

Accurate trajectory-harps for Kähler magnetic fields

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Abstract. In preceding papers we gave estimates on string-lengths, string-cosines and zenith angles of trajectory-harps under the condition that sectional curvatures of the underlying manifold are bounded from above. In this paper we study the cases that equalities hold in these estimates. Refining the previous proofs we give conditions that trajectory-harps are congruent to trajectory-harps on a complex space form.

1. Introduction.

A constant multiple $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ ($\kappa \in \mathbb{R}$) of the Kähler form \mathbb{B}_J on a complete Kähler manifold (M, J) with complex structure J is said to be a *Kähler magnetic field*. We call a smooth curve γ parameterized by its arclength a *trajectory* for \mathbb{B}_κ if it satisfies the differential equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa J\dot{\gamma}$. Since trajectories are geodesics when $\kappa = 0$, and as trajectories induce a dynamical system on the unit tangent bundle of M , we may say that trajectories are generalizations of geodesics (see [1] and also [8], [11]). Moreover, as J is parallel with respect to Riemannian connection ∇ , trajectories are circles. Here, a smooth curve σ parameterized by its arclength is said to be a circle if it satisfies $\nabla_{\dot{\sigma}}\dot{\sigma} = kY$, $\nabla_{\dot{\sigma}}Y = -k\dot{\sigma}$ with a field Y of unit vectors and a nonnegative constant k . We may hence say that trajectories are simplest curves next to geodesics from the viewpoint of Frenet-Serre formula, and are curves which are closely related with complex structure on the underlying manifold. The author therefore intend to study Kähler manifolds by investigating trajectories. In [2], [3], by studying variations of trajectories, he defined magnetic Jacobi fields and gave a result corresponding to Rauch's comparison theorem. As applications of this result, Bai and the author [6], [7] showed comparison theorems on volumes of trajectory-balls and trajectory-spheres.

To show some global properties of the underlying Kähler manifold, he studied trajectory-harps in [4], [5], which are variations of geodesics associated with trajectories. He compared trajectory-harps with those on a complex space form under the condition that sectional curvatures are bounded from above, and showed a result corresponding to Hadamard–Cartan theorem on a Kähler manifold of negative curvature. Unfortunately, being different from Toponogov's comparison theorem on triangles, his proof of a comparison theorem on trajectory-harps was based on *reductio ad absurdum*, hence he could not study the case that equality holds. In this paper, by mixing the arguments in [4], [5]

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and refining them we show that if an equality on string-lengths holds or if that on zenith angles holds in our comparison theorems on trajectory-harps then this trajectory-harp is congruent to that on a complex space form.

2. Trajectory-harps.

On a complete Kähler manifold M , every trajectory for a Kähler magnetic field \mathbb{B}_κ is defined on a whole real line \mathbb{R} . When we restrict a trajectory to a finite closed interval we call it a trajectory-segment, and when we restrict it to a infinite interval like $[0, \infty)$ and $(-\infty, 0]$ we call it a trajectory half-line. Still, for the sake of simplicity, we shall call trajectory-segment and trajectory half-line also a trajectory.

Let $\gamma : [0, T] \rightarrow M$ be a trajectory for a non-trivial Kähler magnetic field \mathbb{B}_κ ($\kappa \neq 0$). That is, it is a trajectory-segment when $T < \infty$ and is a trajectory half-line when $T = \infty$. We suppose $\gamma(t) \neq \gamma(0)$ for $0 < t < T$. A smooth variation $\alpha_\gamma : [0, T] \times \mathbb{R} \rightarrow M$ of geodesics is said to be a *trajectory-harp* associated with γ if it satisfies the following conditions:

- i) $\alpha_\gamma(t, 0) = \gamma(0)$ for all $t \in [0, T]$,
- ii) the curve $s \mapsto \alpha_\gamma(0, s)$ is the geodesic of initial vector $\dot{\gamma}(0)$,
- iii) the curve $s \mapsto \alpha_\gamma(t, s)$ is a geodesic of unit speed which joins $\gamma(0)$ and $\gamma(t)$ when $\gamma(t) \neq \gamma(0)$.

We call the trajectory and each geodesic segment joining two points of the trajectory the *arch* and a *string* of this trajectory-harp, respectively. When the image $\gamma([0, T])$ is contained in the geodesic ball $B_{\iota_p}(p)$ centered at $p = \gamma(0)$ and of radius ι_p of injectivity at p , by joining the unique minimal geodesic of $\gamma(0)$ and $\gamma(t)$ for each t , we obtain a unique trajectory-harp associated with γ . Therefore, when the image $\gamma([0, T])$ is contained in $B_{c_p}(p)$ whose radius is the minimal c_p of first conjugate values at p , we have a trajectory-harp associated with γ . Given a trajectory-harp α_γ associated with a trajectory γ for \mathbb{B}_κ we denote by $l_\gamma(t)$ the length of geodesic joining $\gamma(0)$ and $\gamma(t)$, and call it the *string-length* at t . We set $\delta_\gamma(t) = \langle \dot{\gamma}(t), (\partial\alpha_\gamma/\partial s)(t, l_\gamma(t)) \rangle$ and call it the *string-cosine* at t . It is known that the differential of the string-length is the string-cosine ($l'_\gamma(t) = \delta_\gamma(t)$).

