an affine coordinate system $\{x^1,\dots,x^n\}$ on M and fiber coordinates $\{y^1,\dots,y^n\}$. Then a holomorphic coordinate system$\{z^1,\dots,z^n\}$ on TM is given by

$$z^i = x^i + \sqrt{-1}y^i.$$

For a Riemannian metric g on M we define a Hermitian metric g^T on TM by

$$g^T = \sum_{i,j} g_{ij} dz^i \otimes d\bar{z}^j.$$

It should be remarked that g^T is a Kähler metric if and only if g is a Hessian metric.

Example 1.1.2. (1) Let (D,g) be the pair consisting of the standard affine connection D and the Euclidean metric on $\mathbb{R}^n$. Then (D,g) is a Hessian structure. Indeed, if we set $\varphi(x) = (1/2)\sum_j (x^j)^2$, we have

$$\frac{\partial^2\varphi}{\partial x^i \partial x^j} = \delta_{ij} = g_{ij},$$

where δ_{ij} is the Kronecker delta, that is,

$$\delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j). \end{cases}$$

Moreover, the Kähler metric g^T on $T\mathbb{R}^n \simeq \mathbb{C}^n$ is also the Euclidean metric.

(2) We set $\mathbb{R}_+ = (0,\infty)$. Let D be the standard affine connection, that is, the restriction of the standard affine connection on $\mathbb{R}^n$ to $\mathbb{R}^n_+$. We define a Riemannian metric g on $\mathbb{R}^n_+$ by

$$g_{ij}(x) = \frac{\delta_{ij}}{(x^j)^2}.$$

Then (D,g) is a Hessian structure. Indeed, if we set $\varphi(x) = -\log(x^1 \dots x^n)$, we have

$$\frac{\partial^2\varphi}{\partial x^i \partial x^j} = g_{ij}.$$

When $n=1$, the Kähler metric g^T on $T\mathbb{R}_+ \simeq \mathbb{R}_+ \oplus \sqrt{-1}\,\mathbb{R}$ is the Poincaré metric.

Definition 1.1.3. Let M be a manifold and D a torsion-free affine connection on M. We denote by g a Riemannian metric on M, and by ∇ the Levi-Civita connection of g. We define the *difference tensor* γ of ∇ and D by

$$\gamma = \nabla - D.$$

We denote by $\mathscr{X}(M)$ the space of vector fields on M. Since ∇ and D are torsion-free, it follows that for $X, Y \in \mathscr{X}(M)$

$$\gamma_X Y = \gamma_Y X.$$

It should be remarked that the components $\gamma^i{}_{jk}$ of γ with respect to affine coordinate systems coincide with the Christoffel symbols of ∇.

Definition 1.1.4. Let M be a manifold and D a torsion-free affine connection on M. We denote by g a Riemannian metric on M. We define another affine connection D^* on M as follows:

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D^*_X Z), \quad X, Y, Z \in \mathscr{X}(M).$$

We call D^* the *dual connection* of D with respect to g.

Proposition 1.1.5 ([**1**]). *Let (M, D) be a flat manifold and g a Riemannian manifold on M. Then the following conditions are equivalent.*

(1) *(D, g) is a Hessian structure.*

(2) $(D_X g)(Y, Z) = (D_Y g)(X, Z), \quad X, Y, Z \in \mathscr{X}(M) \quad (\Leftrightarrow \partial g_{jk}/\partial x^i = \partial g_{ik}/\partial x^j)$.

(3) $g(\gamma_X Y, Z) = g(Y, \gamma_X Z), \quad X, Y, Z \in \mathscr{X}(M) \quad (\Leftrightarrow \gamma_{ijk} = \gamma_{jik})$.

(4) $(D_X g)(Y, Z) = 2g(\gamma_X Y, Z), \quad X, Y, Z \in \mathscr{X}(M) \quad (\Leftrightarrow \partial g_{ij}/\partial x^k = 2\gamma_{ijk})$.

(5) $D + D^* = 2\nabla$.

1.2. Koszul forms on flat manifolds.

We introduce Koszul forms which play important roles in Hessian geometry.

Definition 1.2.1. Let (M, D) be a flat manifold and g a Riemannian metric on M. We define a d-closed 1-form α and a symmetric bilinear form β by

$$\alpha = \frac{1}{2} d \log \det[g_{ij}], \quad \beta = D\alpha.$$

Remark that since the changes between affine coordinate systems are affine transformations, α and β are globally well-defined. We call α and β the *first Koszul form* and the *second Koszul form* for (D, g), respectively.

Proposition 1.2.2 ([**1**]). *Let (M, D, g) be a Hessian manifold. Then we have the following equations.*

$$\alpha_i := \alpha\left(\frac{\partial}{\partial x^i}\right) = \sum_r \gamma^r{}_{ri}, \quad \beta_{ij} := \beta\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_r \frac{\partial \gamma^r{}_{ri}}{\partial x^j}.$$

Definition 1.2.3. Let (M, D, g) be a Hessian manifold. If there exists $\lambda \in \mathbb{R}$ such that $\beta = \lambda g$, we call g a *Hesse–Einstein metric.*

It should be remarked that a Hessian metric g on M is a Hesse–Einstein metric if and only if the Kähler metric g^T on TM is a Kähler–Einstein metric ([**1**]).

A convex domain in $\mathbb{R}^n$ which contains no full straight lines is called a *regular convex domain*. By the following theorem, on a regular convex domain there exists a complete Hesse–Einstein metric g which satisfies $g = \beta$. It is called the *Cheng–Yau metric*.

Theorem 1.2.4 ([**3**]). *On a regular convex domain $\Omega \subset \mathbb{R}^n$, there exists a unique convex function φ such that*

$$\begin{cases} \det\left[\dfrac{\partial^2\varphi}{\partial x^i \partial x^j}\right] = e^{2\varphi} & \\ \varphi(x) \to \infty & (x \to \partial\Omega). \end{cases}$$

In addition, the Hessian metric $g = Dd\varphi$ is complete, where D is the standard affine connection on Ω.

Proposition 1.2.5. *The Cheng–Yau metric g defined by Theorem* 1.2.4 *is invariant under affine automorphisms of Ω, where an affine automorphism of Ω is restriction of an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ to Ω which satisfies $A\Omega = \Omega$.*

Proof. An affine transformation A is denoted by

$$Ax = ((Ax)^1, \dots, (Ax)^n), \quad (Ax)^i = \sum_j a^i_j x^j + b^i.$$

We define a function $\tilde{\varphi}$ on Ω by

$$\tilde{\varphi}(x) = \varphi(Ax) + \log|\det[a^i_j]|.$$

Then we have

$$\tilde{\varphi}(x) \to \infty \quad (x \to \infty).$$

Moreover we obtain

$$\frac{\partial^2\tilde{\varphi}}{\partial x^i \partial x^j}(x) = \sum_{k,l} a^k_i a^l_j \frac{\partial^2\varphi}{\partial x^k \partial x^l}(Ax).$$

Hence $\tilde{\varphi}$ is a convex function. Furthermore, it follows that

$$\det\left[\frac{\partial^2\tilde{\varphi}}{\partial x^i \partial x^j}(x)\right] = |\det[a^i_j]|^2 \det\left[\frac{\partial^2\varphi}{\partial x^i \partial x^j}(Ax)\right] = e^{2(\varphi(Ax) + \log|\det[a^i_j]|)} = e^{2\tilde{\varphi}(x)}.$$

Therefore $\tilde{\varphi}$ is also a convex function which satisfies the condition of Theorem 1.2.4. From the uniqueness of the solution we have $\tilde{\varphi} = \varphi$, that is,

$$\varphi(x) = \varphi(Ax) + \log|\det[a^i_j]|.$$

Hence we have

$$g_{ij}(x) = \sum_{k,l} a_i^k a_j^l g_{kl}(Ax).$$

This implies that g is invariant under affine automorphisms. □

EXAMPLE 1.2.6. Let $(\mathbb{R}_+^n, D, g = Dd\varphi)$ be the same as in Example 1.1.2 (2). Then $\varphi(x) = -\log(x^1 \dots x^n)$ satisfies the condition of Theorem 1.2.4.

2. (p, q)-forms on flat manifolds.

Hereafter, we assume that (M, D) is an oriented flat manifold and g is a Riemannian metric on M. In addition, let F be a real line bundle over M endowed with a flat connection D^F and a fiber metric h. Moreover, we denote by s a local frame field on F such that $D^F s = 0$.

