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Alexander invariants of ribbon tangles and planar algebras

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Abstract. Ribbon tangles are proper embeddings of tori and cylinders in the 4-ball B^4, "bounding" 3-manifolds with only ribbon disks as singularities. We construct an Alexander invariant $\mathbf{A}$ of ribbon tangles equipped with a representation of the fundamental group of their exterior in a free abelian group G. This invariant induces a functor in a certain category $\mathbf{R}ib_G$ of tangles, which restricts to the exterior powers of Burau–Gassner representation for ribbon braids, that are analogous to usual braids in this context. We define a circuit algebra $\mathbf{C}ob_G$ over the operad of smooth cobordisms, inspired by diagrammatic planar algebras introduced by Jones [**Jon99**], and prove that the invariant $\mathbf{A}$ commutes with the compositions in this algebra. On the other hand, ribbon tangles admit diagrammatic representations, through welded diagrams. We give a simple combinatorial description of $\mathbf{A}$ and of the algebra $\mathbf{C}ob_G$, and observe that our construction is a topological incarnation of the Alexander invariant of Archibald [**Arc10**]. When restricted to diagrams without virtual crossings, $\mathbf{A}$ provides a purely local description of the usual Alexander poynomial of links, and extends the construction by Bigelow, Cattabriga and the second author [**BCF15**].

1. Introduction.

A *ribbon torus link* is a locally flat embedding of disjoint tori $S^1 \times S^1$ in the ball B^4, bounding locally flat immersed solid tori $S^1 \times D^2$ whose singular sets are finite numbers of *ribbon disks*. These elementary singularities are 4-dimensional analogues of the classical notion of ribbon introduced by Fox [**Fox73**]. Ribbon torus links are considered up to ambient isotopy; the fundamental groups of their complement and the derived Alexander modules provide topological invariants.

In this paper, we consider *ribbon tangles* that are ribbon proper embeddings of disjoint tori and cylinders in the ball B^4. The intersection of a given tangle T with $\partial B^4 = S^3$ is a trivial link L. For a given free abelian group G, tangles are *colored* with a group homomorphism $\varphi\colon H_1(B^4 \setminus T) \to G$. We construct an *Alexander invariant* $\mathbf{A}$ of colored ribbon tangles, lying in the exterior algebra of the homology $\mathbb{Z}[G]$-module $H_1^{\varphi}(S^3 \setminus L)$, twisted by the morphism induced by φ, see Definition 2.4. This invariant coincides with the Alexander polynomial in the case of tangles with only two boundary components, see Proposition 2.6. The construction of $\mathbf{A}$ is based on the *Alexander function* introduced by Lescop [**Les98**]. The proof of the invariance and the main properties follow from algebraic and homological arguments developed in a paper by the second author and Massuyeau [**FM15**].

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Ribbon colored tangles can be *splitted* into morphisms in a category $\mathbf{R}ib_G$. We show that the invariant $\mathbf{A}$ induces a functor from $\mathbf{R}ib_G$ to the category of $\mathbb{Z}[G]$-graded modules, see Theorem 2.10. In the case of *ribbon braids*, that are the analogues of braids in this context, the functor coincides with the exterior powers of the *ad-hoc* colored Burau–Gassner representation, multiplied by a certain relative Alexander polynomial, see Proposition 2.9.

The multiplicativity of $\mathbf{A}$ fits naturally in the context of planar algebras, introduced by Jones [**Jon99**]. We construct an algebra $\mathbf{C}ob_G$ over the colored *cobordisms* operad, see Theorem 3.3. The cobordisms are smooth cylinder in B^4 with a finite collection of disjoint "small" balls removed, and the compositions are given by identification of some boundary components. We observe that cobordisms act on tangles and that $\mathbf{A}$ commutes with this action, see Theorem 3.4.

Ribbon knotted objects admit representations as *broken surfaces* via projections on the 3-dimensional space, and through *welded diagrams* which are representations in the 2-dimensional space. These representations were first introduced by Yajima [**Yaj64**] and Satoh [**Sat00**]. A complete description is given in the papers by Audoux, Bellingeri, Meilhan and Wagner [**ABMW17**], [**Aud16**]. This offers perspectives from both point of view: diagrams permit combinatorial computations of invariants of ribbon knotted objects, and ribbon knotted objects provide topological realizations of welded diagrams. Moreover, a presentation of the group of ribbon knotted objects is obtained from the diagrams by a Wirtinger type algorithm, see [**ABMW17**].

We give a diagrammatic description of the invariant $\mathbf{A}$, see Theorem 4.6 and show that it extends the multivariable Alexander polynomial of virtual tangles developed by Archibald and Bar-Natan. For futher details on these objects, see also [**BNS13**] and [**Pol10**]. The algebra $\mathbf{C}ob_G$ appears as a topological incarnation of a circuit algebra of diagrams $\mathbf{W}eld_\mu$, essentially introduced in [**Arc10**]. The compatibility of $\mathbf{A}$ with this structure allows local calculations in the diagram. If these diagrams do not have any virtual crossing, the construction holds for usual links, and we obtain a purely local description of the usual Alexander polynomial. This extends the construction of the Alexander representation by the second author, Bigelow and Cattabriga, see [**BCF15**].

Our construction arises in the context of defining generalizations of Alexander polynomials to tangle-like objects, in which we can include, in addition to [**BCF15**] concerning usual tangles and [**Arc10**], [**Pol10**] concerning virtual tangles, also the works of Cimasoni and Turaev, through Lagrangian categories [**CT05**], Bigelow [**Big12**] and Kennedy [**Ken13**] which studied diagrammatical invariants of usual tangles, Sartori [**Sar15**], who defined quantum invariants of framed tangles, and Zibrowius in [**Zib16**]. This last one in particular defines an invariant for usual tangles which consists in a finite set of Laurent polynomials, and states, without explicit calculation, that on usual tangles, one can calculate Archibald's invariant from his set of invariants and vice versa. However, nothing seems to appear in the literature about the 4-dimensional case of ribbon tangles.

In Section 2, we recall the definitions of ribbon tangles and construct the invariant $\mathbf{A}$. Section 3 is devoted to the circuit algebra $\mathbf{C}ob_G$ over the cobordism operad, and the properties of $\mathbf{A}$ with respect with the action of cobordisms on tangles. In Section 4, we describe the diagrammatic construction of $\mathbf{A}$ and a circuit algebra $\mathbf{W}eld_\mu$ related

to $\mathbf{C}ob_G$. In Section 5, we compute some examples.

2. The Alexander invariant A.

In this section, G is a free abelian group, and R is the group ring $\mathbb{Z}[G]$.

2.1. Ribbon tangles.

Let m be a positive integer and X a submanifold of the m-dimensional ball B^m. An immersion $Y \subset X$ is *locally flat* if and only if it is locally homeomorphic to a linear subspace $\mathbb{R}^k$ in $\mathbb{R}^m$ for some $k \leq m$, except on ∂X and/or ∂Y, where one of the $\mathbb{R}$ summands should be replaced by $\mathbb{R}_+$. An intersection $Y_1 \cap Y_2 \subset X$ is *flatly transverse* if it is locally homeomorphic to the intersection of two linear subspaces $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k_2}$ in $\mathbb{R}^m$ for some positive integers $k_1, k_2 \leq m$ except on ∂X, ∂Y_1 and/or ∂Y_2, where one of the $\mathbb{R}$ summands is replaced by $\mathbb{R}_+$.

An intersection $D = Y_1 \cap Y_2 \subset S^4$ is a *ribbon disk* if it is homeomorphic to the 2-disk and satisfies: $D \subset \mathring{Y}_1$, $\mathring{D} \subset \mathring{Y}_2$ and ∂D is an essential curve in ∂Y_2. More details on ribbon knotted objects can be found in [**ABMW17**], [**Aud16**]. Denote $I = [0; 1]$.

Definition 2.1. Let L be an oriented trivial link with $2n$ components in $S^3 = \partial B^4$. A *ribbon tangle* T is a locally flat proper embedding in B^4 of oriented disjoint annuli $S^1 \times I$ denoted $A_1, \ldots, A_n$ and disjoint tori $S^1 \times S^1$ denoted $E_1, \ldots, E_m$ such that:

i) There exist locally flat immersed solid tori F_i for $i = 1, \ldots, m$ such that $\partial F_i = E_i$.

ii) $\partial A_i \subset L$ and the orientation induced by A_i on ∂A_i coincides with the given orientation of the two components of L.

iii) There exist n locally flat immersed 3-balls $B_i \simeq B^2 \times I$ such that, for all $i \in \{1, \ldots, n\}$:

$$\partial B_i = A_i \cup_\partial (B^2 \times \{0, 1\}).$$

iv) The singular set of the fillings B_i and F_j is composed by a finite number of ribbon disks.

A *G-colored ribbon tangle* is a pair (T, φ) where T is a ribbon tangle with complement $X_T = B^4 \setminus T$, equipped with a group homomorphism $\varphi\colon H_1(X_T) \to G$.

2.2. Definition of the Alexander invariant A.

Let (X, Y) be a pair of topological spaces. Denote $p\colon \hat{X} \to X$ the maximal abelian cover. For a ring homomorphism $\varphi\colon \mathbb{Z}[H_1(X)] \to R$, we define the twisted chain complex

$$C^\varphi(X, Y) = C(\hat{X}, p^{-1}(Y)) \otimes_{\mathbb{Z}[H_1(X)]} R$$

whose homology is denoted $H_*^\varphi(X, Y; R)$, or simply $H_*^\varphi(X, Y)$.