On a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c , which is a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space \mathbb{C}^n and a complex hyperbolic space $\mathbb{C}H^n(c)$ according as $c > 0$, $c = 0$ and $c < 0$, these functions are expressed explicitly. For a constant C , we define a function $\mathfrak{s}(t; C)$ by

$$\mathfrak{s}(t; C) = \begin{cases} (1/\sqrt{C}) \sin \sqrt{C} t, & \text{when } C > 0, \\ t, & \text{when } C = 0, \\ (1/\sqrt{|C|}) \sinh \sqrt{|C|} t, & \text{when } C < 0. \end{cases}$$

Since a complex space form is a symmetric space of rank one, two trajectories γ_1, γ_2 for $\mathbb{B}_{\kappa_1}, \mathbb{B}_{\kappa_2}$ are congruent to each other if and only if $|\kappa_1| = |\kappa_2|$. Here we say two

smooth curves γ_1, γ_2 are congruent to each other if there are an isometry φ and a constant t_c satisfying $\gamma_2(t) = \varphi \circ \gamma_1(t + t_c)$ for all t . Therefore, string-lengths and string-cosines of trajectory-harps only depend on strengths of Kähler magnetic fields. The string-length $\ell_\kappa(\cdot; c)$ of a trajectory-harp for \mathbb{B}_κ on $\mathbb{C}M^n(c)$ is given by the relationship $\mathfrak{s}(\ell_\kappa(t; c)/2; c) = \mathfrak{s}(t/2; \kappa^2 + c)$, and its string-cosine $\delta_\kappa(t; c)$ is given as

$$\delta_\kappa(t; c) = \begin{cases} \frac{\sqrt{\kappa^2 + c} \cos(\sqrt{\kappa^2 + c} t/2)}{\sqrt{\kappa^2 + c \cos^2(\sqrt{\kappa^2 + c} t/2)}}, & \text{when } \kappa^2 + c > 0, \\ 2/\sqrt{|c|t^2 + 4}, & \text{when } \kappa^2 + c = 0, \\ \frac{\sqrt{|c| - \kappa^2} \cosh(\sqrt{|c| - \kappa^2} t/2)}{\sqrt{|c| \cosh^2(\sqrt{|c| - \kappa^2} t/2) - \kappa^2}}, & \text{when } \kappa^2 + c < 0. \end{cases}$$

Thus, the string-length $\ell_\kappa(\cdot; c)$ is monotone increasing in the interval $[0, \pi/\sqrt{\kappa^2 + c}]$, where we regard $\pi/\sqrt{\kappa^2 + c}$ as infinity when $\kappa^2 + c \leq 0$. We frequently use such a convention throughout of this paper. The function $\delta_\kappa(\cdot; c)$ is monotone decreasing in the interval $[0, 2\pi/\sqrt{\kappa^2 + c}]$.

For the sake of our study we briefly recall comparison theorems on trajectory-harps given in [4], [5]. In [4] we estimate string-lengths under a condition that sectional curvatures are bounded from above. For a trajectory-harp α_γ associated with a trajectory $\gamma : [0, T] \rightarrow M$ for \mathbb{B}_κ and for a constant c , we set $R_\gamma = \sup\{t \mid \delta_\gamma(t) > 0\} (\leq T)$, and set $T_\gamma(c)$ so that $T_\gamma(c) = \min\{t_*\}$ if there is t_* satisfying $0 < t_* \leq T$ and $\ell_\gamma(t_*) = \ell_\kappa(\pi/\sqrt{\kappa^2 + c}; c)$ and $T_\gamma(c) = T$ in other case. We denote by $\tau_\kappa(\cdot; c) : [0, \ell_\kappa(\pi/\sqrt{\kappa^2 + c}; c)] \rightarrow \mathbb{R}$ the inverse function of the function $\ell_\kappa(\cdot; c) : [0, \pi/\sqrt{\kappa^2 + c}] \rightarrow \mathbb{R}$.

PROPOSITION 1 ([4]). *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \rightarrow M$ for a non-trivial Kähler magnetic field \mathbb{B}_κ on a Kähler manifold M whose sectional curvature satisfy $\text{Riem}^M \leq c$ with a constant c . We have the following estimates on string-lengths and string-cosines:*

- (1) $\ell_\gamma(t) \geq \ell_\kappa(t; c)$ for $0 \leq t \leq \min\{R_\gamma, 2\pi/\sqrt{\kappa^2 + c}\}$,
- (2) $\delta_\gamma(t) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $0 \leq t \leq T_\gamma(c)$.

In particular, we have $\min\{R_\gamma, \pi/\sqrt{\kappa^2 + c}\} \geq T_\gamma(c)$.