2.1. (p, q)-forms and fundamental operators.

We denote by $A^{p,q}(M)$ the space of smooth sections of $\wedge^{p,q} := \wedge^p T^*M \otimes \wedge^q T^*M$. An element in $A^{p,q}(M)$ is called a (p, q)-form. For a p-form ω and a q-form η, $\omega \otimes \eta \in A^{p,q}(M)$ is denoted by $\omega \otimes \bar{\eta}$.

Using an affine coordinate system, a (p, q)-form ω is expressed by

$$\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dx^{I_p} \otimes \overline{dx^{J_q}},$$

where

$$I_p = (i_1, \dots, i_p), \quad 1 \le i_1 < \dots < i_p \le n, \quad J_q = (j_1, \dots, j_q), \quad 1 \le j_1 < \dots < j_q \le n,$$
$$dx^{I_p} = dx^{i_1} \wedge \dots \wedge dx^{i_p}, \quad dx^{J_q} = dx^{j_1} \wedge \dots \wedge dx^{j_q}.$$

EXAMPLE 2.1.1. A Riemannian metric g and the second Koszul form β (Definition 1.2.1) are regarded as $(1, 1)$-forms;

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes \overline{dx^j}, \quad \beta = \sum_{i,j} \beta_{ij}\, dx^i \otimes \overline{dx^j}.$$

DEFINITION 2.1.2. We define the exterior product of $\omega \in A^{p,q}(M)$ and $\eta \in A^{r,s}(M)$ by

$$\omega \wedge \eta = \sum_{I_p, J_q, K_r, L_s} \omega_{I_p J_q} \eta_{K_r L_s} dx^{I_p} \wedge dx^{K_r} \otimes \overline{dx^{J_q} \wedge dx^{L_s}},$$

where $\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dx^{I_p} \otimes \overline{dx^{J_q}}$ and $\eta = \sum_{K_r, L_s} \eta_{K_r L_s} dx^{K_r} \otimes \overline{dx^{L_s}}$.

DEFINITION 2.1.3. For $\omega \in A^{r,s}(M)$ we define an exterior product operator $e(\omega) : A^{p,q}(M) \to A^{p+r,q+s}(M)$ by

$$e(\omega)\eta = \omega \wedge \eta.$$

Definition 2.1.4. We denote by $\mathscr{X}(M)$ the set of smooth vector fields on M. For $X \in \mathscr{X}(M)$ we define interior product operators by

$$\begin{aligned} i(X) &: A^{p,q}(M) \to A^{p-1,q}(M), \quad i(X)\omega = \omega(X,\dots;\dots), \\ \bar{i}(X) &: A^{p,q}(M) \to A^{p,q-1}(M), \quad \bar{i}(X)\omega = \omega(\,\dots;X,\dots). \end{aligned}$$

Lemma 2.1.5 ([**2**]). *The following equations hold for $\omega \in A^{p,q}(M)$, $\eta \in A^{p-1,q}(M)$, $\rho \in A^{p,q-1}(M)$ and $X \in \mathscr{X}(M)$.*

$$\begin{aligned} \langle i(X)\omega, \eta\rangle &= \langle \omega, e(\bar{i}(X)g)\eta\rangle, \\ \langle \bar{i}(X)\omega, \rho\rangle &= \langle \omega, e(i(X)g)\rho\rangle, \end{aligned}$$

where $\langle\ ,\ \rangle$ is a fiber metric on $\wedge^p T^*M \otimes \wedge^q T^*M$ induced by g.

Definition 2.1.6. Let $\{E_1,\dots,E_n\}$ be an orthonormal frame field on TM and $\{\theta^1,\dots,\theta^n\}$ the dual frame field of $\{E_1,\dots,E_n\}$. We define $L : A^{p,q}(M) \to A^{p+1,q+1}(M)$ and $\Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M)$ by

$$L := e(g) = \sum_j e(\theta^j)e(\bar{\theta}^j), \quad \Lambda := \sum_j i(E_j)\bar{i}(E_j).$$

We obtain the following from Lemma 2.1.5.

Corollary 2.1.7 ([**2**]). *We have*

$$\langle \Lambda\omega, \eta\rangle = \langle \omega, L\eta\rangle, \quad \textit{for } \omega \in A^{p,q}(M) \textit{ and } \eta \in A^{p-1,q-1}(M).$$

We have the following by a direct calculation.

Proposition 2.1.8 ([**2**]). *We have*

$$[L,\Lambda] = p+q-n, \quad \textit{on } A^{p,q}(M).$$

2.2. Differential operators for (p,q)-forms.
We define two operators ∂ and $\bar{\partial}$ by using the flat connection D.

Definition 2.2.1. We define $\partial : A^{p,q}(M) \to A^{p+1,q}(M)$ and $\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$ by

$$\partial = \sum_i e(dx^i)D_{\partial/\partial x^i}, \quad \bar{\partial} = \sum_i e(\overline{dx^i})D_{\partial/\partial x^i}.$$

Since D is flat, we immediately obtain the following lemma.

Lemma 2.2.2. *We have*

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} = \bar{\partial}\partial.$$

We denote by $A^{p,q}(M,F)$ the space of F-valued (p,q)-forms. Since the transition functions of $\{s\}$ are constant, ∂ and $\bar{\partial}$ are extended on $A^{p,q}(M,F)$ by

$$\partial(s\otimes\omega)=s\otimes\partial\omega,$$
$$\bar{\partial}(s\otimes\omega)=s\otimes\bar{\partial}\omega.$$

DEFINITION 2.2.3. We denote by $A_0^{p,q}(M,F)$ the space of elements of $A^{p,q}(M,F)$ with compact supports. We define the inner product $(\ ,\ )$ on $A_0^{p,q}(M,F)$ by

$$(\omega,\eta)=\int_M\langle\omega,\eta\rangle\, v_g,$$

where $v_g=\sqrt{\det[g_{ij}]}\,dx^1\wedge\cdots\wedge dx^n$, and $\langle\ ,\ \rangle$ is the metric on $F\otimes\wedge^pT^*M\otimes\wedge^qT^*M$ induced by g and h. We set $\|\omega\|=\sqrt{(\omega,\omega)}$.

DEFINITION 2.2.4. We define $A\in A^{1,0}(M)$ and $B\in A^{1,1}(M)$ by

$$A=-\partial\log h(s,s),\quad B=\bar{\partial}A.$$

We call A and B the *first Koszul form* and the *second Koszul form* with respect to the fiber metric h, respectively.

REMARK 2.2.5. Since the transition functions of $\{s\}$ are constant, A and B are globally well-defined.

EXAMPLE 2.2.6. Let α and β be the first Koszul form and the second Koszul form with respect to the Riemannian metric g, respectively. Then the the first Koszul form A_K and the second Koszul form B_K with respect to the fiber metric g on the canonical bundle $K=\wedge^nT^*M$ are given by

$$A_K=2\alpha,\quad B_K=2\beta.$$

The following theorem is an analogue of the Kodaira–Nakano vanishing theorem.

THEOREM 2.2.7 ([**2**]). *Let (M,D) be an oriented n-dimensional compact flat manifold and (F,D^F) be a flat line bundle over M. We set*

$$H_{\bar{\partial}}^{p,q}(M,F)=\frac{\operatorname{Ker}[\bar{\partial}:A^{p,q}(M,F)\to A^{p,q+1}(M,F)]}{\operatorname{Im}[\bar{\partial}:A^{p,q-1}(M,F)\to A^{p,q}(M,F)]}.$$

Assume there exist a fiber metric h on F and a Riemannian metric g on M such that $B+\beta$ is positive definite, where B and β are the second Koszul forms with respect to h and g, respectively. Then we have

$$H_{\bar{\partial}}^{p,q}(M,F)=0,\quad \textit{for } p+q>n.$$

DEFINITION 2.2.8. We define the star operator $\bigstar:A^{p,q}(M)\to A^{n-p,n-q}(M)$ by

$$\omega\wedge\bigstar\eta=\langle\omega,\eta\rangle v_g\otimes\bar{v}_g,\quad \omega,\eta\in A^{p,q}(M),$$

where $\langle\ ,\ \rangle$ is a fiber metric on $\wedge^p T^*M \otimes \wedge^q T^*M$ induced by g. The star operator $\bigstar$ is extended on $A^{p,q}(M,F)$ by

$$\bigstar(s\otimes\omega) = s\otimes\bigstar\omega.$$

Definition 2.2.9. We define $\delta_F : A^{p,q}(M,F) \to A^{p-1,q}(M,F)$ and $\bar{\delta}_F : A^{p,q}(M,F) \to A^{p,q-1}(M,F)$ by

$$\delta_F = (-1)^p \bigstar^{-1}\partial\bigstar + i(X_{A+\alpha}), \quad \bar{\delta}_F = (-1)^q \bigstar^{-1}\bar{\partial}\bigstar + \bar{i}(X_{A+\alpha}),$$

where $\bar{i}(X_{A+\alpha})g = A+\alpha$. The operators will be denoted by δ and $\bar{\delta}$ if (F, D^F, h) is trivial.

