J. Math. Soc. Japan
Vol. 72, No. 1 (2020) pp. 81–118
doi: 10.2969/jmsj/74937493

Small gaps between the set of products of at most two primes

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(Received Apr. 18, 2016)
(Revised Aug. 30, 2018)

Abstract. In this paper, we apply the methods of Maynard and Tao to the set of products of two distinct primes (E_2-numbers). We obtain several results on the distribution of E_2-numbers and primes. Among others, the result of Goldston, Graham, Pintz and Yıldırım on small gaps between m consecutive E_2-numbers is improved.

1. Introduction.

The famous twin prime conjecture asserts that there exist infinitely many prime numbers p for which $p+2$ is also a prime, and this conjecture is widely believed to be true. More generally, about one hundred years ago, Hardy and Littlewood [**9**] conjectured the following, called the *Hardy–Littlewood prime k-tuple conjecture*. Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be a set of k distinct non-negative integers. Then, the number of those n below N such that all of $n + h_1, \ldots, n + h_k$ are primes will be asymptotically

$$\frac{N}{\log^k N} \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}$$

provided that $\nu_p(\mathcal{H}) < p$ for all primes p, where $\nu_p(\mathcal{H})$ denotes the number of residue classes mod p covered by $\mathcal{H}$. In this case we say that the set $\mathcal{H}$ is *admissible*. In particular, the twin prime conjecture is the case $k = 2$ and $\mathcal{H} = \{0, 2\}$. Although this conjecture is still far from our reach, several remarkable results toward it have been established. For example, in a celebrated paper [**1**], Chen proved that there exist infinitely many primes p for which $p + 2$ is either a prime or a product of two primes which are not necessarily distinct.

Recently the studies toward the twin prime conjecture have produced further progress. In 2009, Goldston, Pintz and Yıldırım [**2**] proved that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0, \tag{1.1}$$

where p_n denotes the n-th prime. Their method is called the *GPY sieve*. Moreover, they proved that if primes have the level of distribution θ for some $1/2 < \theta \leq 1$ (see the definition of $BV[\theta, \mathcal{P}]$ below), then

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2010 *Mathematics Subject Classification.* Primary 11N35; Secondary 11N05.
Key Words and Phrases. almost primes, twin prime conjecture, Maynard–Tao method.
This work was partially supported by the JSPS, KAKENHI Grant Number 16K17574.

$$\liminf_{n\to\infty}(p_{n+1}-p_n)<\infty. \tag{1.2}$$

The above assumption seems to be extremely difficult to prove, although it is known that the Bombieri–Vinogradov theorem assures that this is valid for $\theta \le 1/2$. With $\theta = 1/2$ one obtains (1.1). The case $\theta = 1$ is called the Elliott–Halberstam conjecture (EH). Several improvements have been made by the above three authors (see [**3**], [**4**], [**5**]). Among others, their best result on gaps between consecutive primes is that

$$\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\sqrt{\log p_n}(\log\log p_n)^2}<\infty. \tag{1.3}$$

Later, Pintz [**13**] improved this result and obtained

$$\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^{3/7}(\log\log p_n)^{4/7}}<\infty. \tag{1.4}$$

(See also [**6**].) In 2013, a stunning result was established by Zhang [**17**]. He obtained a stronger version of the Bombieri–Vinogradov theorem that is applicable when the moduli are free from large prime divisors, and using this, he proved that

$$\liminf_{n\to\infty}(p_{n+1}-p_n)<7\times 10^7, \tag{1.5}$$

that is, there exist infinitely many consecutive primes for which the gap is at most 7×10^7. The upper bound 7×10^7 has been improved by several experts successively. In the Polymath8a paper [**14**], the right hand side of (1.5) was replaced by 4680. Slightly later, Maynard [**10**] and Tao (private communication with Maynard) invented a refinement of the GPY sieve. In particular, Maynard proved that