The above is an estimate on “lengths” of trajectory-harps. We estimate “fatness” of trajectory-harps in [5]. For a trajectory-harp α_γ associated with a trajectory $\gamma : [0, T] \rightarrow M$ and a constants a, b with $0 \leq a < b \leq T$, we call the restriction of α_γ to $[a, b] \times \mathbb{R}$ a subharp. The length $\vartheta_\gamma(a, b)$ of the curve $[a, b] \ni t \mapsto (\partial\alpha_\gamma/\partial s)(t, 0) \in U_{\gamma(0)}M$ in the unit tangent space is called the *zenith angle* of this subharp. Trivially the angle between two strings are estimated from above as $\angle((\partial\alpha_\gamma/\partial s)(a, 0), (\partial\alpha_\gamma/\partial s)(b, 0)) \leq \vartheta_\gamma(a, b)$.

PROPOSITION 2 ([5]). *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \rightarrow M$ for a non-trivial Kähler magnetic field \mathbb{B}_κ on a Kähler manifold M*

whose sectional curvature satisfy $\text{Riem}^M \leq c$ with a constant c . For arbitrary a, b with $0 \leq a < b \leq T_\gamma(c)$, we set $\hat{a} = \tau_\kappa(\ell_\gamma(a); c)$, $\hat{b} = \tau_\kappa(\ell_\gamma(b); c)$.

(1) the zenith angle of a subharp is estimated as

$$\vartheta_\gamma(a, b) \leq \vartheta_\kappa(\hat{a}, \hat{b}; c) := \cos^{-1} \delta_\kappa(\hat{b}; c) - \cos^{-1} \delta_\kappa(\hat{a}; c);$$

(2) the length of the arch of this subharp is estimated as $b - a \leq \hat{b} - \hat{a}$.

For a trajectory-harp α_γ associated with a trajectory $\gamma : [0, T] \rightarrow M$ and t_0 with $0 < t_0 \leq T$, we set

$$\mathcal{H}_\gamma(t_0) = \{ \alpha_\gamma(t, s) \mid 0 \leq t \leq t_0, 0 \leq s \leq \ell_\gamma(t) \},$$

and call it a *harp-body* of this trajectory-harp.

REMARK 1. Since Propositions 1 and 2 are based on Rauch’s comparison theorem applied to Jacobi fields tangent to harp-bodies, we can weaken the condition on sectional curvatures of the underlying manifold to the condition that sectional curvatures of planes tangent to the harp-body $\mathcal{H}_\gamma(T)$ are not greater than c . Needless to say that if $\text{Riem}^M \leq c$ then every trajectory-harp satisfies this curvature condition on its harp-body.

3. String-lengths.

In this section we study the case that the equality on string-lengths holds in the comparison theorem on trajectory-harps (Propositions 1). In Toponogov’s comparison theorem on triangles we suppose that two triangles have equi-lengths of three edges. Since we only suppose that two trajectory-harps have the same lengths of arches and the same lengths of strings, it seems that our condition is weaker than the condition in Toponogov’s comparison theorem. But we note that string-cosines are derivatives of string-lengths and they give us information on angles between arches and strings. Our main result in this section is the following.

THEOREM 1. Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \rightarrow M$ for \mathbb{B}_κ on a Kähler manifold M . Suppose sectional curvatures of planes tangent to its harp-body $\mathcal{H}_\gamma(T)$ are not greater than c . If $\ell_\gamma(t_0) = \ell_\kappa(t_0; c)$ holds for some t_0 satisfying $0 \leq t_0 \leq T_\gamma(c)$, then the harp-body $\mathcal{H}_\gamma(t_0)$ is totally geodesic, holomorphic and of constant sectional curvature c . In particular $\ell_\gamma(t) = \ell_\kappa(t; c)$ and $\delta_\gamma(t) = \delta_\kappa(t; c)$ hold for $0 \leq t \leq t_0$.

In order to show this, we study the case that the equality on string-cosine holds in Proposition 1.

PROPOSITION 3. Let α_γ be a trajectory-harp for \mathbb{B}_κ on a Kähler manifold M . Suppose sectional curvatures of planes tangent to its harp-body $\mathcal{H}_\gamma(T)$ are not greater than c . If $\delta_\gamma(t_0) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$ holds at some t_0 with $0 < t_0 \leq T_\gamma(c)$, then we have the following:

- 1) The derivatives of string-cosines satisfy $\delta'_\gamma(t_0) = \delta'_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$;
- 2) The vector $(\partial\alpha_\gamma/\partial t)(t_0, s)$ is parallel to $J(\partial\alpha_\gamma/\partial s)(t_0, s)$ for $0 \leq s \leq \ell_\gamma(t_0)$;
- 3) The sectional curvature $\text{Riem}((\partial\alpha_\gamma/\partial t)(t_0, s), (\partial\alpha_\gamma/\partial s)(t_0, s))$ of the tangent plane spanned by $(\partial\alpha_\gamma/\partial t)(t_0, s)$ and $(\partial\alpha_\gamma/\partial s)(t_0, s)$ is equal to c for $0 < s \leq \ell_\gamma(t_0)$.