Let (T, φ) be a G-colored ribbon tangle. The homomorphism φ extends to a ring homomorphism $\varphi\colon \mathbb{Z}[H_1(X_T)] \to R$. For the rest of this section, we set $H = H_1^\varphi(X_T, *)$.

Proposition 2.2. *The R-module H admits a presentation with deficiency n.*

We postpone the proof of Proposition 2.2 to the end of Section 4.1. Consider a presentation of H of the form:

$$H = \langle \gamma_1, \dots, \gamma_{n+q} \mid r_1, \dots, r_q \rangle.$$

Let Γ be the free R-module generated by $\langle \gamma_1, \dots, \gamma_{n+q} \rangle$: the relators $r_1, \dots, r_q$ are words in these generators in Γ. Let us denote $r = r_1 \wedge \cdots \wedge r_q$ and $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_{n+q}$. The Alexander function $\mathcal{A}_T^\varphi \colon \wedge^n H \to R$ is the R-linear map defined by

$$r \wedge \tilde{u} = \mathcal{A}_T^\varphi(u) \cdot \gamma$$

for all $u = u_1 \wedge \cdots \wedge u_n \in \wedge^n H$, where $\tilde{u}_1, \cdots, \tilde{u}_n$ are arbitrary lifts in Γ and $\tilde{u} = \tilde{u}_1 \wedge \cdots \wedge \tilde{u}_n$. Different n-deficient presentations will give rise to Alexander functions that differ only by multiplication by a unit in R. Note that if H is free of rank n, then $\mathcal{A}_T^\varphi$ is a volume form.

Let us consider the $q \times (q+n)$ matrix defined by the presentation of H. If one adds to this matrix the row vectors giving $u_1, \dots, u_n$ in the generators $\gamma_1, \dots, \gamma_{q+n}$, then $\mathcal{A}_T^\varphi(u)$ is the determinant of the resulting $(q+n) \times (q+n)$ matrix.

EXAMPLE 2.3. Suppose that G has rank 2 and is generated by t_1, t_2. Consider the module H whose presentation has generators $\gamma_1, \dots, \gamma_4$ and two relations given by the matrix:

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 - t_1 & t_2 \end{pmatrix}.$$

The values of the Alexander function $\mathcal{A} \colon \wedge^2 H \to R$ are

$$\mathcal{A}(\gamma_1 \wedge \gamma_2) = t_2,\ \mathcal{A}(\gamma_1 \wedge \gamma_3) = 0,\ \mathcal{A}(\gamma_1 \wedge \gamma_4) = 1,$$
$$\mathcal{A}(\gamma_2 \wedge \gamma_3) = -t_2,\ \mathcal{A}(\gamma_2 \wedge \gamma_4) = 1 - t_1,\ \mathcal{A}(\gamma_3 \wedge \gamma_4) = 1.$$

Let $H_\partial = H_1^\varphi(S^3 \setminus L, *)$, which is the free R-module of rank $2n$, generated by the meridians of L. Let $m_\partial \colon H_\partial \to H$ be induced by the inclusion map $S^3 \setminus L \hookrightarrow X_T$. For short, for a given $z \in \wedge^n H_\partial$, we use the notation $m_\partial z$ for $\wedge^n m_\partial(z)$.

DEFINITION 2.4. The element $\mathbf{A}(T, \varphi)$ of $\wedge^n H_\partial$ is the (colored) isotopy invariant defined by the following property:

$$\forall z \in \wedge^n H_\partial,\ \mathcal{A}_T^\varphi(m_\partial z) = \omega_\partial(\mathbf{A}(T, \varphi) \wedge z) \tag{2.1}$$

where ω_∂ is a volume form on H_∂.

2.3. The Alexander polynomial of a $(1-1)$-tangle.

Given a finitely generated R-module H, and $k \geq 0$, the *k-th Alexander polynomial* of H is the greatest common divisor of all minors of order $(m-k)$ in a $q \times m$ presentation matrix of H. This invariant of H, denoted $\Delta_k(H) \in R$, is defined up to multiplication by a unit of R.

DEFINITION 2.5. The *Alexander polynomial* $\Delta^\varphi(T) \in R$ of a G-colored ribbon tangle (T,φ) is $\Delta_0\left(H_1^\varphi(X_T)\right)$.

Similarly to classical knot theory, T is a $(1-1)$-*tangle* if $n=1$. The components of T in B^4 consist of m tori and a cylinder whose boundary is a 2-component trivial link L in $S^3=\partial B^4$. Let x_1 and x_2 be the meridians of the components of L. Note that in X_T, both x_1 and $-x_2$ are homologous to the meridian x of the cylinder. We use the same notations x_1 and x_2 for the homology classes of their lifts in $H_1^\varphi(S^3\setminus L,*)$.

PROPOSITION 2.6. *Let (T,φ) be a G-colored $(1-1)$-tangle, such that φ is not trivial. Denote $t=\varphi(x)$ and let r be the rank of $\varphi(H_1(X_T))$. Then the element $\mathbf{A}(T,\varphi)$ of H_∂ is given by*

$$\mathbf{A}(T,\varphi)=\begin{cases}(t-1)\Delta^\varphi(T)\cdot(x_1-x_2) & \text{if } r\geq 2,\\ \Delta^\varphi(T)\cdot(x_1-x_2) & \text{if } r=1.\end{cases}$$

It is worth noticing that, up to a unit in R, the result is independent of the order chosen on the components of L. For the reader's convenience, we give a short proof of Proposition 2.6. More detailed arguments can be found in [**FM15**, Section 3].

PROOF. Denote $H = H_1^\varphi(X_T,*)$. From the long exact sequence of the pair $(X_T,*)$:

$$0 \longrightarrow H_1^\varphi(X_T) \longrightarrow H \longrightarrow H_0^\varphi(*) \longrightarrow H_0^\varphi(X_T) \longrightarrow 0,$$

we deduce that $\mathrm{Tors}H_1^\varphi(X_T)\simeq \mathrm{Tors}H$. Moreover, $\mathrm{rk}H=\mathrm{rk}H_1^\varphi(X_T)+1$. This implies that

$$\Delta^\varphi(T)=\Delta_1(H).$$

Let A be the matrix of a presentation of $H=\langle\gamma_1,\dots,\gamma_{q+1}\mid r_1,\dots,r_q\rangle$, and $\mathcal{A}^\varphi$ be the related Alexander function. We have

$$\forall z_1,\dots,z_{q+1}\in R,\ \mathcal{A}^\varphi(z_1\gamma_1+\cdots+z_{q+1}\gamma_{q+1})=\sum_{i=1}^{q+1}\det(A_i)z_i,$$

where A_i is A with the i-th column removed. Hence,

$$\Delta_1(H)=\gcd\mathcal{A}^\varphi(H)=\gcd\{\varphi(h);\ h\in H\}.$$

If $\mathcal{A}^\varphi=0$, then $\Delta_\varphi(T)=0$. Consider now the connecting homomorphism $\partial_*\colon H\to H_0^\varphi(*)\simeq R$. If $\mathcal{A}^\varphi\neq 0$, then rank $(H)=1$ and two linear maps $H\otimes_R\mathbb{Q}R\to\mathbb{Q}R$ are linearly dependent, where $\mathbb{Q}R$ is the fraction field of R. Then there exist elements P,Q in R such that for each $h\in H$, $\mathcal{A}^\varphi(h)=(P/Q)\partial_*(h)$. Hence

$$\mathcal{A}^\varphi(h)=\Delta^\varphi(T)\cdot\frac{\partial_*(h)}{\gcd\partial_*(H)}.$$

For a loop γ based in $*$ with lift $\hat{\gamma}$, one has $\partial_*\hat{\gamma} = \varphi(\gamma) - 1$. Hence, $\gcd \partial_*(H)$ is equal to 1 if $r \geq 2$ and is equal to $t-1$ if $r = 1$. We deduce that for all $h \in H$

$$\mathcal{A}^\varphi(h) = \begin{cases} (t-1)\Delta^\varphi(T) & \text{if } r \geq 2, \\ \Delta^\varphi(T) & \text{if } r = 1. \end{cases}$$

Let ω_∂ be the volume form on H_∂ relative to the choice of the meridians x_1, x_2. By definition, $\mathbf{A}(T,\varphi)$ verifies $\mathcal{A}^\varphi_T(m_\partial z) = \omega_\partial(\mathbf{A}(T,\varphi) \wedge z)$. Since $m_\partial(x_1) = x$ and $m_\partial(x_2) = -x$, we obtain the result from

$$\mathcal{A}^\varphi(x) = \omega_\partial(\mathbf{A}(T,\varphi) \wedge x_1) = -\omega_\partial(\mathbf{A}(T,\varphi) \wedge x_2). \qquad \square$$

2.4. The Burau functor.

We begin with defining ribbon cobordisms in $B^3 \times I$. Let L_0 and L_1 be two trivial links in the interior of B^3. Assign a sign to each component of the links. A *ribbon cobordism* from L_0 to L_1 is a collection S of ribbon annuli and tori in $B^3 \times I$ whose boundaries are L_0 in $B^3 \times \{0\}$ and L_1 in $B^3 \times \{1\}$, with according signs. Two cobordisms are equivalent if there is an ambient isotopy fixing the boundary circles L_0 and L_1.