Proposition 2.2.10 ([**2**]). *The operators δ_F and $\bar{\delta}_F$ are the adjoint operators of ∂ and $\bar{\partial}$ with respect to the inner product $(\ ,\ )$ respectively, that is, for $\omega \in A^{p,q}(M,F)$, $\eta \in A_0^{p-1,q}(M,F)$ and $\rho \in A_0^{p,q-1}(M,F)$ we have*

$$(\delta_F\omega, \eta) = (\omega, \partial\eta), \quad (\bar{\delta}_F\omega, \rho) = (\omega, \bar{\partial}\rho).$$

Definition 2.2.11. We define the connection $\mathscr{D}$ and $\overline{\mathscr{D}}$ on $\wedge^p T^*M \otimes \wedge^q T^*M$ as follows: For $\omega \in A^p(M)$ and $\eta \in A^q(M), X \in \mathscr{X}(M)$

$$\begin{aligned}\mathscr{D}_X(\omega\otimes\bar{\eta}) &= 2\gamma_X\omega\otimes\bar{\eta} + D_X(\omega\otimes\bar{\eta}),\\ \overline{\mathscr{D}}_X(\omega\otimes\bar{\eta}) &= 2\omega\otimes\overline{\gamma_X\eta} + D_X(\omega\otimes\bar{\eta}),\end{aligned}$$

where $\gamma = \nabla - D$ and ∇ is the Levi-Civita connection of g (cf. Definition 1.1.3).

The following lemma follows from Proposition 1.1.5.

Lemma 2.2.12 ([**2**]). *The following conditions are equivalent.*

(1) *(D,g) is a Hessian structure.*

(2) $\partial g = 0 \quad (\Leftrightarrow \bar{\partial}g = 0)$.

(3) $\mathscr{D}g = 0 \quad (\Leftrightarrow \overline{\mathscr{D}}g = 0)$.

Let D^* be the dual connection of D with respect to g (cf. Definition 1.1.4). We obtain the following from Proposition 1.1.5.

Lemma 2.2.13. *Let (D,g) be a Hessian structure. Then we have*

$$\begin{aligned}\mathscr{D}_X(\omega\otimes\bar{\eta}) &= D^*_X\omega\otimes\bar{\eta} + \omega\otimes\overline{D_X\eta},\\ \overline{\mathscr{D}}_X(\omega\otimes\bar{\eta}) &= D_X\omega\otimes\bar{\eta} + \omega\otimes\overline{D^*_X\eta},\end{aligned}$$

for $\omega \in A^p(M)$ and $\eta \in A^q(M), X \in \mathscr{X}(M)$.

When (D,g) is a Hessian structure, the operators ∂, $\bar{\partial}$, δ_F and $\bar{\delta}_F$ are expressed with $\mathscr{D}$ and $\bar{\mathscr{D}}$.

Proposition 2.2.14 ([**2**]). *Let (D, g) be a Hessian structure. Then we have*

$$\partial = \sum_j e(\theta^j)\mathscr{D}_{E_j}, \quad \bar\partial = \sum_j e(\bar\theta^j)\overline{\mathscr{D}}_{E_j},$$
$$\delta_F = -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j} + i(X_{A+\alpha}), \quad \bar\delta_F = -\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} + \bar{i}(X_{A+\alpha}),$$

where $\bar{i}(X_{A+\alpha})g = A + \alpha$.

2.3. The new differential operator ∂'_F.

We introduce the operator ∂'_F which is not defined in [**2**].

Definition 2.3.1. We define the differential operator $\partial'_F : A^{p,q}(M, F) \to A^{p+1,q}(M, F)$ by

$$\partial'_F = \partial - e(A + \alpha).$$

The operator will be denoted by ∂' if (F, D^F, h) is trivial.

Theorem 2.3.2. *We have*

$$(\partial'_F)^2 = 0, \quad \partial'_F\bar\partial - \bar\partial\partial'_F = e(B + \beta).$$

Proof. We obtain

$$\begin{aligned}(\partial'_F)^2 &= (\partial - e(A+\alpha))(\partial - e(A+\alpha))\\ &= \partial^2 - e(\partial(A+\alpha)) + e(A+\alpha)\partial - e(A+\alpha)\partial + e(A+\alpha)e(A+\alpha)\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\partial'_F\bar\partial &= (\partial - e(A+\alpha))\bar\partial = \partial\bar\partial - e(A+\alpha)\bar\partial,\\ \bar\partial\partial'_F &= \bar\partial(\partial - e(A+\alpha)) = \bar\partial\partial - e(B+\beta) - e(A+\alpha)\bar\partial.\end{aligned}$$

Hence

$$\partial'_F\bar\partial - \bar\partial\partial'_F = e(B + \beta).$$

□

Definition 2.3.3. We define $\delta'_F : A^{p,q}(M, F) \to A^{p-1,q}(M, F)$ by

$$\delta'_F = \delta_F - i(X_{A+\alpha}),$$

where $\bar{i}(X)g = A + \alpha$. The operator will be denoted by δ' if (F, D^F, h) is trivial.

We obtain the following Corollaries from Lemma 2.1.5, Propositions 2.2.10 and 2.2.14.

Corollary 2.3.4. *The operator δ'_F is the adjoint operator of ∂'_F with respect to* *the inner product* (,), *that is, for* $\omega \in A^{p,q}(M,F)$, $\eta \in A_0^{p-1,q}(M,F)$ *we have*

$$(\delta'_F\omega, \eta) = (\omega, \partial'_F\eta).$$

COROLLARY 2.3.5. *Let* (D,g) *be a Hessian structure. Then we have*

$$\delta'_F = -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j}.$$

The following theorem is an analogue of the Kähler identities.

THEOREM 2.3.6. *Let* (D,g) *be a Hessian structure. Then we have*

$$\begin{aligned}\Lambda\partial'_F + \partial'_F\Lambda = -\bar{\delta}_F, \quad \Lambda\bar{\partial} + \bar{\partial}\Lambda = -\delta'_F,\\ L\delta'_F + \delta'_F L = -\bar{\partial}, \quad L\bar{\delta}_F + \bar{\delta}_F L = -\partial'_F.\end{aligned}$$

PROOF. It follows from Corollary 2.3.5, Proposition 2.2.12 and 2.2.14 that

$$\begin{aligned}\delta'_F L &= -\sum_j i(E_j)\overline{\mathscr{D}}_{E_j}L = -\sum_j i(E_j)L\overline{\mathscr{D}}_{E_j}\\ &= -\sum_{j,k} i(E_j)e(\theta^k)e(\bar{\theta}^k)\overline{\mathscr{D}}_{E_j}\\ &= -\sum_{j,k} e(\bar{\theta}^k)(\delta_j^k - e(\theta^k)i(E_j))\overline{\mathscr{D}}_{E_j}\\ &= -\sum_j e(\bar{\theta}^j)\overline{\mathscr{D}}_{E_j} + \sum_k e(\bar{\theta}^k)e(\theta^k)\sum_j i(E_j)\overline{\mathscr{D}}_{E_j}\\ &= -\bar{\partial} - L\delta'_F.\end{aligned}$$

Similarly, we have

$$-\sum_j \bar{i}(E_j)\mathscr{D}_{E_j}L = -\partial + L\sum_j \bar{i}(E_j)\mathscr{D}_{E_j}.$$

Moreover,

$$\begin{aligned}\bar{i}(X_{A+\alpha})L &= \bar{i}(X_{A+\alpha})\sum_k e(\theta^k)e(\bar{\theta}^k)\\ &= \sum_k e(\theta^k)\{(A+\alpha)(E_k) - e(\bar{\theta}^k)\bar{i}(X_{A+\alpha})\}\\ &= e(A+\alpha) - L\bar{i}(X_{A+\alpha}).\end{aligned}$$

Hence it follows from Proposition 2.2.14 that

$$\bar{\delta}_F L = \Big(-\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} + \bar{i}(X_{A+\alpha})\Big)L$$

$$
\begin{aligned}
&= -(\partial - e(A+\alpha)) - L\Big(-\sum_j \bar{i}(E_j)\mathscr{D}_{E_j} + \bar{i}(X_{A+\alpha})\Big) \\
&= -\partial'_F - L\bar{\delta}_F.
\end{aligned}
$$

We have the other equalities by taking the adjoint operators. □

DEFINITION 2.3.7. We define the Laplacians $\Box'_F$ and $\bar{\Box}$ with respect to ∂'_F and $\bar{\partial}$ by

$$\Box'_F = \partial'_F\delta'_F + \delta'_F\partial'_F, \quad \bar{\Box}_F = \bar{\partial}\bar{\delta}_F + \bar{\delta}_F\bar{\partial}.$$

The Laplacians will be denoted by $\Box'$ and $\bar{\Box}$ if (F, D^F, h) is trivial.