PROOF. We put $Z_t(s) = (\partial\alpha_\gamma/\partial t)(t, s)$, which is a Jacobi field along a string $s \mapsto \alpha_\gamma(t, s)$. By direct computation we have

$$\delta'_\gamma(t) = \kappa \left\langle J\dot{\gamma}(t), \frac{\partial\alpha_\gamma}{\partial s}(t, \ell_\gamma(t)) \right\rangle + \left\langle \dot{\gamma}(t), (\nabla_{\partial\alpha_\gamma/\partial s} Z_t)(\ell_\gamma(t)) \right\rangle. \tag{3.1}$$

The first term of the right hand side of (3.1) is estimated as

$$\kappa \left\langle J\dot{\gamma}(t), \frac{\partial\alpha_\gamma}{\partial s}(t, \ell_\gamma(t)) \right\rangle \geq -|\kappa| \sqrt{1 - \delta_\gamma(t)^2}. \tag{3.2}$$

Since $\gamma(t) = \alpha_\gamma(t, \ell_\gamma(t))$, we have $Z_t(\ell_\gamma(t)) = \dot{\gamma}(t) - \delta_\gamma(t)(\partial\alpha_\gamma/\partial s)(t, \ell_\gamma(t))$. In particular, we have $\|Z_t(\ell_\gamma(t))\|^2 = 1 - \delta_\gamma(t)^2$. As $Z_t(s)$ is orthogonal to $(\partial\alpha_\gamma/\partial s)(t, s)$, the second term of the right hand side of (3.1) is expressed as

$$\left\langle \dot{\gamma}(t), (\nabla_{\partial\alpha_\gamma/\partial s} Z_t)(\ell_\gamma(t)) \right\rangle = \{1 - \delta_\gamma(t)^2\} \times \frac{\langle Z_t(\ell_\gamma(t)), (\nabla_{\partial\alpha_\gamma/\partial s} Z_t)(\ell_\gamma(t)) \rangle}{\|Z_t(\ell_\gamma(t))\|^2}.$$

We take a trajectory-harp $\hat{\alpha}_{\hat{\gamma}}$ associated with a trajectory $\hat{\gamma}$ for \mathbb{B}_κ on $\mathbb{C}M^n(c)$. We put $\hat{Z}_t(s) = (\partial\hat{\alpha}_{\hat{\gamma}}/\partial t)(t, s)$. Since $t_0 \leq T_\gamma(c)$, we have $\ell_\gamma(t_0) \leq \ell_\kappa(\pi/\sqrt{\kappa^2 + c}; c) \leq \pi/\sqrt{\kappa^2 + c}$. As $\delta_\gamma(t_0) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$, Rauch's comparison theorem (see [9], [10], for example) shows that

$$\begin{aligned} & \left\langle \dot{\gamma}(t_0), (\nabla_{\partial\alpha_\gamma/\partial s} Z_{t_0})(\ell_\gamma(t_0)) \right\rangle \\ & \geq \{1 - \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)^2\} \\ & \quad \times \frac{\langle \hat{Z}_{\tau_\kappa(\ell_\gamma(t_0); c)}(\ell_\kappa(t_0; c)), (\nabla_{\partial\hat{\alpha}_{\hat{\gamma}}/\partial s} \hat{Z}_{\tau_\kappa(\ell_\gamma(t_0); c)})(\ell_\kappa(t_0; c)) \rangle}{\|\hat{Z}_{\tau_\kappa(\ell_\gamma(t_0); c)}(\ell_\kappa(t_0; c))\|^2} \\ & = \left\langle \dot{\hat{\gamma}}(\tau_\kappa(\ell_\gamma(t_0); c); c), (\nabla_{\partial\hat{\alpha}_{\hat{\gamma}}/\partial s} \hat{Z}_{\tau_\kappa(\ell_\gamma(t_0); c)})(\ell_\kappa(t_0; c)) \right\rangle. \end{aligned} \tag{3.3}$$

By (3.2) and (3.3) we have

$$\delta'_\gamma(t_0) \geq \delta'_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c) = \frac{d}{dt} \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c) \Big|_{t=t_0}$$

because $\ell'_\gamma(t_0) = \delta_\gamma(t_0) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c) = \ell'_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$. If we suppose $\delta'_\gamma(t_0) > (d/dt)\delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)|_{t=t_0}$, as we have $\delta_\gamma(t) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $0 \leq t \leq T_\gamma(c)$, we find that $\delta_\gamma(t_0) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$ leads us to a contradiction. Hence we see $\delta'_\gamma(t_0) = \delta'_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$.