The category $\mathcal{R}ib$ of ribbon cobordisms is the category whose: objects are sequences ε of signs ± 1 of length n, corresponding to trivial links with n components in B^3, such that ε is a sequence of signs affected to the components; morphisms $\varepsilon_0 \to \varepsilon_1$ are the equivalence classes of ribbon cobordisms between L_0 and L_1. The *degree* of a morphism $\varepsilon_0 \to \varepsilon_1$ is $\delta = (n_1 - n_0)/2$. The composition $S \circ S'$ of two ribbon cobordisms S and S' in $\mathcal{R}ib$ is defined by identifying $(B^3 \times \{1\}, S')$ to $(B^3 \times \{0\}, S)$.

The category $\mathcal{R}ib$ can be refined to the category of colored ribbon cobordisms $\mathcal{R}ib_G$. An object is a pair (ε, φ), where $\varphi\colon H_1(B^3 \setminus L) \to G$. A morphism $(\varepsilon_0, \varphi_0) \to (\varepsilon_1, \varphi_1)$ is a pair (S, φ) such that $\varphi \circ m_i = \varphi_i$ for $i = 0, 1$, if m_i are induced by the inclusions $X_{L_i} \hookrightarrow X_S$, where $X_{L_i} = (B^3 \times \{i\}) \setminus L_i$ and $X_S = (B^3 \times I) \setminus S$.

Let $gr\mathcal{M}od_G$ be the category of $\mathbb{Z}$-graded R-modules, whose morphisms are graded R-linear maps of arbitrary degree, up to multiplication by an element of $\pm G$. We now define a projective functor $\rho\colon \mathcal{R}ib_G \longrightarrow gr\mathcal{M}od_G$, which will be called the *Burau functor*.

We fix a base point $*$ in B^3. Given a trivial link L with n components in B^3, and a homomorphism $\varphi\colon H_1(B^3 \setminus L) \to G$, we denote by M the free R-module of $H_1^\varphi(X_L, *; R)$ of rank n. In particular, for $i = 0, 1$, taken L_i as before on $B^3 \times \{i\}$, we denote by M_i the free R-module $H_1^{\varphi_i}(X_{L_i}, *; R)$ of rank n_i. The map

$$\rho\colon \mathcal{R}ib_G \to gr\mathcal{M}od_G$$

is defined by the following construction. Let (ε, φ) be an object of $\mathcal{R}ib_G$, corresponding to a trivial link L. The image by ρ of (ε, φ) is

$$\rho(\varepsilon, \varphi) = \wedge M$$

the exterior algebra of the free R-module M. To a morphism $(S, \varphi)\colon (\varepsilon_0, \varphi_0) \to (\varepsilon_1, \varphi_1)$, we associate a R-linear map

$$\rho(S,\varphi)\colon \wedge M_0 \longrightarrow \wedge M_1,$$

of degree δ as follows. Let J be the interval in $\partial B^3 \times I$, which connects the base points of the bottom and top balls. By Proposition 2.2, the R-module $H_1^\varphi(X_S, J; R)$ admits a presentation with deficiency $d = (n_0 + n_1)/2$. Consider the Alexander function $\mathcal{A}_S^\varphi : \wedge^d H_1^\varphi(X_S, J; R) \to R$. For any integer $k \geq 0$, the image $\rho(S,\varphi)(x) \in \wedge^{k+\delta} M_1$ of any $x \in \wedge^k M_0$ is defined by the following property

$$\forall y \in \wedge^{d-k} M_1,\ \mathcal{A}_S^\varphi(\wedge^k m_0(x) \wedge \wedge^{d-k} m_1(y)) = \omega_1\left(\rho(S,\varphi)(x) \wedge y\right),$$

where ω_1 is a volume form on M_1.

THEOREM 2.7. *The map ρ is a degree preserving functor $\mathcal{R}ib_G \to gr\mathcal{M}od_G$.*

The proof of Theorem 2.7 follows word by word the proof of [**FM15**, Theorem I]. Note that the deficiency of the presentation of $H_1^\varphi(X_S, J; R)$ depends only on n_0 and n_1.

A G-colored *ribbon tube* (S,φ) is a morphism such that the inclusions m_0 and m_1 induce isomorphisms in homology (with integer coefficients). Ribbon tubes are analogous to string links; the links L_0 and L_1 have the same number of components, and S has no toric component. For a fixed φ, the set of G-colored ribbon tubes forms a monoid $\mathcal{T}_\varphi$. Following [**KLW01**, Proposition 2.1], one proves that m_0 and m_1 induce isomorphisms

$$(m_i)_*\colon H^\varphi(X_{L_i}; QR) \longrightarrow H^\varphi(X_S; QR), \text{ for } i = 0, 1$$

where QR is the quotient field of R. Set H^φ to be $H^\varphi(X_{L_0}; QR) = H^\varphi(X_{L_1}; QR)$. Set $L = L_0 = L_1$. The composition $m_1^{-1} \circ m_0$ is an automorphism of $H^\varphi(X_L; QR)$.

DEFINITION 2.8. The *colored Burau representation* is the monoid homomorphism

$$r^\varphi\colon \mathcal{T}_\varphi \to \mathrm{Aut}(H^\varphi).$$

Let $\Delta^\varphi(X_T, X_L) = \Delta_0\left(H_1^\varphi(X_T, X_L; R)\right)$ be the Alexander polynomial of the pair (X_T, X_L), see Section 2.3.

PROPOSITION 2.9. *For any G-colored ribbon tube $(S,\varphi) \in \mathcal{T}_\varphi$, we have*

$$\rho(S,\varphi) = \Delta^\varphi(X_T, X_L) \cdot \wedge r^\varphi :\ \wedge H^\varphi \to \wedge H^\varphi.$$

In particular, if the ribbon tube (S,φ) is *monotone* with respect to the coordinate in I in $B^4 \times I$, it is called a *ribbon braid* (see [**ABMW17**], [**Dam17**]). In this case $\Delta^\varphi(X_T, X_L) = 1$ and $\rho(S,\varphi)$ coincide with the exterior powers of r^φ. The proof of Proposition 2.9 can be obtained similarly to [**FM15**, Proposition 7.2], or in the monotone case, to [**BCF15**, Section 3.1].

Let (T,φ) be a G-colored ribbon tangle in B^4. Let $L = L_0 \cup L_1$ be a splitting of L into two disjoint (trivial) links, and let B_0 and B_1 be two 3-balls such that

$$S^3 = B_0 \cup_{S^2 \times \{0\}} S^2 \times [0;1] \cup_{S^2 \times \{1\}} B_1$$

and $L_i \subset B_i$ for $i = 0, 1$. Let φ_i be induced by φ on $H_1(B_i \setminus L_i)$, and ε_i be sequences of signs according to the co-orientations of the components of T. Then (T, φ) is *splitted* as a morphism $(\tilde{T}, \varphi)\colon (\varepsilon_0, \varphi_0) \to (\varepsilon_1, \varphi_1)$. Note that $2n = n_0 + n_1$.

THEOREM 2.10. *Let (T, φ) be a G-colored ribbon tangle, and $(\tilde{T}, \varphi)\colon (\varepsilon_0, \varphi_0) \to (\varepsilon_1, \varphi_1)$ be a splitting of $(\mathcal{T}, \varphi)$ in $\mathcal{R}ib_G$. There is an isomorphism, well-defined up to a unit in R,*

$$\wedge^n H_\partial \to \mathrm{Hom}_R(\wedge M_0, \wedge M_1)$$

sending $\mathbf{A}(T, \varphi)$ to $\oplus_k (-1)^{k(n_0-k)} \rho_k(\tilde{T}, \varphi)$, where ρ_k is the k-component of ρ.

An explicit example is given in Remark 5.1.

PROOF. The decomposition $H_\partial = M_0 \oplus M_1$ induces a natural isomorphism

$$\wedge^n H_\partial \simeq \bigoplus_{k=0}^{n} \big(\wedge^k M_0 \otimes \wedge^{n-k} M_1 \big).$$

The element $\mathbf{A}(T, \varphi) \in \wedge^n H_\partial$ decomposes as $\sum_k \mathbf{A}_k(T, \varphi)$, where $\mathbf{A}_k(T, \varphi) \in \wedge^k M_0 \otimes \wedge^{n-k} M_1$. Suppose now that k is fixed; the element $\mathbf{A}_k(T, \varphi)$ might not be decomposable. There exist a finite sequence of element $A_0^l \in \wedge^k M_0$ and $A_1^l \in \wedge^{n-k} M_1$ (depending on k, and (T, φ)) such that $\mathbf{A}_k(T, \varphi) = \sum_l A_0^l \otimes A_1^l$. Let ω_0 be a volume form $\wedge^{n_0} M_0 \to R$. There is an isomorphism

$$\wedge^k M_0 \otimes \wedge^{n-k} M_1 \simeq \mathrm{Hom}_R(\wedge^{n_0-k} M_0, \wedge^{n-k} M_1)$$

sending $\mathbf{A}_k(T, \varphi)$ to

$$x \mapsto \sum_l \omega_0 \left(x \wedge A_0^l\right) \cdot A_1^l.$$

We now show that this morphism coincides with $\rho_{n_0-k}(\tilde{T}, \varphi)$. Let $x \in \wedge^{n_0-k} M_0$. Note that we have $n_0 - k + \delta = n - k$ and the morphism has degree δ. Consider a volume form $\omega_1\colon \wedge^{n_1} M_1 \to R$ and the sum $\omega_\partial = \omega_0 \otimes \omega_1\colon \wedge^n H_\partial \to R$. Let $\mathcal{A}^\varphi$ be the Alexander function related to a presentation of $H_1^\varphi(X_T, *; R)$ of deficiency n. Since $m_\partial = m_0 \oplus m_1$ we have, for all y in $\wedge^{k+\delta} M_1$,

$$\begin{aligned}
\mathcal{A}^\varphi(\wedge^{n_0-k} m_0(x) \wedge \wedge^{k+\delta} m_1(y)) &= \omega_\partial(\mathbf{A}(T, \varphi) \wedge x \wedge y) \\
&= \omega_\partial(\mathbf{A}_k(T, \varphi) \wedge x \wedge y) \\
&= \sum_l \omega_0(A_0^l \wedge x) \cdot \omega_1(A_1^l \wedge y) \\
&= \omega_1\big(\sum_l \omega_0(A_0^l \wedge x) \cdot A_1^l \wedge y \big).
\end{aligned}$$