The following theorem is an analogue of the Kodaira–Nakano identity.

THEOREM 2.3.8. *Let (D, g) be a Hessian structure. Then we have*

$$\bar{\Box}_F = \Box'_F + [e(\beta + B), \Lambda].$$

PROOF. It follows from Theorems 2.3.2 and 2.3.6 that

$$
\begin{aligned}
\bar{\Box}_F &= \bar{\partial}\bar{\delta}_F + \bar{\delta}_F\bar{\partial} = -\bar{\partial}(\Lambda\partial'_F + \partial'_F\Lambda) - (\Lambda\partial'_F + \partial'_F\Lambda)\bar{\partial} \\
&= (\Lambda\bar{\partial} + \delta'_F)\partial'_F - \bar{\partial}\partial'_F\Lambda - \Lambda\partial'_F\bar{\partial} + \partial'_F(\bar{\partial}\Lambda + \delta'_F) \\
&= \delta'_F\partial'_F + \partial'_F\delta'_F + (\partial'_F\bar{\partial} - \bar{\partial}\partial'_F)\Lambda - \Lambda(\partial'_F\bar{\partial} - \bar{\partial}\partial'_F) \\
&= \Box'_F + [e(B+\beta), \Lambda].
\end{aligned}
$$
□

3. Vanishing theorems of L^2-cohomology groups.

We introduce L^2-cohomology groups on flat manifolds and some vanishing theorems.

3.1. L^2-cohomology groups on flat manifolds.

We denote by $L^2(M, F \otimes \wedge^{p,q})$ the completion of $A_0^{p,q}(M, F)$ with respect to the L^2-inner product $(\ ,\ )$ induced by g and h. The space $L^2(M, F \otimes \wedge^{p,q})$ is identified with the space of square-integrable sections of $F \otimes \wedge^{p,q}$.

DEFINITION 3.1.1. For $\omega \in L^2(M, F \otimes \wedge^{p,q})$ we define $\bar{\partial}\omega$ and $\bar{\delta}_F\omega$ as follows:

$$
\begin{aligned}
(\bar{\partial}\omega, \eta) &= (\omega, \bar{\delta}_F\eta), \quad \text{for } \eta \in A_0^{p,q+1}(M, F), \\
(\bar{\delta}_F\omega, \rho) &= (\omega, \bar{\partial}\rho), \quad \text{for } \rho \in A_0^{p,q-1}(M, F).
\end{aligned}
$$

In general, we cannot say $\bar{\partial}\omega \in L^2(M, F \otimes \wedge^{p,q+1})$ and $\bar{\delta}_F\omega \in L^2(M, F \otimes \wedge^{p,q-1})$. We set

$$
\begin{aligned}
&W(M, F \otimes \wedge^{p,q}) \\
&\quad = \{\omega \in L^2(M, F \otimes \wedge^{p,q}) \mid \bar{\partial}\omega \in L^2(M, F \otimes \wedge^{p,q+1}), \bar{\delta}_F\omega \in L^2(M, F \otimes \wedge^{p,q-1})\}, \\
&D(M, F \otimes \wedge^{p,q}) = \{\omega \in L^2(M, F \otimes \wedge^{p,q}) \mid \bar{\partial}\omega \in L^2(M, F \otimes \wedge^{p,q+1})\}.
\end{aligned}
$$

In addition, we define the norm $\| \ \|_W$ on $W(M, F \otimes \wedge^{p,q})$ by

$$\|\omega\|_W = \|\omega\| + \|\bar{\partial}\omega\| + \|\bar{\delta}_F\omega\|, \quad \omega \in W(M, F \otimes \wedge^{p,q}).$$

The space $W(M, F \otimes \wedge^{p,q})$ is complete with respect to $\| \ \|_W$.

Proposition 3.1.2 ([**4**]). *If g is complete, the space $A_0^{p,q}(M, F)$ is dense in $W(M, F \otimes \wedge^{p,q})$ with respect to the L^2-norm $\| \ \|_W$.*

Definition 3.1.3. We define the L^2*-cohomology group* of (p, q)-type by

$$L^2 H_{\bar{\partial}}^{p,q}(M, F) = \frac{\mathrm{Ker}[\bar{\partial} : D(M, F \otimes \wedge^{p,q}) \to D(M, F \otimes \wedge^{p,q+1})]}{\overline{\mathrm{Im}[\bar{\partial} : D(M, F \otimes \wedge^{p,q-1}) \to D(M, F \otimes \wedge^{p,q})]}},$$

where $\overline{\mathrm{Im}[\bar{\partial} : D(M, F \otimes \wedge^{p,q-1}) \to D(M, F \otimes \wedge^{p,q})]}$ is the closure of $\mathrm{Im}[\bar{\partial} : D(M, F \otimes \wedge^{p,q-1}) \to D(M, F \otimes \wedge^{p,q})]$ with respect to the L^2-norm $\| \ \|$.

3.2. Vanishing theorems of Kodaira–Nakano type.

In this section we show vanishing theorems of Kodaira–Nakano type.

Lemma 3.2.1. *Assume g is a Hessian metric and $B + \beta$ is positive definite. For the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of the matrix $[\sum_k g^{ik}(B + \beta)_{kj}]$, we set $b_q = \sum_{j=1}^q \lambda_j$. Then we have*

$$\|\bar{\partial}\omega\|^2 + \|\bar{\delta}_F\omega\|^2 \geq \|b_q^{1/2}\omega\|^2, \quad \textit{for } \omega \in A_0^{n,q}(M, F).$$

Proof. By Theorem 2.3.8 we obtain

$$\begin{aligned} \|\bar{\partial}\omega\|^2 + \|\bar{\delta}_F\omega\|^2 &= (\bar{\Box}_F\omega, \omega) \\ &= (\Box'_F\omega, \omega) + ([e(B + \beta), \Lambda]\omega, \omega) \\ &\geq ([e(B + \beta), \Lambda]\omega, \omega) \\ &= (e(B + \beta)\Lambda\omega, \omega). \end{aligned}$$

Hence it is sufficient to show $(e(B + \beta)\Lambda\omega, \omega) \geq \|b_q^{1/2}\omega\|^2$.

We take the orthonormal frame field $\{E_1, \ldots, E_n\}$ on TM, where the matrix $[(B + \beta)(E_i, E_j)]$ is diagonal. We set $\mu_j = (B + \beta)(E_j, E_j)$. Using the dual frame field $\{\theta^1, \ldots, \theta^n\}$ of $\{E_1, \ldots, E_n\}$, $\omega \in A_0^{n,q}(M, F)$ is denoted by

$$\omega = \sum_{J_q} \omega_{J_q} \otimes \bar{\theta}^{J_q}, \quad \omega_{J_q} \in A_0^n(M, F).$$

Then

$$\begin{aligned} e(B + \beta)\Lambda\omega &= \sum_j \mu_j e(\theta^j) e(\bar{\theta}^j) \sum_k i(E_k)\bar{i}(E_k) \sum_{J_q} \omega_{J_q} \otimes \bar{\theta}^{J_q} \\ &= \sum_{j, J_q} \mu_j \omega_{J_q} \otimes e(\bar{\theta}^j)\bar{i}(E_j)\bar{\theta}^{J_q} \end{aligned}$$

$$= \sum_{J_q} \sum_{j \in J_q} \mu_j \omega_{J_q} \otimes \bar{\theta}^{J_q}.$$

Therefore

$$\begin{aligned}(e(B+\beta)\Lambda\omega, \omega) &= \int_M \sum_{J_q} \sum_{j \in J_q} \mu_j \langle \omega_{J_q}, \omega_{J_q} \rangle v_g \\ &\geq \int_M \sum_{J_q} b_q \langle \omega_{J_q}, \omega_{J_q} \rangle v_g = \|b_q^{1/2}\omega\|^2.\end{aligned}$$