We now go back to inequalities (3.2) and (3.3). Since we have $\delta'_\gamma(t_0) = \delta'_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$, we see equalities hold in these equations. By the equality in (3.2) we find that $\dot{\gamma}(t_0)$ is contained in the plane in $T_{\gamma(t_0)}M$ which is spanned by $(\partial\alpha_\gamma/\partial s)(t_0, \ell_\gamma(t_0))$ and $J(\partial\alpha_\gamma/\partial s)(t_0, \ell_\gamma(t_0))$, hence find that $Z_{t_0}(\ell_\gamma(t_0))$ is parallel to $J(\partial\alpha_\gamma/\partial s)(t_0, \ell_\gamma(t_0))$. The equality in (3.3) shows that the sectional curvature satisfies $\text{Riem}((\partial\alpha_\gamma/\partial t)(t_0, s), (\partial\alpha_\gamma/\partial s)(t_0, s)) = c$ and that the vector field $Z_{t_0}/\|Z_{t_0}\|$ is parallel along the string $s \mapsto \alpha_\gamma(t_0, s)$ for $0 \leq s \leq \ell_\gamma(t_0)$. Thus we get the conclusion. \square

PROOF OF THEOREM 1. By the assumption we have $t_0 = \tau_\kappa(\ell_\gamma(t_0); c)$. As we have $\tau_\kappa(\ell_\gamma(0); c) = 0$, we find by using Proposition 1 that

$$t_0 = \int_0^{t_0} \frac{d}{dt} \tau_\kappa(\ell_\gamma(t); c) dt = \int_0^{t_0} \frac{\delta_\gamma(t)}{\delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)} dt \geq \int_0^{t_0} dt = t_0.$$

Hence we obtain $\delta_\gamma(t) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $0 \leq t \leq t_0$. Thus, we find by Proposition 3 that $\text{Riem}((\partial\alpha_\gamma/\partial t)(t, s), (\partial\alpha_\gamma/\partial s)(t, s)) = c$ and $(\partial\alpha_\gamma/\partial t)(t, s) = \psi(t, s)J(\partial\alpha_\gamma/\partial s)(t, s)$ with a smooth function $\psi(t, s)$ for $0 \leq t \leq t_0$ and $0 \leq s \leq \ell_\gamma(t)$. As $s \mapsto \alpha_\gamma(t, s)$ is a geodesic for each t , this expression of $\partial\alpha_\gamma/\partial t$ shows that $\mathcal{H}_\gamma(t_0)$ is totally geodesic. \square

REMARK 2. Proposition 3 and the proof of Theorem 1 show the following on a trajectory-harp α_γ associated with a trajectory $\gamma : [0, T] \rightarrow M$ for \mathbb{B}_κ . Suppose sectional curvatures of planes tangent to its harp-body $\mathcal{H}_\gamma(T)$ are not greater than c . If its string-cosine satisfies $\delta_\gamma(t) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $0 \leq t \leq t_0$, then the harp-body $\mathcal{H}_\gamma(t_0)$ is totally geodesic, holomorphic and of constant sectional curvature c .

We note that Proposition 1 does not tell on the relationship of string-cosines at the same arch-length t of trajectory-harps. The author hence does not have idea whether the condition $\delta_\gamma(t_0) = \delta_\kappa(t_0; c)$ tells on feature of the trajectory-harp or not.

To study more on congruency of trajectory-harps we here introduce some indicator functions. Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \rightarrow M$ for a Kähler magnetic field \mathbb{B}_κ on M . For a constant c , we set $S_\gamma(c)$ so that $S_\gamma(c) = \min\{s_*\}$ if there is s_* satisfying $0 < s_* \leq T$ and $\ell_\kappa(s_*; c) = R_\gamma$, and $S_\gamma(c) = T$ in other case. We denote by $\tau_\gamma : [0, \ell_\gamma(R_\gamma)] \rightarrow \mathbb{R}$ the inverse function of $\ell_\gamma : [0, R_\gamma] \rightarrow \mathbb{R}$. We define three functions $L_\gamma(\cdot; c) : [0, T_\gamma(c)] \rightarrow \mathbb{R}$, $U_\gamma(\cdot; c) : [0, S_\gamma(c)] \rightarrow \mathbb{R}$ and $D_\gamma(\cdot; c) : [0, T_\gamma(c)] \rightarrow \mathbb{R}$ by

$$\begin{aligned} L_\gamma(t; c) &= \int_0^t \delta_\kappa(\tau_\kappa(\ell_\gamma(u); c); c) du, \\ U_\gamma(u; c) &= \int_0^u \delta_\gamma(\tau_\gamma(\ell_\kappa(t; c))) dt, \\ D_\gamma(t; c) &= 1 + \int_0^t \delta'_\kappa(\tau_\kappa(\ell_\gamma(u); c); c) du. \end{aligned}$$

We call $L_\gamma(\cdot; c)$ and $U_\gamma(\cdot; c)$ the c -inner tuning length and c -outer tuning length of this trajectory-harp, respectively. In view of the estimate on string-cosines in Proposition 1,

one can easily see that these functions are closely related to string-lengths. We call $D_\gamma(\cdot; c)$ the c -tuning cosine of this trajectory-harp,

PROPOSITION 4. *Let α_γ be a trajectory-harp for \mathbb{B}_κ associated with a trajectory $\gamma : [0, T] \rightarrow M$. If sectional curvatures of planes tangent to $\mathcal{H}_\gamma(T)$ are not greater than c , then we have the following.*