$$\liminf_{n\to\infty}(p_{n+1}-p_n)\le 600. \tag{1.6}$$

They also proved the existence of the bounded gaps between m-consecutive primes for any fixed $m \ge 2$. One of the remarkable points is that their method is relatively quite simple, compared with Zhang's, and it is very convenient to extend or generalize to other situations. The current world record of the small gaps between primes is accomplished by the Polymath project [**15**], in which the upper bound

$$\liminf_{n\to\infty}(p_{n+1}-p_n)\le 246 \tag{1.7}$$

is obtained unconditionally. Moreover, it is proved that the right hand side of (1.7) may be replaced by 6 if we assume a strong form of the Elliott–Halberstam conjecture and that is the limit of this method.

In this paper, we treat the integers expressed by products of two distinct primes, called the E_2-*numbers* in [**7**], together with the prime numbers. In the papers [**7**], [**8**], Goldston, Graham, Pintz and Yıldırım investigated the distribution of E_2-numbers. We denote by q_n the n-th E_2-numbers. That is, $q_1 = 6, q_2 = 10, q_3 = 14, q_4 = 15, \ldots$. Using the GPY sieve, they proved that

$$\liminf_{n\to\infty}(q_{n+1}-q_n) \le 6. \tag{1.8}$$

Moreover, they proved that if the E_2-numbers have the level of distribution θ for some $0<\theta<1$, then for any sufficiently large $\rho \in \mathbf{N}$,

$$\liminf_{n\to\infty}(q_{n+\rho}-q_n) \le \rho(1+o(1))\exp\left(-\gamma+\frac{\rho}{2\theta}\right) \tag{1.9}$$

holds. Later Thorne [**16**] generalized their results to the set of products of r distinct primes for any $r\ge 2$. He applied the result to some related problems in number theory, for example, divisibility of class numbers, nonvanishing of L-functions at the central point, and triviality of ranks of elliptic curves.

The purpose of this paper is to apply the method of Maynard [**10**] and Tao to the distribution of E_2-numbers. Their multi-dimensional sieve enables us to establish rather small gaps between consecutive several E_2-numbers. In particular, the estimate (1.9) can be remarkably improved. We denote by $\mathcal{E}_2$ the set of all E_2-numbers. We denote by $\mathcal{P}$ the set of all prime numbers, and put $\mathcal{P}_2=\mathcal{P}\cup\mathcal{E}_2$. For a sufficiently large natural number N, we define

$$\beta(n) := \begin{cases} 1 & (n=p_1p_2,\ Y<p_1\le N^{1/2}<p_2) \\ 0 & (\text{otherwise}), \end{cases} \tag{1.10}$$

where $Y=N^{\eta}$, $1 \ll \eta < 1/4$. Throughout this paper, the implicit constants might be dependent on this η. (We will not necessarily mention this fact every time.) Next we define

$$\begin{gathered} \pi^{\flat}(N) := \sharp\{p\in\mathcal{P} \mid N\le p<2N\}, \\ \pi^{\flat}(N;q,a) := \sharp\{p\in\mathcal{P} \mid N\le p<2N, p\equiv a \ (\mathrm{mod}\ q)\}, \\ \pi_{\beta}(N) := \sum_{N<n\le 2N}\beta(n), \quad \pi_{\beta,q}(N) := \sum_{\substack{N<n\le 2N\\ (n,q)=1}}\beta(n), \\ \pi_{\beta}(N;q,a) := \sum_{\substack{N<n\le 2N\\ n\equiv a (\mathrm{mod}\ q)}}\beta(n). \end{gathered}$$

We write the following hypotheses:

Hypothesis 1. ($BV[\theta,\mathcal{P}]$). For any $\epsilon>0$, the estimate

$$\sum_{q\le N^{\theta-\epsilon}}\mu^2(q)\max_{(a,q)=1}\left|\pi^{\flat}(N;q,a)-\frac{\pi^{\flat}(N)}{\varphi(q)}\right| \ll_A \frac{N}{\log^A N} \quad (N\to\infty) \tag{1.11}$$

holds for any $A>0$.