Hence $\sum_l \omega_0(x\wedge A_0^l)\cdot A_1^l = (-1)^{k(n_0-k)}\rho_{n_0-k}(\tilde{T},\varphi)(x)$. Note that if, for cosmetic reasons, one prefers that $\mathbf{A}_k(T,\varphi)$ induces $(-1)^{k(n_0-k)}\rho_k(\tilde{T},\varphi)(x)$ instead, one might compose with Hodge duality $\wedge^k M_0 \to \wedge^{n_0-k} M_0$. □

3. The circuit algebra of colored cobordisms.

In this section, we introduce the circuit algebra Cob_G. The algebraic structure is inspired by [**Jon99**], [**Pol10**], see also [**Ken13**], [**Arc10**].

DEFINITION 3.1. Let $B = B_0$ be a 4-ball and $B_1, \dots, B_p$ be disjoint 4-balls in the interior of B. For every $i \in \{0, \dots, p\}$, let L_i (with $L = L_0$) be a trivial oriented link with $2n_i$ ($n = n_0$) components in $S_i^3 = \partial B_i$ (with $S^3 = S_0^3$). A *cobordism* C is a disjoint union of locally flat proper embedded annuli in $B \setminus \{\mathring{B}_1, \dots, \mathring{B}_p\}$, whose boundary are the links L_i, with the conditions of Definition 2.1 but without singularities.

DEFINITION 3.2. Let C' and C'' be two cobordisms such that B_i' is a ball of C' with $n_i' = n''$. The *composition* $C' \circ_i C''$ is the cobordism obtained with the identification of $B'' = B_0''$ with B_i'.

As in the previous section, G is a fixed free abelian group with group ring R. A *G-colored* cobordism is a pair (C,φ) where C is a cobordism with complement X_C, equipped with a group homomorphism $\varphi\colon H_1(B^4 \setminus C) \to G$. The orientation-preserving diffeomorphism classes of G-colored cobordisms with composition of compatible cobordisms, form an operad denoted $\mathcal{C}_G$.

Let $\mathcal{H}om_G$ be the operad of tensor powers of R-modules and R-linear applications, considered up to an element of $\pm G$. The composition in $\mathcal{H}om_G$ is induced by the usual composition of maps. We construct the circuit algebra Cob_G as a morphism of operads from $\mathcal{C}_G$ to $\mathcal{H}om_G$.

Let (L,φ) be a G-colored oriented trivial link with k components in S^3, with complement $X_L = S^3 \setminus L$. The group homomorphism $\varphi\colon H_1(S^3 \setminus L) \to G$ induces a ring homomorphism denoted also $\varphi\colon \mathbb{Z}[H_1(S^3 \setminus L)] \to R$. Let $*$ be a base point on S^3. The R-module $H_1^\varphi(S^3 \setminus L, *; R)$ is free of rank k, generated by the meridians of L.

Let now (C,φ) be a G-colored cobordism, with complement X_C. For $i = 1, \dots, p$, let $*$ and $*_i$ be base points in the boundary of B_i and J_i be intervals (whose interiors are disjoint, and disjoint from C) connecting $*$ to $*_i$. Note that the union of the J_i is contractible. The homomorphism φ induces a ring homomorphism $\mathbb{Z}[H_1(X_C)] \to R$ and also denoted φ. The inclusion $m_i\colon S_i^3 \setminus L_i \hookrightarrow X_T$ induces $\varphi_i\colon \mathbb{Z}[H_1(S_i^3 \setminus L_i)] \to R$. Set $H = H_1^\varphi(X_C, J)$, $H_\partial = H_1^\varphi(S^3 \setminus L, *)$, and $H_{\partial_i} = H_1^{\varphi_i}(S^3 \setminus L_i, *_i)$ for $i = 1, \dots, p$. Note that H is free of rank $r = n + n_1 + \cdots + n_p$ and H_{∂_i} are free of rank n_i. Let ω_C^φ be a volume form $\omega_C^\varphi\colon \wedge^r H \to R$, and $\omega_\partial\colon \wedge^{2n} H_\partial \to R$. For $i = 1, \dots, p$, we denote again $m_i\colon H_{\partial_i} \to H$ the map induced by the inclusion. Let $m\colon \otimes_i \wedge^{n_i} H_{\partial_i} \to \wedge^{n_1+\cdots+n_p} H$ be defined as $m = \wedge^{n_1} m_1 \wedge \cdots \wedge \wedge^{n_p} m_p$.

To the cobordism (C,φ) we associate

$$\Upsilon_{C,\varphi}\colon \bigotimes_{i=1}^{p} \wedge^{n_i} H_{\partial_i} \to \wedge^n H_\partial$$

such that, for $x \in \otimes_i\big(\wedge^{n_i} H_{\partial_i}\big)$,

$$\omega_C^{\varphi}(m(x) \wedge m_\partial(y)) = \omega_\partial(\Upsilon_{C,\varphi}(x) \wedge y),\ \forall y \in \wedge^n H_\partial. \tag{3.1}$$

Cob_G denotes the object that associates to a pair (L,φ) the homology module $H_1^{\varphi}(S^3 \setminus L, *; R)$, and to each pair (C,φ) the linear map $\Upsilon_{C,\varphi}$.

THEOREM 3.3. *Cob_G is a circuit algebra.*

PROOF. We show that Υ commutes with the composition of cobordisms. Let (C',φ') and (C'',φ'') be two cobordisms with

$$\Upsilon_{C',\varphi'}\colon \bigotimes_{k=1}^{p'} \wedge^{k'_i} H_{\partial'_k} \to \wedge^{n'} H'_\partial, \quad \text{and} \quad \Upsilon_{C'',\varphi''}\colon \bigotimes_{l=1}^{p''} \wedge^{n''_l} H_{\partial''_l} \to \wedge^{n''} H_{\partial''}.$$

Let φ be the coloring induced by φ' and φ'' on $C' \circ_i C''$. Then, we have to prove that for all $u_k \in \wedge_i^k H_{\partial'_k}$ with $k = 1,\dots,p'$ and $k \neq i$ and all v_l for $l = 1,\dots,p''$,

$$\Upsilon_{C'\circ_i C'',\varphi}\big(u_1 \otimes \cdots \otimes (\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}\big) = \Upsilon_{C',\varphi'}\big(u_1 \otimes \cdots \otimes \Upsilon_{C'',\varphi''}(\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}\big).$$

Let $H' = H_1^{\varphi'}(X_{T'}, J')$ and $H'' = H_1^{\varphi''}(X_{T''}, J'')$ be the (free) homology modules of the exteriors of the cobordisms. Let $\alpha_1,\dots,\alpha_{2n''}$ be a basis of $H_{\partial''} \simeq H_{\partial'_i}$. Consider presentations of H' and H'':

$$H'' = \langle m_{\partial''}\alpha_1,\dots,m_{\partial''}\alpha_{2n''},\beta_1,\dots,\beta_k \,|\, \rho_1,\dots,\rho_s\rangle,$$
$$H' = \langle m'_i\alpha_1,\dots,m'_i\alpha_{2n''},\zeta_1,\dots,\zeta_l \,|\, r_1,\dots,r_t\rangle.$$

Applying Mayer–Vietoris theorem to $X_T = X_{T'} \cup X_{T''}$, we obtain that the (free) module H is generated by

$$m_{\partial''}\alpha_1,\dots,m_{\partial''}\alpha_{2n''},m'_i\alpha_1,\dots,m'_i\alpha_{2n''},\beta_1,\dots,\beta_k,\zeta_1,\dots,\zeta_l \tag{3.2}$$

subject to the relations $\rho_1,\dots,\rho_s,r_1,\dots,r_t,m_{\partial''}\alpha_1 - m'_i\alpha_1,\dots,m_{\partial''}\alpha_{2n''} - m'_i\alpha_{2n''}$. Let ω' and ω'' be volume forms on H' and H'', and $\omega_{\partial'}$ be the form on $H_{\partial'}$. Let ω be the form on H induced by ω' and ω''. For the computation below, we introduce the notation

$$u \wedge_i v = u_1 \wedge \cdots \wedge u_{i-1} \wedge (v_1 \wedge \cdots \wedge v_{p''}) \wedge u_{i+1} \wedge \cdots \wedge u_{p'}.$$

We want to show that, for all $y \in \wedge^{n'} H_{\partial'}$,

$$\begin{aligned}&\omega_{\partial'}\big(\Upsilon_{C'\circ C''}(u_1 \otimes \cdots \otimes (\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}) \wedge y\big)\\ &\quad= \omega_{\partial'}\big(\Upsilon_{C'}(u_1 \otimes \cdots \otimes \Upsilon_{C''}(\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}) \wedge y\big).\end{aligned}$$