□

MAIN THEOREM 1. *Let (M, D, g) be an oriented n-dimensional complete Hessian manifold and (F, D^F) a flat line bundle over M. We denote by h a fiber metric on F. Assume $B+\beta$ is positive definite, where B and β are the second Koszul forms with respect to fiber metric h and Hessian metric g respectively. For $q \geq 1$ let b_q be the same as in Lemma 3.2.1. Then for all $v \in L^2(M, F \otimes \wedge^{n,q})$ such that $\bar{\partial}v = 0$ and $b_q^{-1/2}v \in L^2(M, F \otimes \wedge^{n,q})$, there exists $u \in L^2(M, F \otimes \wedge^{n,q-1})$ such that*

$$\bar{\partial}u = v, \quad \|u\| \leq \|b_q^{-1/2}v\|.$$

In particular, if there exists $\varepsilon > 0$ such that $B+\beta-\varepsilon g$ is positive semi-definite, we have

$$L^2 H_{\bar{\partial}}^{n,q}(M, F) = 0, \quad \text{for } q \geq 1.$$

PROOF. The theorem can be shown by applying the method as in complex analysis in several variables (cf. [**5**, Lemma 4.1.1]) to the case of Hessian manifolds.

We set $\operatorname{Ker}\bar{\partial} = \{\omega \in L^2(M, F \otimes \wedge^{n,q}) \mid \bar{\partial}\omega = 0\}$. Since $\operatorname{Ker}\bar{\partial}$ is a closed subspace in $L^2(M, F \otimes \wedge^{n,q})$, we have

$$L^2(M, F \otimes \wedge^{n,q}) = \operatorname{Ker}\bar{\partial} \oplus (\operatorname{Ker}\bar{\partial})^{\perp},$$

where $(\operatorname{Ker}\bar{\partial})^{\perp}$ is the orthogonal complement of $\operatorname{Ker}\bar{\partial}$. A (n, q)-form $\omega \in L^2(M, F \otimes \wedge^{p,q})$ is expressed by

$$\omega = \omega_1 + \omega_2, \quad \omega_1 \in \operatorname{Ker}\bar{\partial}, \quad \omega_2 \in (\operatorname{Ker}\bar{\partial})^{\perp}.$$

For $\eta \in A_0^{n,q-1}(M, F)$, we have

$$(\bar{\delta}_F\omega, \eta) = (\omega, \bar{\partial}\eta) = 0,$$

and so

$$\bar{\delta}_F\omega_2 = 0.$$

Since $v \in \operatorname{Ker}\bar{\partial}$ by assumption, we obtain

$$|(v, \omega)|^2 = |(v, \omega_1)|^2 = |(b_q^{-1/2}v, b_q^{1/2}\omega_1)|^2 \leq \|b_q^{-1/2}v\|^2 \|b_q^{1/2}\omega_1\|^2.$$

Assume $\omega \in W(M, F \otimes \wedge^{n,q})$. Then

$$\bar{\partial}\omega_1 = 0, \quad \bar{\delta}_F\omega_1 = \bar{\delta}_F\omega \in L^2(M, F \otimes \wedge^{n,q-1}),$$

and so $\omega_1 \in W(M, F \otimes \wedge^{n,q})$. Hence by Proposition 3.1.2, ω_1 satisfies the inequality in Lemma 3.2.1:

$$\|b_q^{1/2}\omega_1\|^2 \leq \|\bar{\partial}\omega_1\|^2 + \|\bar{\delta}_F\omega_1\|^2 = \|\bar{\delta}_F\omega_1\|^2 = \|\bar{\delta}_F\omega\|^2 < \infty.$$

Therefore for $\omega \in W(M, F \otimes \wedge^{n,q})$ we have

$$|(v,\omega)|^2 \leq \|b_q^{-1/2}v\|^2\|\bar{\delta}_F\omega\|^2 < \infty.$$

By this inequality a linear functional $\lambda : \bar{\delta}_F W(M, F \otimes \wedge^{n,q}) \ni \bar{\delta}_F\omega \mapsto (v,\omega) \in \mathbb{R}$ is well-defined and the operator norm C is

$$C \leq \|b_q^{-1/2}v\| < \infty.$$

We set $\operatorname{Ker}\bar{\delta}_F = \{\omega \in L^2(M, F \otimes \wedge^{n,q}) \mid \bar{\delta}_F\omega = 0\}$. $\operatorname{Ker}\bar{\delta}_F$ is also a closed subspace in $L^2(M, F \otimes \wedge^{n,q})$ and

$$L^2(M, F \otimes \wedge^{n,q}) = \operatorname{Ker}\bar{\delta}_F \oplus (\operatorname{Ker}\bar{\delta}_F)^{\perp},$$

where $(\operatorname{Ker}\bar{\delta}_F)^{\perp}$ is the orthogonal complement of $\operatorname{Ker}\bar{\delta}_F$. In the same way we have $(\operatorname{Ker}\bar{\delta}_F)^{\perp} \subset \operatorname{Ker}\bar{\partial}$ and for $\hat{\omega} \in (\operatorname{Ker}\bar{\delta}_F)^{\perp} \cap W(M, F \otimes \wedge^{n,q})$,

$$\|b_q^{1/2}\hat{\omega}\|^2 \leq \|\bar{\partial}\hat{\omega}\|^2 + \|\bar{\delta}_F\hat{\omega}\|^2 = \|\bar{\delta}_F\hat{\omega}\|^2.$$

Let $\{\eta_k\} \subset \bar{\delta}_F W(M, F \otimes \wedge^{n,q})$ be a Cauchy sequence with respect to the norm $\| \;\|$ on $L^2(M, F \otimes \wedge^{n,q-1})$. Each η_k is denoted by

$$\eta_k = \bar{\delta}_F\hat{\omega}_k, \quad \hat{\omega}_k \in (\operatorname{Ker}\bar{\delta}_F)^{\perp} \cap W(M, F \otimes \wedge^{n,q}),$$

and by the said inequality $\{\hat{\omega}_k\}$ is also a Cauchy sequence with respect to the norm $\| \;\|$ on $L^2(M, F \otimes \wedge^{n,q})$. This implies $\{\hat{\omega}_k\}$ is a Cauchy sequence with respect to the norm $\| \;\|_W$ on $W(M, F \otimes \wedge^{n,q})$. Hence by completeness of $W(M, F \otimes \wedge^{n,q})$ with respect to $\| \;\|_W$, we have

$$\hat{\omega}_k \to \hat{\omega} \in W(M, F \otimes \wedge^{n,q}) \quad (k \to \infty),$$

and

$$\eta_k \to \bar{\delta}_F\hat{\omega} \quad (k \to \infty).$$

Therefore, $\bar{\delta}_F W(M, F \otimes \wedge^{n,q})$ is a closed space of $L^2(M, F \otimes \wedge^{p,q-1})$ with respect to the norm $\| \;\|$.

From the above, by applying Riesz representation theorem to the linear functional $\lambda : \bar{\delta}_F W(M, F \otimes \wedge^{n,q}) \to \mathbb{R}$, there exists $u \in \bar{\delta}_F W(M, F \otimes \wedge^{n,q})$ such that

$$\begin{cases}\lambda(\eta)=(u,\eta), \quad \eta\in\bar{\delta}_F W(M,F\otimes\wedge^{n,q}) \\ \|u\|=C\leq\|b_q^{-1/2}v\|.\end{cases}$$

By the first equation, for all $\omega\in A_0^{n,q}(M,F)$ we have

$$(v,\omega)=\lambda(\bar{\delta}_F\omega)=(u,\bar{\delta}_F\omega),$$

and so

$$\bar{\partial}u=v.$$

This implies the first assertion.

Suppose there exists $\varepsilon>0$ such that $B+\beta-\varepsilon g$ is positive semi-definite. Then by the definition of b_q, $b_q\geq\varepsilon q$. Hence for all $v\in L^2(M,F\otimes\wedge^{n,q})$ we obtain

$$\int_M\langle b_q^{-1/2}v,b_q^{-1/2}v\rangle v_g\leq(\varepsilon q)^{-1}\int_M\langle v,v\rangle v_g<\infty,$$

that is,

$$b_q^{-1/2}v\in L^2(M,F\otimes\wedge^{n,q}).$$

This implies the second assertion. □

The following theorem corresponds to Theorem 2.2.7 in the case of complete Hessian manifolds.