- (1) $\ell_\kappa(t; c) \geq L_\gamma(t; c)$ for $0 \leq t \leq T_\gamma(c)$.
- (2) If the equality $\ell_\kappa(t_0; c) = L_\gamma(t_0; c)$ holds at some t_0 with $0 < t_0 \leq T_\gamma(c)$, then the harp-body $\mathcal{H}_\gamma(t_0)$ is totally geodesic, holomorphic and of constant sectional curvature c . In particular, $\ell_\gamma(t) = \ell_\kappa(t; c)$ and $\delta_\gamma(t) = \delta_\kappa(t; c)$ hold for $0 \leq t \leq t_0$.
- (3) $U_\gamma(u; c) \geq \ell_\kappa(u; c)$ for $0 \leq u \leq \min\{T, \pi/\sqrt{\kappa^2 + c}\}$.
- (4) If the equality $U_\gamma(u_0; c) = \ell_\kappa(u_0; c)$ holds at some u_0 with $0 < u_0 \leq \min\{T, \pi/\sqrt{\kappa^2 + c}\}$, then the harp-body $\mathcal{H}_\gamma(\tau_\gamma(\ell_\kappa(u_0; c))) = \mathcal{H}_\gamma(u_0)$ is totally geodesic, holomorphic and of constant sectional curvature c . In particular, $\ell_\gamma(u) = \ell_\kappa(u; c)$ and $\delta_\gamma(t) = \delta_\kappa(t; c)$ hold for $0 \leq u \leq u_0$.

PROOF. By Proposition 1 and Remark 1, we have $\ell_\gamma(u) \geq \ell_\kappa(u; c)$ for $0 \leq u \leq T_\gamma(c)$. As $\tau_\kappa(\cdot; c)$ is monotone increasing and $\delta_\kappa(\cdot; c)$ is monotone decreasing, we find

$$\delta_\kappa(\tau_\kappa(\ell_\gamma(u); c); c) \leq \delta_\kappa(\tau_\kappa(\ell_\kappa(u; c); c); c) = \delta_\kappa(u; c).$$

Therefore we obtain $L_\gamma(t; c) \leq \int_0^t \delta_\kappa(u; c) du = \ell_\kappa(t; c)$ for $0 \leq t \leq T_\gamma(c)$. The condition $\ell_\kappa(t_0; c) = L_\gamma(t_0; c)$ shows that $\delta_\kappa(u; c) = \delta_\kappa(\tau_\kappa(\ell_\gamma(u); c); c)$ for $0 \leq u \leq t_0$. If we suppose $\ell_\gamma(t_1) > \ell_\kappa(t_1; c)$ at some t_1 with $0 < t_1 < t_0$, we have

$$\delta_\kappa(t_1; c) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_1); c); c) < \delta_\kappa(\tau_\kappa(\ell_\kappa(t_1; c); c); c) = \delta_\kappa(t_1; c),$$

which is a contradiction. Therefore we obtain $\ell_\gamma(t) = \ell_\kappa(t; c)$ for $0 \leq t \leq t_0$. We obtain the second assertion from Theorem 1.

Next we study c -outer tuning lengths. As we have $\delta_\gamma(u) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(u); c); c)$ for $0 \leq u \leq T_\gamma(c)$, by putting $t = \tau_\kappa(\ell_\gamma(u); c)$ we have $u = \tau_\gamma(\ell_\kappa(t; c))$, hence have $\delta_\gamma(\tau_\gamma(\ell_\kappa(t; c))) \geq \delta_\kappa(t; c)$ for $0 \leq t \leq \min\{T, \pi/\sqrt{\kappa^2 + c}\}$. Therefore, we obtain $U_\gamma(u; c) \geq \int_0^u \delta_\kappa(t; c) dt = \ell_\kappa(u; c)$ for $0 \leq u \leq \min\{T, \pi/\sqrt{\kappa^2 + c}\}$. The condition $U_\gamma(u_0; c) = \ell_\kappa(u_0; c)$ shows that $\delta_\gamma(\tau_\gamma(\ell_\kappa(t; c))) = \delta_\kappa(t; c)$ for $0 \leq t \leq u_0$. That is, we have $\delta_\gamma(u) = \delta_\kappa(\tau_\kappa(\ell_\gamma(u); c); c)$ for $0 \leq u \leq \tau_\gamma(\ell_\kappa(u_0; c))$. We obtain the fourth assertion by Remark 2. □

REMARK 3. It is known that $\delta'_\gamma(0) = 0$ and $\delta''_\gamma(0) = -\kappa^2/4$ for a trajectory-harp α_γ for \mathbb{B}_κ (see [4]). Therefore δ_γ is monotone decreasing on some interval $[0, \epsilon]$. On this interval, as $\ell_\gamma(t) \geq \ell_\kappa(t; c)$, we find $U_\gamma(t; c) \geq \ell_\gamma(t)$.