We have

$$\omega_{\partial'}\big(\Upsilon_{C'\circ C''}(u_1 \otimes \cdots \otimes (\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}) \wedge y\big) \cdot m_{\partial''}\alpha \wedge \beta \wedge m'_i\alpha \wedge \zeta$$

$$
\begin{aligned}
&= \omega((m'u \wedge_i m''v) \wedge m_{\partial'} y) \cdot m_{\partial''}\alpha \wedge \beta \wedge m'_i\alpha \wedge \zeta \\
&= \rho \wedge r \wedge (m_{\partial''}\alpha - m'_i\alpha) \wedge (m'u \wedge_i m''v) \wedge m_{\partial'} y \\
&= \sum_Q (-1)^{|Q|}\epsilon_Q \cdot \rho \wedge r \wedge m_{\partial''}\alpha_Q \wedge m'_i\alpha_{\bar{Q}} \wedge (m'u \wedge_i m''v) \wedge m_{\partial'} y,
\end{aligned}
$$

where the sum is taken over all subsets $Q \subset \{1, \dots, 2n''\}$ of cardinal n''. The number t of relations r_i can be chosen arbitrarily to be even and $m_{\partial''}\alpha_Q \wedge m'_i\alpha_{\bar{Q}} \wedge (m'u \wedge_i m''v)$ coincides with $m_{\partial''}\alpha_Q \wedge m''v \wedge (m'u \wedge_i m'_i\alpha_{\bar{Q}})$ up to a sign depending only on p'' and the n'_i, the sum coincides up to a sign with

$$
\sum_Q (-1)^{|Q|}\epsilon_Q \cdot \big(\rho \wedge m_{\partial''}\alpha_Q \wedge m''v) \wedge (r \wedge (m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'} y)
$$

whose summands are equal to

$$
(-1)^{|Q|}\epsilon_Q \cdot \omega''(m_{\partial''}\alpha_Q \wedge m''v) \cdot \omega'((m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'} y) \cdot (m_{\partial''}\alpha \wedge \beta \wedge m'_i\alpha \wedge \zeta).
$$

It follows that, up to a sign,

$$
\begin{aligned}
&\omega_{\partial'}\big(\Upsilon_{C'\circ C''}(u_1 \otimes \cdots \otimes (\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}) \wedge y\big) \\
&\quad = \sum_Q (-1)^{|Q|}\epsilon_Q \cdot \omega''(m_{\partial''}\alpha_Q \wedge m''v) \cdot \omega'\big((m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'} y\big) \\
&\quad = \omega'\big(\sum_Q (-1)^{|Q|}\epsilon_Q \cdot \omega''(m_{\partial''}\alpha_Q \wedge m''v) \cdot (m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'} y\big) \\
&\quad = \omega'\big(\sum_Q (-1)^{|Q|}\epsilon_Q \cdot \omega_{\partial''}(\Upsilon_{C''}(v_1 \otimes \cdots \otimes v_{p''}) \wedge \alpha_Q) \cdot (m'u \wedge_i m'_i\alpha_{\bar{Q}}) \wedge m_{\partial'} y\big) \\
&\quad = \omega'\big((m'u \wedge_i \big[\sum_Q (-1)^{|Q|}\epsilon_Q \cdot \omega_{\partial''}(\Upsilon_{C''}(v_1 \otimes \cdots \otimes v_{p''}) \wedge \alpha_Q)\big] m'_i\alpha_{\bar{Q}}\big) \wedge m_{\partial'} y\big) \\
&\quad = \omega'\big((m'u \wedge_i m'_i(\Upsilon_{C''}(v_1 \otimes \cdots \otimes v_{p''}))\big) \wedge m'_\partial y\big) \\
&\quad = \omega_{\partial'}\big(\Upsilon_{C'}(u_1 \otimes \cdots \otimes \Upsilon_{C''}(\otimes_1^{p''} v_l) \otimes \cdots \otimes u_{p'}) \wedge y\big).
\end{aligned}
$$

□

3.1. Action of cobordisms on ribbon tangles.

Given a cobordism C and a collection of ribbon tangles $T_1, \cdots, T_p$, one may create a new ribbon tangle, if $n(T_i) = n_i$ for all $i = 1, \dots, p$, by gluing each T_i into the internal ball B_i of C. The action of G-colored cobordisms on G-colored ribbon tangles is defined once the colorings coincide on the boundary components. The following theorem states that the invariant $\mathbf{A}$ respects the structure of circuit algebra Cob_G.

THEOREM 3.4. *Let (T, ψ) be the G-colored ribbon tangle obtained by gluing the G-colored ribbon tangles $(T_1, \varphi_1), \cdots, (T_p, \varphi_p)$ to a G-colored cobordism (C, φ). The following equality holds*:

$$
\mathbf{A}(T, \psi) = \mathbf{\Upsilon}_{C,\varphi}\big(\mathbf{A}(T_1, \varphi_1) \otimes \cdots \otimes \mathbf{A}(T_p, \varphi_p)\big) \in \wedge^n H_\partial.
$$

PROOF. For $i = 1, \cdots, p$, consider a presentation of $H_{T_i} = H_1^{\varphi_i}(X_{T_i}, *_i)$ of the

form

$$H_{T_i} = \langle m_{\partial_i}\gamma_1^i, \dots, m_{\partial_i}\gamma_{2n_i}^i, \beta_1^i, \dots, \beta_{k_i}^i \mid \rho_1^i, \dots, \rho_{s_i}^i \rangle$$

and let $\mathcal{A}_{T_i}^{\varphi_i}$ be the Alexander function related to this presentation. Let $H_C = H_1^{\varphi}(X_C, J)$ be the free module, with volume form ω_C^{φ} associated to the presentation

$$H_C = \langle m_1\gamma_1^1, \dots, m_1\gamma_{2n_1}^1, \dots, m_p\gamma_1^p, \dots, m_p\gamma_{2n_p}^p, \alpha_1, \dots, \alpha_l \mid r_1, \dots, r_t \rangle.$$

By successive Mayer–Vietoris arguments, the module $H_T = H_1^{\varphi}(X_T, *)$ admits a presentation with generators of the form $m_{\partial_i}\gamma_j^i$, $m_i\gamma_j^i$, for $j = 1, \dots, 2n_i$ and $i = 1, \dots, p$, and $\beta_1^i, \dots, \beta_{k_i}^i, \alpha_1, \dots, \alpha_l$. They are subject to the relations $m_{\partial_i}\gamma_j^i - m_i\gamma_j^i$, for $j = 1, \dots, 2n_i$ and $i = 1, \dots, p$, and $\rho_1^i, \dots, \rho_{s_i}^i$ for $i = 1, \dots, p$ and $r_1, \dots, r_t$. One has $\mathbf{A}(T, \psi) = \Upsilon_{C,\varphi}\big(\mathbf{A}(T_1, \varphi_1) \otimes \cdots \otimes \mathbf{A}(T_p, \varphi_p)\big)$ if and only if

$$\forall z \in \textstyle\bigwedge^n H_\partial,\ \mathbf{A}(T, \psi) \wedge z = \Upsilon_{C,\varphi}\big(\mathbf{A}(T_1, \varphi_1) \otimes \cdots \otimes \mathbf{A}(T_p, \varphi_p)\big) \wedge z.$$

By Equation (2.1), $\omega_\partial(\mathbf{A}(T, \psi) \wedge z) = \mathcal{A}_T^{\psi}\big(m_\partial z\big)$. For short, we write $\mathbf{A}(T_i)$ for $\mathbf{A}(T_i, \varphi_i)$, for $i = 1, \dots, p$. By definition of the Alexander function $\mathcal{A}_T^{\psi}$,

$$\begin{aligned}
&\mathcal{A}_T^{\psi}\big(m_\partial z\big) \cdot m_{\partial_*}\gamma \wedge m_*\gamma \wedge \beta \wedge \alpha = \rho \wedge r \wedge (m_{\partial_i}^i - m_i\gamma^i) \wedge m_\partial z \\
&\quad = \sum_{Q_1, \dots, Q_p} (-1)^{|Q|}\epsilon_Q \cdot \rho \wedge r \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \cdots \wedge m_{\partial_p}\gamma_{Q_p}^p \wedge m_1\gamma_{\bar{Q}_1}^1 \wedge \cdots \wedge m_p\gamma_{\bar{Q}_p}^p \wedge m_\partial z,
\end{aligned} \tag{3.3}$$

where the sum is taken over the subsets $Q_i \subset \{1, \dots, 2n_i\}$ of cardinal n_i (the other terms vanish). $\bar{Q}_i$ denotes the complement of Q_i, $|Q| = n_1 + \cdots + n_p$ and ϵ_Q the signature of the permutation $Q_1 \cdots Q_p \bar{Q}_1 \cdots \bar{Q}_p$, where the elements of $Q_1 \cdots Q_p$ in increasing order are followed by the elements of $\bar{Q}_1 \cdots \bar{Q}_p$ in increasing order. Moreover, we can decide arbitrarily that the number t of relations in the presentation of H_C is even, and get (3.3) to be equal to:

$$\begin{aligned}
&\sum_{Q_1, \dots, Q_p} (-1)^{|Q|}\epsilon_Q \cdot (\rho \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \cdots \wedge m_{\partial_p}\gamma_{Q_p}^p) \wedge (r \wedge m_1\gamma_{\bar{Q}_1}^1 \wedge \cdots \wedge m_p\gamma_{\bar{Q}_p}^p \wedge m_\partial z) \\
&= \sum (-1)^{|Q|}\epsilon_Q \cdot (\rho \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \cdots \wedge m_{\partial_p}\gamma_{Q_p}^p) \cdot \omega_C^{\varphi}(m_1\gamma_{\bar{Q}_1}^1 \wedge \cdots \wedge m_p\gamma_{\bar{Q}_p}^p \wedge m_\partial z) \\
&\qquad \cdot (m_*\gamma \wedge \alpha).
\end{aligned}$$