Main Theorem 2. *Let (M,D,g) be an oriented n-dimensional complete Hessian manifold and (F,D^F) a flat line bundle over M. We denote by h a fiber metric on F. Assume that there exists $\varepsilon>0$ such that $B+\beta=\varepsilon g$ where B and β are the second Koszul forms with respect to fiber metric h and Hessian metric g respectively. Then for $p+q>n$ and all $v\in L^2(M,F\otimes\wedge^{p,q})$ such that $\bar{\partial}v=0$, there exists $u\in L^2(M,F\otimes\wedge^{p,q-1})$ such that*

$$\bar{\partial}u=v,\quad\|u\|\leq\{\varepsilon(p+q-n)\}^{-1/2}\|v\|.$$

In particular, we have

$$L^2H_{\bar{\partial}}^{p,q}(M,F)=0,\quad\text{for } p+q>n.$$

Proof. By Proposition 2.1.8, on $A^{p,q}(M,F)$ we have

$$[e(B+\beta),\Lambda]=\varepsilon[L,\Lambda]=\varepsilon(p+q-n).$$

Hence by Theorem 2.3.8, for all $\omega\in A_0^{p,q}(M,F)$ we obtain

$$\|\bar{\partial}\omega\|^2+\|\bar{\delta}_F\omega\|^2\geq\varepsilon(p+q-n)\|\omega\|^2.$$

Then the assertions are proved similarly to Main Theorem 1. □

Corollary 3.2.2. *Let $(\mathbb{R}^n, g)$ be the Euclidean space, D be the canonical affine connection on $\mathbb{R}^n$, and $(F = \mathbb{R}^n \times \mathbb{R}, D^F)$ be the trivial flat line bundle on $\mathbb{R}^n$. In addition, we define a fiber metric h on F by*

$$h(s,s) = e^{-\varphi},$$

where $\varphi(x) = (1/2)\sum_i (x^i)^2$ and $s : \mathbb{R}^n \ni x \mapsto (x,1) \in F$. Then for $q \geq 1$ and $v \in L^2(\mathbb{R}^n, F \otimes \wedge^{p,q})$ such that $\bar{\partial} v = 0$, there exists $u \in L^2(\mathbb{R}^n, F \otimes \wedge^{p,q-1})$ such that

$$\bar{\partial} u = v, \quad \|u\| \leq q^{-1/2}\|v\|.$$

In particular, we have

$$L^2 H^{p,q}_{\bar{\partial}}(\mathbb{R}^n, F) = 0, \quad \text{for } p \geq 0 \text{ and } q \geq 1.$$

Proof. The Hessian metric $g = Dd\varphi$ is complete and the second Koszul forms with respect to h and g are

$$B = -\partial\bar{\partial} \log h(s,s) = \partial\bar{\partial}\varphi = g, \quad \beta = \frac{1}{2}\partial\bar{\partial}\det[\delta_{ij}] = 0.$$

Hence by Main Theorem 2, for $p = n$ we obtain the assertion.

Next, we consider the case of $p = 0$. For $v \in L^2(\mathbb{R}^n, F \otimes \wedge^{0,q})$ we set

$$\hat{v} = dx^1 \wedge \cdots \wedge dx^n \otimes v.$$

Then we have $\hat{v} \in L^2(\mathbb{R}^n, F \otimes \wedge^{n,q})$ and $\|\hat{v}\| = \|v\|$. Since $\bar{\partial} v = 0$ and $\bar{\partial}\hat{v} = 0$ are equivalent, by Main Theorem 2 there exists $\hat{u} \in L^2(\mathbb{R}^n, F \otimes \wedge^{n,q-1})$ such that $\bar{\partial}\hat{u} = \hat{v}$ and $\|\hat{u}\| \leq q^{-1/2}\|\hat{v}\|$. Here $\hat{u}$ can be expressed as

$$\hat{u} = dx^1 \wedge \cdots \wedge dx^n \otimes u, \quad u \in L^{0,q-1}(\mathbb{R}^n, g, F, h),$$

and so

$$dx^1 \wedge \cdots \wedge dx^n \otimes \bar{\partial} u = \bar{\partial}\hat{u} = \hat{v} = dx^1 \wedge \cdots \wedge dx^n \otimes v.$$

Therefore, we have $\bar{\partial} u = v$. Moreover, we obtain

$$\|u\| = \|\hat{u}\| \leq q^{-1/2}\|\hat{v}\| = q^{-1/2}\|v\|.$$

Hence the assertion for $p = 0$ follows.

Finally, for $p \geq 1$, $v \in L^2(\mathbb{R}^n, F \otimes \wedge^{p,q})$ can be expressed as

$$v = \sum_{I_p} dx^{I_p} \otimes v_{I_p}, \quad I_p = (i_1, \ldots, i_p), \quad 1 \leq i_1 < \cdots < i_p \leq n, \quad v_{I_p} \in L^2(\mathbb{R}^n, F \otimes \wedge^{0,q}),$$

and we have

$$\|v\|^2 = \sum_{I_p} \|v_{I_p}\|^2.$$

If $\bar\partial v = 0$, for all I_p we obtain $\bar\partial v_{I_p} = 0$. Hence by the case of $p = 0$, there exists $\{u_{I_p}\} \subset L^2(\mathbb{R}^n, F \otimes \wedge^{0,q-1})$ such that $\bar\partial u_{I_p} = v_{I_p}$ and $\|u_{I_p}\| \le q^{-1/2}\|v_{I_p}\|$. Here we set

$$u = \sum_{I_p} dx^{I_p} \otimes u_{I_p}.$$

Then we have

$$\bar\partial u = \sum_{I_p} dx^{I_p} \otimes \bar\partial u_{I_p} = \sum_{I_p} dx^{I_p} \otimes v_{I_p} = v,$$
$$\|u\|^2 = \sum_{I_p} \|u_{I_p}\|^2 \le \sum_{I_p} q^{-1}\|v_{I_p}\|^2 = q^{-1}\|v\|^2.$$

This completes the proof. □

Corollary 3.2.3. *Let $\Omega \in \mathbb{R}^n$ be a regular convex domain, D be the canonical affine connection on Ω, g be the Cheng–Yau metric defined by Theorem* 1.2.4. *Then for $p+q > n$ and $v \in L^2(\Omega, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\Omega, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \le (p+q-n)^{-1/2}\|v\|.$$

In particular, we have

$$L^2 H^{p,q}_{\bar\partial}(\Omega) = 0, \quad \textit{for } p+q > n.$$

Proof. Since g is complete and $\beta = g$, the assertion follows from Main Theorem 2. □

Let $\Omega \in \mathbb{R}^{n-1}$ be a regular convex domain and we set $V = \{(ty, t) \in \mathbb{R}^n \mid y \in \Omega, t > 0\}$. Let $\tilde D$ be the canonical affine connection on V and $\tilde g$ be the Cheng–Yau metric on $(V, \tilde D)$ defined by Theorem 1.2.4. In addition, we define an action $\rho : \mathbb{Z} \to GL(V)$ by

$$\rho(k)x = e^k x, \quad k \in \mathbb{Z}, \quad x \in V.$$

Then we have $\mathbb{Z}\backslash V \simeq \Omega \times S^1$. Moreover, by Proposition 1.2.5 this action preserves $(\tilde D, \tilde g)$ and so a Hessian structure (D, g) on $\Omega \times S^1$ is defined by projecting $(\tilde D, \tilde g)$ on $\Omega \times S^1$. The Hessian metric g is complete and the second Koszul form with respect to g is equal to g. Hence the following corollary follows from Main Theorem 2.

Corollary 3.2.4. *Let $(\Omega \times S^1, D, g)$ be as above. Then for $p+q > n$ and $v \in L^2(\Omega \times S^1, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\Omega \times S^1, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \le (p+q-n)^{-1/2}\|v\|.$$

In particular, we have

$$L^2 H^{p,q}_{\bar\partial}(\Omega \times S^1) = 0, \quad \textit{for } p+q > n.$$

3.3. L^2-cohomology groups on regular convex cones.

A regular convex domain Ω in $\mathbb{R}^n$ is said to be a *regular convex cone* if, for any x in Ω and any positive real number λ, λx belongs to Ω. In this section we show vanishing theorems for regular convex cones with the Cheng–Yau metrics which differ from Corollary 3.2.3.