PROPOSITION 5. *Let α_γ be a trajectory-harp for \mathbb{B}_κ associated with a trajectory $\gamma : [0, T] \rightarrow M$. If sectional curvatures of planes tangent to $\mathcal{H}_\gamma(T)$ are not greater than c , then we have the following.*

- (1) $D_\gamma(t; c) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $0 \leq t \leq T_\gamma(c)$.
- (2) If the equality $D_\gamma(t_0; c) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$ holds at some t_0 with $0 < t_0 \leq T_\gamma(c)$, then the harp-body $\mathcal{H}_\gamma(t_0)$ is totally geodesic, holomorphic and of constant sectional curvature c . In particular, $\ell_\gamma(t) = \ell_\kappa(t; c)$ and $\delta_\gamma(t) = \delta_\kappa(t; c)$ hold for $0 \leq t \leq t_0$.

PROOF. Since $\delta_\kappa(\cdot; c)$ is monotone decreasing, we have

$$\begin{aligned} D_\gamma(t; c) &\geq 1 + \int_0^t \delta'_\kappa(\tau_\kappa(\ell_\gamma(u); c); c) \times \frac{\delta_\gamma(u)}{\delta_\kappa(\tau_\kappa(\ell_\gamma(u); c); c)} du \\ &= 1 + \int_0^{\tau_\kappa(\ell_\gamma(t); c)} \delta'_\kappa(s; c) ds = \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c). \end{aligned}$$

Thus, if $D_\gamma(t_0; c) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t_0); c); c)$ holds, then we have $\delta_\gamma(t) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $0 \leq t \leq t_0$. We hence get the conclusion by Remark 2. □

4. Zenith angles.

In this section we study congruency of trajectory-harps through zenith angles. We consider the cases that equalities hold in Proposition 2.

THEOREM 2. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \rightarrow M$ for \mathbb{B}_κ on a Kähler manifold M . Suppose sectional curvatures of planes tangent to $\mathcal{H}_\gamma(T)$ are not greater than c . We take a subharp $\alpha_\gamma|_{[a, b] \times \mathbb{R}}$ with $0 \leq a < b \leq T_\gamma(c)$ and set $\hat{a} = \tau_\kappa(\ell_\gamma(a); c)$, $\hat{b} = \tau_\kappa(\ell_\gamma(b); c)$.*

- (1) If $\vartheta_\gamma(a, b) = \cos^{-1} \delta_\kappa(\hat{b}; c) - \cos^{-1} \delta_\kappa(\hat{a}; c)$ holds, then the harp-body $\mathcal{H}_\gamma(a, b) = \{\alpha_\gamma(t, s) \mid a \leq t \leq b, 0 \leq s \leq \ell_\gamma(t)\}$ of this subharp is totally geodesic, holomorphic and of constant sectional curvature c . Moreover, $\ell_\gamma(t) = \ell_\kappa(t; c)$ and $\delta_\gamma(t) = \delta_\kappa(t; c)$ hold for $a \leq t \leq b$.
- (2) If the length of the arch satisfies $b - a = \hat{b} - \hat{a}$, we have the same conclusion as that in (1).

PROOF. We use the same notations as in the proof of Proposition 3. By Rauch’s comparison theorem we have $\|Z_t(s)\| \geq \|(\nabla_{\partial\alpha_\gamma/\partial s} Z_t)(0)\| \mathfrak{s}(s; c)$. Thus by Proposition 1 we have

$$\begin{aligned} \vartheta_\gamma(a, b) &\leq \int_a^b \frac{\|Z_t(\ell_\gamma(t))\|}{\mathfrak{s}(\ell_\gamma(t); c)} dt = \int_a^b \frac{\sqrt{1 - \delta_\gamma(t)^2}}{\mathfrak{s}(\ell_\gamma(t); c)} dt \\ &\leq \int_a^b \frac{\sqrt{1 - \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)^2}}{\mathfrak{s}(\ell_\gamma(t); c)} dt = \int_a^b \frac{\|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}(\ell_\gamma(t))\|}{\mathfrak{s}(\ell_\gamma(t); c)} dt \\ &\leq \int_a^b \frac{\|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}(\ell_\gamma(t))\|}{\mathfrak{s}(\ell_\gamma(t); c)} \frac{\delta_\gamma(t)}{\delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)} dt \end{aligned}$$

$$= \int_{\hat{a}}^{\hat{b}} \frac{\|\widehat{Z}_u(\ell_\kappa(u; c))\|}{\mathfrak{s}(\ell_\kappa(u; c); c)} du = \vartheta_{\hat{\gamma}}(\hat{a}, \hat{b}) = \vartheta_\kappa(\hat{a}, \hat{b}; c).$$

Thus, if $\vartheta_\gamma(a, b) = \vartheta_\kappa(\hat{a}, \hat{b}; c)$ holds, then we have $\delta_\gamma(t) = \delta_\kappa(\tau_\kappa(\delta_\gamma(t); c); c)$ for $a \leq t \leq b$. Hence we get the first assertion.