Moreover, since $s_1, \dots, s_p$ can also be supposed even,

$$\begin{aligned}
\rho \wedge m_{\partial_1}\gamma_{Q_1}^1 \wedge \cdots \wedge m_{\partial_p}\gamma_{Q_p}^p &= (\rho^1 \wedge m_{\partial_1}\gamma_{Q_1}^1) \wedge \cdots \wedge (\rho^p \wedge m_{\partial_p}\gamma_{Q_p}^p) \\
&= \mathcal{A}_{T_1}^{\varphi_1}(m_{\partial_1}\gamma_{Q_1}^1) \cdots \mathcal{A}_{T_p}^{\varphi_p}(m_{\partial_p}\gamma_{Q_p}^p) \cdot m_{\partial_*}\gamma \wedge \beta \\
&= \omega_{\partial_1}(\mathbf{A}(T_1) \wedge \gamma_{Q_1}^1) \cdots \omega_{\partial_p}(\mathbf{A}(T_p) \wedge \gamma_{Q_p}^p) \cdot m_{\partial_*}\gamma \wedge \beta.
\end{aligned}$$

Then $\mathcal{A}_T^{\psi}\big(m_\partial z\big)$ coincides with the sum over $Q_1 \cdots Q_p$ of the summands

$$(-1)^{|Q|}\epsilon_Q \cdot \omega_{\partial_1}(\mathbf{A}(T_1)\wedge\gamma^1_{Q_1})\cdots\omega_{\partial_p}(\mathbf{A}(T_p)\wedge\gamma^p_{Q_p})\cdot\omega^{\varphi}_C(m_{\bar{Q}_1}\gamma^1_{\bar{Q}_1}\wedge\cdots\wedge m_{\bar{Q}_p}\gamma^p_{\bar{Q}_p}\wedge m_\partial z)$$
$$= \omega^{\varphi}_C\big(\sum(-1)^{|Q|}\epsilon_Q\cdot\omega_{\partial_1}(\mathbf{A}(T_1)\wedge\gamma^1_{Q_1})\cdots\omega_{\partial_p}(\mathbf{A}(T_p)\wedge\gamma^p_{Q_p})$$
$$\cdot\, m_{\bar{Q}_1}\gamma^1_{\bar{Q}_1}\wedge\cdots\wedge m_{\bar{Q}_p}\gamma^p_{\bar{Q}_p}\wedge m_\partial z\big).$$

Note that for all $i=1,\dots,p$ and $x\in\wedge^{n_i}H_{\partial_i}$, we have

$$x=\sum_{|Q_i|=n_i}\epsilon_{\bar{Q}}\omega_{\partial_i}(x\wedge\gamma^i_{Q_i})\cdot\gamma^i_{\bar{Q}_i}=\sum_{|Q_i|=n_i}(-1)^{|Q_i|}\epsilon_{Q_i}\omega_{\partial_i}(x\wedge\gamma^i_{Q_i})\cdot\gamma^i_{\bar{Q}_i}.$$

Moreover, $(-1)^{|Q|}=(-1)^{|Q_1|}\cdots(-1)^{|Q_p|}$ and $\epsilon_Q\epsilon_{Q_1}\cdots\epsilon_{Q_p}$ do not depend on $Q_1,\dots,Q_p$ but only on $n_1=|Q_1|,\dots,n_p=|Q_p|$. Hence, up to a sign

$$\begin{aligned}\mathcal{A}^{\psi}_T\big(m_\partial z\big)&=\omega^{\varphi}_C(m_1\mathbf{A}(T_1)\wedge\cdots\wedge m_p\mathbf{A}(T_p)\wedge m_\partial z)\\&=\omega^{\varphi}_C(m(\mathbf{A}_C(\Upsilon(T_1)\otimes\cdots\otimes\mathbf{A}(T_p)))\wedge m_\partial z)\\&=\omega_\partial(\Upsilon_C(\mathbf{A}(T_1)\otimes\cdots\otimes\mathbf{A}(T_p))\wedge z).\end{aligned}$$

□

4. A diagrammatic description.

Broken surfaces are locally flat immersions in the 3-ball B^3 of disjoint annuli and tori, whose singularities consist of a finite number of circles. Any ribbon tangle in B^4 can be projected onto a broken surface in a suitable sense. These projections can be viewed as a way to represent ribbon tangles similarly to diagrams in usual knot theory. Conversely, every broken surface is the projection of a ribbon tangle. This gives a correspondence between ribbon tangles and broken surfaces (in fact, of a certain type, called *symmetric*).

Welded diagrams are a quotient of virtual diagrams under a certain set of moves, see for example [**Aud16**]. To each welded tangle one may associate a symmetric broken surface diagram, see [**Sat00**] and [**Yaj62**]. This define a map $Tube$ sending any welded tangle to the ribbon tangle associated to the symmetric broken surface resulting from the preceding construction.

For general ribbon knotted objects, such as ribbon tangles, ribbon tubes [**ABMW17**] and knotted spheres [**Yan69**], the Tube map is well defined and surjective, but its injectivity is still an open question [**BD16**]. However, this is an isomorphisms on ribbon braids and extended ribbon braids [**BH13**], [**Dam17**].

A *combinatorial* fundamental group can be defined for welded tangle diagrams, with the Wirtinger method (virtual crossings are simply ignored). This group coincides with the fundamental group of the complement of an associated ribbon tangle in B^4 [**Sat00**], [**Yaj64**], [**ABMW17**]. Then, two welded tangle representing the same ribbon tangle have isomorphic fundamental groups (and this isomorphism sends meridian to meridian).

Let D be the unit disk in $\mathbb{C}$, and for any positive integer n, let $x_1,\dots,x_{2n}$ be a fixed ordered set of points in ∂D.

Definition 4.1. Let n be a positive integer. A *welded tangle* on n-strands, or *welded n-tangle* is a proper immersion τ of an oriented 1-manifold in D. It consists of some copies of the circle and n copies of the unit interval whose boundary is $\{x_1,\dots,x_{2n}\}$.

The singular set of τ is a finite number of transversal double points equipped with a partial order on the preimages. By convention, this order is specified by erasing a small neighbourhood of the lowest preimage, or by pointing the crossing if the preimages are not comparable. If the preimages of a crossing are comparable, then the crossing is said *classical*; otherwise it is said virtual. Moreover, a classical crossing is *positive* if the basis made of the tangent vectors of the highest and lowest preimages is positive; otherwise, it is *negative*.

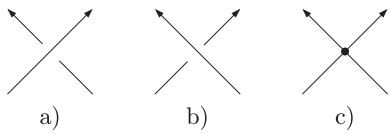

Figure 1. Positive, negative and virtual crossings.

Let μ be a positive integer. A *μ-colored* welded tangle is a pair (τ, ψ) where τ is a welded tangle and ψ is a map from the set of strands and circles to the set $\{t_1, \dots, t_\mu\}$. Two welded colored tangles are equivalent if they are related by *generalized Reidemeister moves* (see [**Kam07**, pp. 445 and 454]), respecting the coloring.

4.1. Computation of A.

Let (τ, ψ) be a μ-colored welded tangle. It decomposes into a finite union of disjoint oriented arcs. Label the crossings with (formal) letters, and each arc with the same letter as the crossing it begins at. If an arc connects points on the border of τ without meeting any crossing, we use the convention of Figure 3. We construct a matrix $M^\psi(\tau)$ with coefficients in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ where the rows are indexed by crossings (positive, negative and virtual) and points interrupting arcs, and the columns by the arcs.

- Fill row corresponding to each positive and negative crossing as shown in Figure 2,

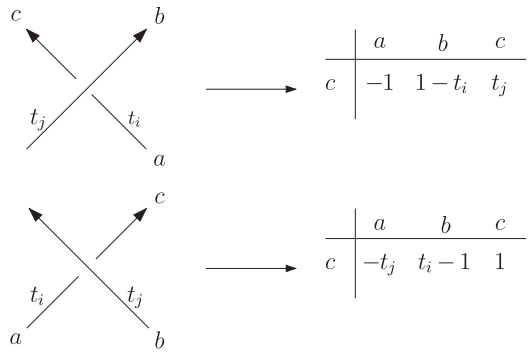

Figure 2. The rule to fill the matrix $M^\psi(T)$, where t_i and t_j are not necessarily different. If $b = a$ or $b = c$ we add the contributions.

- At each point on the diagram, fill the row as shown in Figure 3.

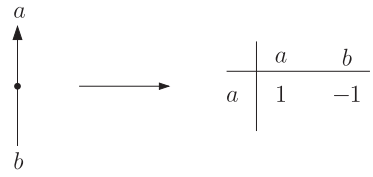

Figure 3. Rule for arcs that don't begin at crossings.

The other entries of the rows are zero. Virtual crossings can be ignored or considered as divided arcs. Notice that, after some Reidemeister moves of type I, one might suppose that there every arc begins at a crossing, and the receipt of Figure 3 becomes useless to construct the matrix $M^\psi(\tau)$.