Proposition 3.3.1. *Let* $(\Omega, D, g = Dd\varphi)$ *be a regular convex cone in* $\mathbb{R}^n$ *with the Cheng–Yau metric* (*Theorem* 1.2.4). *Then we have the following equations.*

(1) $\displaystyle\sum_j x^j \frac{\partial \varphi}{\partial x^j} = -n.$

(2) $\displaystyle \operatorname{grad}\varphi = -\sum_j x^j \frac{\partial}{\partial x^j}.$

(3) $\displaystyle\sum_k x^k \gamma_{ijk} = -g_{ij}.$

Proof. By the proof of Proposition 1.2.5, for $t > 0$ and $x \in \Omega$ we have

$$\varphi(tx) = \varphi(x) - n\log t.$$

Then we obtain

$$\sum_j x^j \frac{\partial \varphi}{\partial x^j} = \left.\frac{d}{dt}\right|_{t=1} \varphi(tx) = -n.$$

Taking the derivative of both sides with respect to x^i we have

$$\frac{\partial \varphi}{\partial x^i} + \sum_j x^j \frac{\partial^2 \varphi}{\partial x^i \partial x^j} = 0. \tag{$*$}$$

Since $\partial^2\varphi/\partial x^i \partial x^j = g_{ij}$ we obtain

$$\operatorname{grad}\varphi = \sum_{i,j} g^{ij} \frac{\partial \varphi}{\partial x^j}\frac{\partial}{\partial x^i} = -\sum_j x^j \frac{\partial}{\partial x^j}.$$

Equation $(*)$ is equivalent to

$$\frac{\partial \varphi}{\partial x^j} + \sum_k x^k g_{jk} = 0.$$

Taking the derivative of both sides with respect to x^i and applying Proposition 1.1.5 we have

$$g_{ij} + g_{ij} + \sum_k 2x^k \gamma_{ijk} = 0,$$

that is,

$$\sum_k x^k \gamma_{ijk} = -g_{ij}. \qquad \square$$

We set $H = \sum_j x^j \partial/\partial x^j$ $(= -\operatorname{grad}\varphi)$ and denote by $\mathscr{L}_H$ Lie differentiation with respect to H.

PROPOSITION 3.3.2. *For $\sigma \in A^p(\Omega)$ we have*

$$\mathscr{L}_H \sigma = D_H \sigma + p\sigma.$$

PROOF. For $X \in \mathscr{X}(\Omega)$ we obtain

$$D_X H = X,$$

and so

$$[H, X] = D_H X - D_X H = D_H X - X.$$

Then for $X_1, \dots, X_p \in \mathscr{X}(\Omega)$ we have

$$\begin{aligned}
&(\mathscr{L}_H \sigma)(X_1, \dots, X_p) \\
&= H\sigma(X_1, \dots, X_p) - \sum_i \sigma(X_1, \dots, [H, X_i], \dots, X_p) \\
&= H\sigma(X_1, \dots, X_p) - \sum_i \sigma(X_1, \dots, D_H X_i, \dots, X_p) + p\sigma(X_1, \dots, X_p) \\
&= (D_H \sigma)(X_1, \dots, X_p) + p\sigma(X_1, \dots, X_p). \qquad \square
\end{aligned}$$

By Cartan's formula we have the following.

COROLLARY 3.3.3. *For $\omega \in A^{p,q}(\Omega)$ we have*

$$\begin{aligned}
(\partial i(H) + i(H)\partial)\omega &= D_H \omega + p\omega, \\
(\bar\partial \bar\imath(H) + \bar\imath(H)\bar\partial)\omega &= D_H \omega + q\omega.
\end{aligned}$$

MAIN THEOREM 3. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng–Yau metric. Then for $p > q \geq 1$ and all $v \in L^2(\Omega, \wedge^{p,q})$ such that $\bar\partial v = 0$, there exists $u \in L^2(\Omega, \wedge^{p,q-1})$ such that*

$$\bar\partial u = v, \quad \|u\| \leq (p-q)^{-1/2}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\Omega, \wedge^{p,0})$ satisfies $\bar\partial v = 0$, then $v = 0$. In particular, we have

$$L^2 H^{p,q}_{\bar\partial}(\Omega) = 0, \quad \textit{for } p > q.$$

PROOF. As a corollary of Theorem 2.3.6 we obtain

$$\Lambda\partial + \partial\Lambda = -\bar\delta + \bar\imath(X_\alpha), \quad \Lambda\bar\partial + \bar\partial\Lambda = -\delta + i(X_\alpha).$$

Then we have

$$\begin{aligned}
\bar{\partial}\bar{\delta} &= \bar{\partial}(-\Lambda\partial - \partial\Lambda + \bar{i}(X_\alpha)) \\
&= (\Lambda\bar{\partial} + \delta - i(X_\alpha))\partial - \bar{\partial}\partial\Lambda + \bar{\partial}\bar{i}(X_\alpha) \\
&= \delta\partial - i(X_\alpha)\partial + \bar{\partial}\bar{i}(X_\alpha) + \Lambda\bar{\partial}\partial - \bar{\partial}\partial\Lambda, \\
\bar{\delta}\bar{\partial} &= (-\Lambda\partial - \partial\Lambda + \bar{i}(X_\alpha))\bar{\partial} \\
&= -\Lambda\partial\bar{\partial} + \partial(\bar{\partial}\Lambda + \delta - i(X_\alpha)) + \bar{i}(X_\alpha)\bar{\partial} \\
&= \partial\delta - \partial i(X_\alpha) + \bar{i}(X_\alpha)\bar{\partial} - \Lambda\partial\bar{\partial} + \partial\bar{\partial}\Lambda,
\end{aligned}$$

and so

$$\bar{\square} = \square - (\partial i(X_\alpha) + i(X_\alpha)\partial) + (\bar{\partial}\bar{i}(X_\alpha) + \bar{i}(X_\alpha)\bar{\partial}),$$

where $\square = \partial\delta + \delta\partial$.

Since φ is the solution of the equation in Theorem 1.2.4,

$$X_\alpha = \operatorname{grad}\varphi = -H.$$

Hence by Corollary 3.3.3,

$$\bar{\square} = \square + p - q.$$

Therefore, for $\omega \in A_0^{p,q}(\Omega)$ we obtain

$$\|\bar{\partial}\omega\|^2 + \|\bar{\delta}\omega\|^2 \geq (p-q)\|\omega\|^2.$$

Then the assertions are proved similarly to Main Theorem 1. $\square$

We have the following from Main Theorem 3 and Corollary 3.2.3.

Corollary 3.3.4. *Let $(\Omega, D, g = Dd\varphi)$ be a regular convex cone in $\mathbb{R}^n$ with the Cheng–Yau metric. Then we have*

$$L^2H_{\bar{\partial}}^{p,q}(\Omega) = 0, \quad \text{for } p+q > n \text{ or } p > q.$$

3.4. L^2-cohomology groups on $\mathbb{R}^n_+$.

The Cheng–Yau metric on a regular convex cone $\mathbb{R}^n_+$ is $g = -Dd\log(x^1 \cdots x^n) = \sum_i (dx^i/x^i)^2$. We can apply Corollary 3.3.4 to $(\mathbb{R}^n_+, D, g)$. However, we have a stronger vanishing theorem.

Main Theorem 4. *For $p \geq 1$, $q \geq 1$ and $v \in L^2(\mathbb{R}^n_+, \wedge^{p,q})$ such that $\bar{\partial}v = 0$, there exists $u \in L^2(\mathbb{R}^n_+, \wedge^{p,q-1})$ such that*

$$\bar{\partial}u = v, \quad \|u\| \leq p^{-1/2}\|v\|.$$

In the case of $p > q = 0$, if $v \in L^2(\mathbb{R}^n_+, \wedge^{p,0})$ satisfies $\bar{\partial}v = 0$, then $v = 0$. In particular, we have

$$L^2 H^{p,q}_{\bar\partial}(\mathbb{R}^n_+) = 0, \quad \textit{for } p \geq 1 \textit{ and } q \geq 0.$$

In this section we show Main Theorem 4. For the canonical coordinate $x = (x^1, \ldots, x^n)$ on $\mathbb{R}^n_+$, we set $t = (t^1, \ldots, t^n) = (\log x^1, \ldots, \log x^n)$.

LEMMA 3.4.1. *The following equations hold.*

(1) $g\left(\dfrac{\partial}{\partial t^i}, \dfrac{\partial}{\partial t^j}\right) = \delta_{ij}$.