As we have

$$\begin{aligned} b - a &= \int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b \|\dot{\gamma}(\tau_\kappa(\ell_\gamma(t); c); c)\| dt \\ &\leq \int_a^b \|\dot{\gamma}(\tau_\kappa(\ell_\gamma(t); c); c)\| \frac{\delta_\gamma(t)}{\delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)} dt = \int_{\hat{a}}^{\hat{b}} \|\dot{\gamma}(u)\| du = \hat{b} - \hat{a}, \end{aligned}$$

the condition shows that $\delta_\gamma(t) = \delta_\kappa(\tau_\kappa(\delta_\gamma(t); c); c)$ for $a \leq t \leq b$. Hence we get the second assertion. \square

In the second assertion of Theorem 2 we suppose that lengths of two strings and the length of arch are the same as the corresponding subharp in a complex space form. We may hence say that this assertion corresponds to the comparison theorem on geodesic triangles.

In [5], to show “fatness” of trajectory-harps, we studied sector-arcs of harp-sectors. Given a trajectory-harp $\alpha_\gamma : [0, T] \times \mathbb{R} \rightarrow M$ and constants a, b with $0 < a < b \leq T$, we consider the restriction $\alpha_\gamma^{a,b} = \alpha_\gamma|_{[a,b] \times [0, \ell_\gamma(a)]}$ and call it a harp-sector. Considering its sector-arc $[a, b] \ni t \mapsto \alpha_\gamma(t, \ell_\gamma(t))$, we denote its length by $sl_\gamma(a, b)$. When sectional curvatures of planes tangent to the harp-body $\mathcal{H}_\gamma(T)$ are not greater than c , we have

$$\mathfrak{s}(\ell_\gamma(a); c) \angle \left(\frac{\partial \alpha_\gamma}{\partial s}(a, 0), \frac{\partial \alpha_\gamma}{\partial s}(b, 0) \right) \leq sl_\gamma(a, b) \leq \mathfrak{s}(\ell_\gamma(a); c) \vartheta_\kappa(\hat{a}, \hat{b}; c)$$

with $\hat{a} = \tau_\kappa(\ell_\gamma(a); c)$, $\hat{b} = \tau_\kappa(\ell_\gamma(b); c)$.

PROPOSITION 6. *Let α_γ be a trajectory-harp associated with a trajectory $\gamma : [0, T] \rightarrow M$ for \mathbb{B}_κ on a Kähler manifold M . Suppose sectional curvatures of planes tangent to $\mathcal{H}_\gamma(T)$ are not greater than c . We take its harp-sector $\alpha_\gamma^{a,b}$ with $0 \leq a < b \leq T_\gamma(c)$. If $sl_\gamma(a, b) = \mathfrak{s}(\ell_\gamma(a); c) \vartheta_\kappa(\hat{a}, \hat{b}; c)$ holds, then the harp-body $\mathcal{H}_\gamma(a, b)$ is totally geodesic, holomorphic and of constant sectional curvature c . Moreover, $\ell_\gamma(t) = \ell_\kappa(t; c)$ for $a \leq t \leq b$.*

PROOF. We use the same notations as in the proof of Proposition 3. As we have

$$\|Z_t(\ell_\gamma(t))\| = \sqrt{1 - \delta_\gamma(t)^2} \leq \sqrt{1 - \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)^2} = \|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}(\ell_\gamma(t))\|,$$

Rauch’s comparison theorem guarantees that $\|Z_t(\ell)\| \leq \|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}(\ell)\|$ for $0 \leq \ell \leq \ell_\gamma(t)$. Thus we have

$$sl_\gamma(a, b) = \int_a^b \|Z_t(\ell_\gamma(t))\| dt \leq \int_a^b \|\widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}(\ell_\gamma(a))\| dt$$

$$\begin{aligned} &\leq \int_a^b \left\| \widehat{Z}_{\tau_\kappa(\ell_\gamma(t); c)}(\ell_\gamma(a)) \right\| \frac{\delta_\gamma(t)}{\delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)} dt \\ &= \int_{\hat{a}}^{\hat{b}} \left\| \widehat{Z}_u(\ell_\gamma(a)) \right\| du = \mathfrak{s}(\ell_\gamma(a); c) \vartheta_\kappa(\hat{a}, \hat{b}; c). \end{aligned}$$

Hence the equality shows that $\delta_\gamma(t) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for $a \leq t \leq b$. We therefore get the conclusion. \square

If $s\ell_\gamma(a, b) = \mathfrak{s}(\ell_\gamma(a); c) \angle((\partial\alpha_\gamma/\partial s)(a, 0), (\partial\alpha_\gamma/\partial s)(b, 0))$ holds under the assumption of $\text{Riem}^M \leq c$, we find by Rauch's comparison theorem that $\{\alpha_\gamma(t, s) \mid a \leq t \leq b, 0 \leq s \leq \ell_\gamma(a)\}$ is of constant sectional curvature c and $\{(\partial\alpha_\gamma/\partial s)(t, 0) \mid a \leq t \leq b\}$ is contained in a plane of $T_{\gamma(0)}M$.

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