REMARK 4.2. Let p be the number of internal arcs of τ. Since τ has $2n$ arcs connected to the boundary, the total number of arcs is $2n+p$. One easily observes that the matrix $M^\psi(\tau)$ has size $(p+n)\times(p+2n)$.

DEFINITION 4.3. Let (τ,ψ) be a μ-colored welded n-tangle and H_∂ be the module of rank $2n$ freely generated by the set of marked points $\{x_1,\dots,x_{2n}\}$. The invariant α is defined to be

$$\alpha(\tau,\psi)=\sum_I |M^\psi(\tau)_I|\cdot x_I \in \bigwedge^n H_\partial,$$

where the sum is taken for all subset $I\subset\{1,\dots,2n\}$ of n elements, $|M^\psi(\tau)_I|$ is the determinant of the $(n+p)$-minor of $M^\psi(\tau)$ corresponding to the columns indexed by the internal arcs and the columns relative to the arcs indexed by I, and x_I is the wedge product of the generators x_i with $i\in I$.

A computation shows that $\alpha(\tau,\psi)$ is invariant by generalized Reidemeister moves, up to multiplication by a unit. Otherwise the invariance is simply a consequence of Theorem 4.6 below.

EXAMPLE 4.4. Consider the tangle τ given by one positive crossing, see Figure 1. The matrix $M^\psi(\tau)$ coincides with the matrix of Example 2.3. The module H_∂ is generated by $x_1,\dots,x_4$ and

$$\alpha(\tau,\psi)=t_2x_3\wedge x_4+x_2\wedge x_3-t_2x_1\wedge x_4+(t_1-1)x_1\wedge x_3+x_1\wedge x_2\in\bigwedge^2 H_\partial.$$

Let τ be a welded tangle. $\pi(\tau)$ denotes the group defined by the Wirtinger method (ignoring the virtual crossings). Then, there is a system of generators of $\pi(\tau)$ in one-to-one correspondance with the arcs of τ. In particular, a μ-coloring of a welded tangle τ can be viewed as a group homomorphism ψ from $\pi(\tau)$ to the free abelian group $\langle t_1,\dots,t_\mu\rangle_{\text{abel}}$ generated by $t_1,\dots,t_\mu$.

The following proposition follows directly from the results of Satoh and Yajima [**Sat00**], [**Yaj64**].

PROPOSITION 4.5. *Let τ be a welded tangle. For any ribbon tangle T such that T is the image of τ by the Tube map, there is an isomophism*

$$\pi(\tau) \simeq \pi_1(B^4 \setminus T)$$

sending arcs of τ to meridians of T.

THEOREM 4.6. *Let μ be a positive integer and G be a free abelian group of rank μ. Let (τ, ψ) be a μ-colored welded tangle and (T, φ) be a G-colored ribbon tangle, such that T is the image of τ by the Tube map. Suppose that there are generators $t_1, \dots, t_\mu$ of G such that the following diagram commutes*:

$$\begin{array}{ccc} \pi_1(B^4 \setminus T) & \xrightarrow{\varphi} & G \\ \simeq\downarrow & & \downarrow\simeq \\ \pi(\tau) & \xrightarrow{\psi} & \langle t_1\rangle \times \cdots \times \langle t_\mu\rangle. \end{array}$$

Then,

$$\mathbf{A}(T, \varphi) = \alpha(\tau, \psi) \in \wedge^n H_\partial.$$

Welded string links are string links with possibly virtual crossings; for details on these last ones see [**KLW01**]. Through the Tube map, ribbon tubes can be described by welded string links, see [**ABMW17**, Section 3.3]. By Theorem 2.10, Proposition 2.9 and Theorem 4.6, the invariant α induces the usual (generalisation of) colored Burau representation—or Gassner, if the coloring is maximal—on (welded) string links [**Bar05**].

PROOF. The points of $\tau \cap \partial B^2$ are in one-to-one correspondence with the component of the trivial link L in $T \cap \partial B^4$, and H_∂ is generated by $x_1, \dots, x_{2n}$. By Proposition 4.5 and Fox calculus, the matrix $M^\psi(\tau)$ is a presentation matrix of the R-module $H_1^\varphi(X_T, *)$, viewed as a $\mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$-module through the choice of generators of G. Then $M^\psi(\tau)$ is used to compute $\mathcal{A}^\varphi$. By definition, for all $I \subset \{1, \dots, 2n\}$ with cardinal n:

$$\omega_\partial(\mathbf{A}(T, \varphi) \wedge x_I) = \mathcal{A}^\varphi(m_\partial x_I).$$

To calculate $\mathcal{A}^\varphi(m_\partial x_I)$, we consider the matrix $M^\psi(\tau)$, add n row vectors giving the element $mx_{i_1}, \dots, mx_{i_n}$ and compute the determinant of the resulting square matrix. Hence adding mx_{i_j} corresponds to add the $p + n + j^{\text{th}}$ row $(0, \dots, 0, 1, 0, \dots, 0)$ where 1 is at position $p + j$. We obtain $\mathcal{A}^\varphi(m_\partial x_I) = \epsilon_{\bar{I}} |M^\psi(\tau)_{\bar{I}}|$, where $\bar{I}$ is the complement of I and $\epsilon_{\bar{I}}$ is the signature of the permutation $I\bar{I}$ (where the elements of I in increasing order are followed by the elements of $\bar{I}$ in increasing order). Finally, we get

$$\mathbf{A}(T, \varphi) = \sum_I \epsilon_{\bar{I}} \cdot \omega_\partial(\mathbf{A}(T, \varphi) \wedge x_I) \cdot x_{\bar{I}} = \sum_I |M^\psi(\tau)_{\bar{I}}| \cdot x_{\bar{I}} = \alpha(\tau, \psi). \qquad \square$$

REMARK 4.7. The invariant α coincides up to a unit with the invariant of virtual tangles introduced by Archibald [**Arc10**]. It is worth mentioning that using a specific

canonical choice for the marked points x_i, her construction is well-defined, not only up to multiplication by a unit (through the multiplication by a correction term).

PROOF OF PROPOSITION 2.2. As observed in the proof of Theorem 4.6, the matrix $M^{\psi}(\tau)$ is a presentation matrix of $H_1^{\varphi}(X_T, *)$, whose size is $(p+n) \times (p+2n)$, see Remark 4.2. We get a presentation of deficiency n. □

4.2. The circuit algebra $\mathbf{Weld}_{\mu}$.

DEFINITION 4.8. A *circuit p-diagram* consists of the following data:

- The unit disk $D = D_0$ in $\mathbb{C}$ together with a finite set of disjoint subdisks $D_1, \dots, D_p$ in the interior of D. For every i in $\{0, \dots, p\}$, each D_i have $2n_i$ distinct marked points with sign on its boundary (with $n = n_0$), and a base point $*$ on the boundary of each disk.
- A finite set of embedded oriented arcs whose boundary are marked points in the D_i. They may cross each other along virtual crossings *only*. Each marked point is the boundary point of some string —which meets the corresponding disk transversally— and the sign is coherent with the orientation.

Circuit diagrams encode only the matching of marked points, see Figure 4.

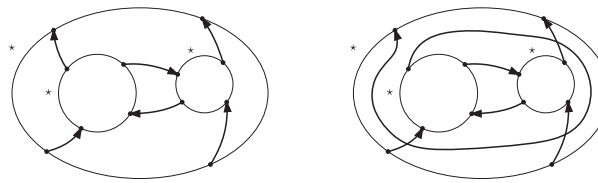

Figure 4. Two equivalent circuit diagrams.

DEFINITION 4.9. Consider a circuit p'-diagram P' and a circuit p''-diagram P'' such that D'_i is a disk of P' with $n'_i = n''$, for some $i \in \{1, \dots, p'\}$. If the signs of the marked points match, we define the diagram $P = P' \circ_i P''$ by rescaling via isotopy the tangle P'' so that the boundary of D'' is identified with the boundary of D'_i, and making its marked and base points coincide to those of D'_i. Then D'_i is removed to obtain $P' \circ_i P''$. This operation is well defined since the starting points eliminate any rotational ambiguity.

A *μ-colored* circuit diagram is a pair (P, ψ) where P is a circuit diagram and ψ is a map from the set of arcs of P to the set $\{t_1, \cdots, t_\mu\}$. Two μ-colored circuit diagrams can be composed once the coloring match on the boundary components. The μ-colored circuit diagrams form an operad $\mathcal{D}_\mu$. Let $S = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ be the Laurent polynomial ring. Let $\mathcal{H}om_\mu$ be the operad of tensor products of S-modules and S-linear maps. The circuit algebra Weld_μ is constructed as a morphism from the operad $\mathcal{D}_\mu$ to $\mathcal{H}om_\mu$ as follows.

Consider the unit circle with a base point and a set of marked points $X = \{x_1, \dots, x_{2k}\}$, for $k \geq 0$ (with a sign). To this data, we associate the module $\wedge^k H_\partial$,

where H_∂ is the free S-module of rank $2k$ generated by X. Let (P,ψ) be a μ-colored circuit diagram, and $M = \{c_1,\dots,c_q\}$ be the set of curves of P. Consider the free module H generated by M, and the volume form $\omega\colon \wedge^q H \to S$ related to this basis. For $i = 1,\dots,p$, H_{∂_i} denotes the module associated to the boundary circle ∂D_i and H_∂ the module associated to ∂D. Let $m_i\colon H_{\partial_i} \to H$ be the morphims defined by $m_i(x_j) = \mathrm{sign}(x_j)c_j$ if $x_j \in \partial c_j$. The mophism m_∂ is defined similarly. Let ω_{∂_i} be the volume form on H_{∂_i} related to the generating system of points of the circle ∂D_i. Set $m = \otimes_i(\wedge^{n_i} m_{\partial_i})$. To the colored diagram (P,ψ) we associate

$$\gamma_{P,\psi}\colon \bigotimes_{i=1}^{p} \wedge^{n_i} H_{\partial_i} \to \wedge^n H_\partial$$

such that, for $x \in \otimes_i\big(\wedge^{n_i} H_{\partial_i}\big)$,

$$\omega_C^\varphi(m(x)\wedge m_\partial(y)) = \omega_\partial(\gamma_{P,\psi}(x)\wedge y),\ \forall y \in \wedge^n H_\partial.$$

Then, we can prove Proposition 4.10 below, by repeating the arguments of the proof of Theorem 3.3.