(2) $D_{\partial/\partial t^i} \dfrac{\partial}{\partial t^j} = \delta_{ij} \dfrac{\partial}{\partial t^j}, \quad D^*_{\partial/\partial t^i} \dfrac{\partial}{\partial t^j} = -\delta_{ij} \dfrac{\partial}{\partial t^j}$.

(3) $D_{\partial/\partial t^i} dt^j = -\delta_i{}^j dt^j, \quad D^*_{\partial/\partial t^i} dt^j = \delta_i{}^j dt^j$.

(4) $\alpha = -\sum_j dt^j, \quad$ *where α is the first Koszul form for (D, g).*

LEMMA 3.4.2. *On $(\mathbb{R}^n_+, D, g)$ we have*

$$\bar\delta = -\sum_j \mathscr{D}_{\partial/\partial t^j} \bar{i}\left(\frac{\partial}{\partial t^j}\right).$$

PROOF. By Proposition 2.2.14, Lemmas 3.4.1 and 2.2.13 we obtain

$$\begin{aligned}
\bar\delta &= -\sum_j \bar{i}\left(\frac{\partial}{\partial t^j}\right)\mathscr{D}_{\partial/\partial t^j} - \bar{i}\left(\sum_j \frac{\partial}{\partial t^j}\right) \\
&= -\sum_j \left\{\bar{i}\left(\frac{\partial}{\partial t^j}\right)\mathscr{D}_{\partial/\partial t^j} + \bar{i}\left(D_{\partial/\partial t^j}\frac{\partial}{\partial t^j}\right)\right\} \\
&= -\sum_j \mathscr{D}_{\partial/\partial t^j} \bar{i}\left(\frac{\partial}{\partial t^j}\right).
\end{aligned}$$

□

PROPOSITION 3.4.3. *Let $\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dt^{I_p} \otimes \overline{dt^{J_q}} \in A^{p,q}(\mathbb{R}^n_+)$. Then we have*

$$\bar\Box\, \omega = \sum_{I_p, J_q} (\Delta + p)\, \omega_{I_p J_q} dt^{I_p} \otimes \overline{dt^{J_q}},$$

where $\Delta = -\sum_j (\partial/\partial t^j)^2$.

PROOF. It is sufficient to show the equation when $\omega = f\, dt^{I_p} \otimes \overline{dt^{J_q}}$. For a multi-index $J_q = (j_1, \ldots, j_q)$, $j_1 < \cdots < j_q$, we define $J_{n-q} = (j_{q+1}, \ldots, j_n)$, $j_{q+1} < \cdots < j_n$, where (J_q, J_{n-q}) is a permutation of $(1, \ldots, n)$. By Lemmas 3.4.1 and 3.4.2 we obtain

$$\bar\partial\omega = \sum_{i \in J_{n-q}} \frac{\partial f}{\partial t^i}\, dt^{I_p} \otimes \overline{dt^i} \wedge \overline{dt^{J_q}} - \sum_{i \in I_p \cap J_{n-q}} f\, dt^{I_p} \otimes \overline{dt^i} \wedge \overline{dt^{J_q}},$$

$$\bar\delta\omega = -\sum_{j \in J_q} \mathscr{D}_{\partial/\partial t^j}\left(f\, dt^{I_p} \otimes \bar{i}\left(\frac{\partial}{\partial t^j}\right)\overline{dt^{J_q}}\right)$$

$$= -\sum_{j\in J_q}\frac{\partial f}{\partial t^j}\,dt^{I_p}\otimes\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)\overline{dt^{J_q}} - \sum_{j\in I_p\cap J_q} f\,dt^{I_p}\otimes\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)\overline{dt^{J_q}},$$

$$\begin{aligned}
\bar{\delta}\bar{\partial}\omega = &-\sum_{i\in J_{n-q}}\sum_{j\in J_q\cup\{i\}}\frac{\partial^2 f}{\partial t^i\partial t^j}\,dt^{I_p}\otimes\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)(\overline{dt^i}\wedge\overline{dt^{J_q}})\\
&-\sum_{i\in J_{n-q}}\sum_{j\in I_p\cap(J_q\cup\{i\})}\frac{\partial f}{\partial t^j}\,dt^{I_p}\otimes\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)(\overline{dt^i}\wedge\overline{dt^{J_q}})\\
&+\sum_{i\in I_p\cap J_{n-q}}\sum_{j\in J_q\cup\{i\}}\frac{\partial f}{\partial t^j}\,dt^{I_p}\otimes\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)(\overline{dt^i}\wedge\overline{dt^{J_q}})\\
&+\sum_{i\in I_p\cap J_{n-q}}\sum_{j\in I_p\cap(J_q\cup\{i\})} f\,dt^{I_p}\otimes\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)(\overline{dt^i}\wedge\overline{dt^{J_q}}),
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}\bar{\delta}\omega = &-\sum_{j\in J_q}\sum_{i\in J_{n-q}\cup\{j\}}\frac{\partial^2 f}{\partial t^i\partial t^j}\,dt^{I_p}\otimes\overline{dt^i}\wedge\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)\overline{dt^{J_q}}\\
&+\sum_{j\in J_q}\sum_{i\in I_p\cap(J_{n-q}\cup\{j\})}\frac{\partial f}{\partial t^j}\,dt^{I_p}\otimes\overline{dt^i}\wedge\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)\overline{dt^{J_q}}\\
&-\sum_{j\in I_p\cap J_q}\sum_{i\in J_{n-q}\cup\{j\}}\frac{\partial f}{\partial t^i}\,dt^{I_p}\otimes\overline{dt^i}\wedge\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)\overline{dt^{J_q}}\\
&+\sum_{j\in I_p\cap J_q}\sum_{i\in I_p\cap(J_{n-q}\cup\{j\})} f\,dt^{I_p}\otimes\overline{dt^i}\wedge\bar{i}\Big(\frac{\partial}{\partial t^j}\Big)\overline{dt^{J_q}}.
\end{aligned}$$

We denote by $(\bar{\delta}\bar{\partial}\omega)_k$ and $(\bar{\partial}\bar{\delta}\omega)_k$ the k-th terms of $\bar{\delta}\bar{\partial}\omega$ and $\bar{\partial}\bar{\delta}\omega$ respectively, where $k=1,2,3,4$. Then we have

$$\begin{aligned}
(\bar{\delta}\bar{\partial}\omega)_1+(\bar{\partial}\bar{\delta}\omega)_1 &= -\sum_{j=1}^{n}\Big(\frac{\partial}{\partial t^j}\Big)^2 f\,dt^{I_p}\otimes\overline{dt^{J_q}},\\
(\bar{\delta}\bar{\partial}\omega)_2+(\bar{\partial}\bar{\delta}\omega)_3 &= -\sum_{j\in I_p}\frac{\partial f}{\partial t^j}\,dt^{I_p}\otimes\overline{dt^{J_q}},\\
(\bar{\delta}\bar{\partial}\omega)_3+(\bar{\partial}\bar{\delta}\omega)_2 &= \sum_{j\in I_p}\frac{\partial f}{\partial t^j}\,dt^{I_p}\otimes\overline{dt^{J_q}},\\
(\bar{\delta}\bar{\partial}\omega)_4+(\bar{\partial}\bar{\delta}\omega)_4 &= \sum_{j\in I_p} f\,dt^{I_p}\otimes\overline{dt^{J_q}} = pf\,dt^{I_p}\otimes\overline{dt^{J_q}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.4.4. *For $\omega\in A_0^{p,q}(\mathbb{R}^n_+)$ we have*

$$\|\bar{\partial}\omega\|^2+\|\bar{\delta}\omega\|^2\geq p\|\omega\|^2.$$

Proof. A (p,q)-form $\omega \in A_0^{p,q}(\mathbb{R}_+^n)$ is expressed by $\omega = \sum_{I_p, J_q} \omega_{I_p J_q} dt^{I_p} \otimes \overline{dt^{J_q}}$. By Lemma 3.4.1 and Proposition 3.4.3 we obtain

$$\begin{aligned} \|\bar\partial\omega\|^2 + \|\bar\delta\omega\|^2 &= (\bar\Box\,\omega, \omega) \\ &= \sum_{I_p, J_q} (\Delta\,\omega_{I_p J_q}, \omega_{I_p J_q}) + p\|\omega\|^2 \\ &\geq p\|\omega\|^2. \end{aligned}$$

□

Using the above, we have Main Theorem 4 similarly to Main Theorem 1.

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