PROPOSITION 4.10. *Weld_μ is a circuit algebra.*

Note that Weld_μ is similar to half densities introduced by Archibald, see [**Arc10**] for the definition of half densities and [**Dam16**] for the explicit correspondence. The morphism $\gamma_{P,\psi}$ could be written as the interior product relative to a subset corresponding to interior arcs of P.

The Tube map and the choice of a set of generators $\{t_1,\dots,t_\mu\}$ of G induce a surjective morphism of algebras

$$\mathrm{Weld}_\mu \longrightarrow \mathbf{C}ob_G.$$

Given a colored circuit diagram P and a collection of colored tangle diagrams $\tau_1,\dots,\tau_p$, one may create a new tangle, if the data on the boundaries and the colorings match, by gluing τ_i into the internal disk D_i of P. Similarly to Theorem 3.4, the invariant α commutes with this action of μ-colored circuit diagrams on μ-colored welded tangles. Indeed, one has the following proposition.

PROPOSITION 4.11. *Let (τ,ψ) be the μ-colored welded tangle obtained by gluing the μ-colored welded tangles $(\tau_1,\psi_1),\cdots,(\tau_p,\psi_p)$ to a μ-colored circuit diagram (P,χ). The following equality holds*:

$$\alpha(\tau,\psi) = \gamma_{P,\chi}\big(\alpha(\tau_1,\psi_1)\otimes\cdots\otimes\alpha(\tau_p,\psi_p)\big) \in \wedge^n H_\partial.$$

5. Examples.

Consider the tangle diagram τ, given in Figure 5. We let $G = \mathbb{Z} =< t >$ and ψ be the coloring sending all arcs of τ to t. In this section, we compute $\alpha(\tau,\psi)$ in different ways. By Theorem 4.6, this computes the value of $\mathbf{A}(T,\varphi)$ the image of (τ,ψ) by the

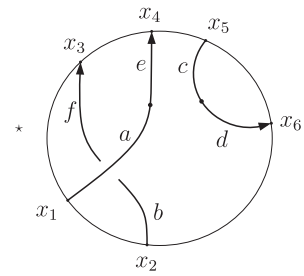

Figure 5. A welded tangle τ.

Tube map.

First, we compute $\alpha(\tau,\psi)$ directly. Label the arcs of τ with letters a to f as in Figure 5. We obtain the matrix

$$M^{\psi}(\tau) = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1-t & -1 & 0 & 0 & 0 & t \end{pmatrix} \end{array}.$$

The $\mathbb{Z}[t^{\pm 1}]$-module H_∂ is free, generated by $x_1,\dots,x_6$ and $\alpha(\tau,\psi) \in \bigwedge^3 H_\partial$ is given by

$$\begin{aligned} \alpha(\tau,\psi) = &- x_1\wedge x_2\wedge x_5 + x_1\wedge x_2\wedge x_6 + x_2\wedge x_5\wedge x_4 + (t-1)x_1\wedge x_5\wedge x_4 \\ &- tx_1\wedge x_5\wedge x_3 + (1-t)x_1\wedge x_6\wedge x_4 + tx_1\wedge x_6\wedge x_3 + tx_6\wedge x_4\wedge x_3 \\ &- x_2\wedge x_6\wedge x_4 - tx_5\wedge x_4\wedge x_3. \end{aligned}$$

We now consider (τ,ψ) as the composition of the circuit diagram (P,ψ) with (σ,ψ), see Figure 6. We have to compute $\gamma_{P,\psi}\colon \bigwedge^2 H_{\partial_1} \to \bigwedge^3 H_\partial$ (here $p=1$). Let H be the free

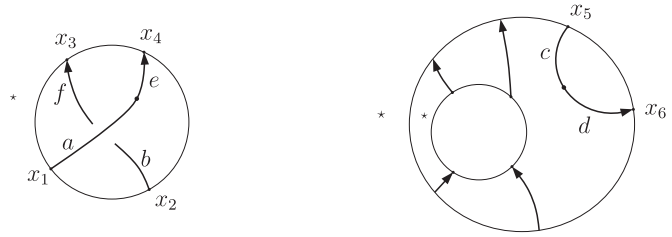

Figure 6. A welded tangle σ and a circuit diagram P.

module generated by the curves of σ, labelled a,b,e,f. We have that $H_{\partial_1} = \langle x_1,\dots,x_4\rangle$ and $H_\partial = \langle x_1,\dots,x_6\rangle$. Using the volume form on H related to the choice of the basis a,b,c,e,d,f, and the maps induced by the inclusions $m_1\colon H_{\partial_1} \to H$ and $m_\partial\colon H_\partial \to H$, we obtain

$$\gamma_{C,\psi}(x_i\wedge x_j) = x_i\wedge x_j\wedge (x_6 - x_5),\ \forall\ i,j = 1,\dots,4,$$

and

$$M^{\psi}(\sigma) = \begin{array}{c} \begin{array}{cccc} a & b & e & f \end{array} \\ \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1-t & t \end{pmatrix} \end{array}.$$

We get

$$\alpha(\sigma,\psi) = x_1 \wedge x_2 + (t-1)x_1 \wedge x_4 - tx_1 \wedge x_3 + x_2 \wedge x_4 - tx_3 \wedge x_4.$$

The composition $\alpha(\tau,\psi) = \gamma_{P,\psi}(\alpha(\sigma,\psi))$ gives the result. Finally, we consider τ as the composition of the tangle $\sigma \otimes \beta$ with the circuit Q, see Figure 7. Here $p = 2$, and

$$\gamma_{Q,\psi} \colon \textstyle\bigwedge^2 H_{\partial_1} \otimes H_{\partial_2} \to \bigwedge^3 H_{\partial}.$$

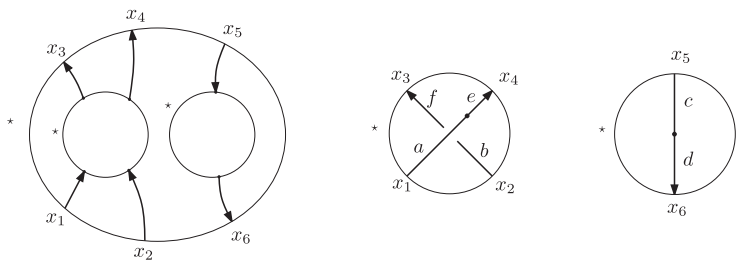

Figure 7. A circuit diagram Q representing a cobordism and two welded tangles σ and β.

As previously $\alpha(\sigma,\psi) = x_1 \wedge x_2 + (t-1)x_1 \wedge x_4 - tx_1 \wedge x_3 + x_2 \wedge x_4 - tx_3 \wedge x_4 \in \bigwedge^2 H_{\partial_1}$ and $\gamma(\beta) = x_6 - x_5 \in H_{\partial_2}$. The composition $\alpha(\tau,\psi) = \gamma_{Q,\psi}\,(\alpha(\sigma,\psi) \otimes \alpha(\beta,\psi))$ gives the result again.

Remark 5.1. In the sense of Subsection 2.4, there is a splitting of the tangle σ to an (oriented) braid σ_1 in B_2. We have

$$M_0 = \langle x_1, x_2 \rangle \text{ and } M_1 = \langle x_3, x_4 \rangle,$$

and $\alpha(\sigma,\psi) \in \bigwedge^2 H_{\partial_1} \simeq \bigwedge^2 (M_0 \oplus M_1) \simeq \bigoplus_{k=0}^{2} \bigwedge^k M_0 \otimes \bigwedge^{2-k} M_1$, similarly to the proof of Theorem 2.10. Then $\alpha(\sigma,\psi)$ decomposes as:

$$\big(-tx_3 \wedge x_4\big) \oplus \big((t-1)x_1 \otimes x_4 - tx_1 \otimes x_3 + x_2 \otimes x_4\big) \oplus \big(x_1 \wedge x_2\big).$$

Let $\omega_0 \colon \Lambda^2 M_0 \to R$ be the volume form related to the basis $\langle x_1, x_2 \rangle$. For $k = 1$, the element $(t-1)x_1 \otimes x_4 - tx_1 \otimes x_3 + x_2 \otimes x_4 \in M_0 \otimes M_1$ induces the morphism $M_0 \to M_1$:

$$x \mapsto (t-1)\omega_0(x \wedge x_1) \cdot x_4 - t\,\omega_0(x \wedge x_1) \cdot x_3 + \omega_0(x \wedge x_2) \cdot x_4.$$

The image of x_1 is x_4 and the image of x_2 is $(1-t)x_4 + tx_3$. This corresponds to the Burau representation. Similalrly, the other values of k give k^{th}-exterior powers of Burau,

up to a sign.

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