Hypothesis 2. ($BV[\theta,\mathcal{E}_2]$). We fix an arbitrary $0<\eta<1/4$ in the definition of the function β. For any $\epsilon>0$, the estimate

$$\sum_{q\le N^{\theta-\epsilon}} \mu^2(q) \max_{(a,q)=1} \left| \pi_\beta(N;q,a) - \frac{\pi_{\beta,q}(N)}{\varphi(q)} \right| \ll_A \frac{N}{\log^A N} \quad (N\to\infty) \tag{1.12}$$

holds for any $A>0$.

We say that the set $\mathcal{P}$ (resp. $\mathcal{E}_2$) has *level of distribution* θ if $BV[\theta,\mathcal{P}]$ (resp. $BV[\theta,\mathcal{E}_2]$) holds. The *Bombieri–Vinogradov theorem* asserts that $BV[\theta,\mathcal{P}]$ is valid for $\theta = 1/2$. Motohashi [**12**] proved that $BV[\theta,\mathcal{E}_2]$ also holds for $\theta = 1/2$. The *Elliott–Halberstam conjecture* asserts that $BV[\theta,\mathcal{P}]$ will be valid for $\theta = 1$, and we expect that $BV[\theta,\mathcal{E}_2]$ will be valid for the same value. Hence we call $BV[1,\mathcal{P}]$ (resp. $BV[1,\mathcal{E}_2]$) the Elliott–Halberstam conjecture for $\mathcal{P}$ (resp. $\mathcal{E}_2$).

The main theorems of this paper are as follows:

Theorem 1.1. *Assume that the sets $\mathcal{P}$ and $\mathcal{E}_2$ have level of distribution $\theta>0$. Then, for any $\epsilon>0$, there exists $\rho_\epsilon>0$ such that for any integer $\rho>\rho_\epsilon$, the inequality*

$$\liminf_{n\to\infty}(q_{n+\rho}-q_n) \le \exp\left(\frac{(2+\epsilon)\rho}{3\theta\log\rho}\right) \tag{1.13}$$

holds. In particular, unconditionally we have

$$\liminf_{n\to\infty}(q_{n+\rho}-q_n) \le \exp\left(\frac{2(2+\epsilon)\rho}{3\log\rho}\right) \tag{1.14}$$

for any $\rho>\rho_\epsilon$.

Theorem 1.2. *For any admissible set $\mathcal{H}=\{h_1,h_2,\ldots,h_6\}$, there exist infinitely many n such that at least three of $n+h_1, n+h_2,\ldots,n+h_6$ are in $\mathcal{P}_2$.*

The set $\mathcal{H}=\{0,4,6,10,12,16\}$ is an admissible set with six elements. Hence if we denote by r_n the n-th element of $\mathcal{P}_2=\mathcal{P}\cup\mathcal{E}_2$, unconditionally we have

$$\liminf_{n\to\infty}(r_{n+2}-r_n) \le 16. \tag{1.15}$$

If we assume the Elliott–Halberstam conjecture for both $\mathcal{P}$ and $\mathcal{E}_2$, far stronger results can be obtained:

Theorem 1.3. *Assume the Elliott–Halberstam conjecture for $\mathcal{P}$ and $\mathcal{E}_2$. Then, there exist infinitely many n such that all of $n, n+2, n+6$ are in $\mathcal{P}_2$. In particular,*

$$\liminf_{n\to\infty}(r_{n+2}-r_n) \le 6. \tag{1.16}$$

We note that Maynard [**11**] unconditionally proved that $n(n+2)(n+6)$ has at most seven prime factors infinitely often. Theorem 1.3 is regarded as a (conditional) improvement of his theorem. Finally,

Theorem 1.4. *Assume the Elliott–Halberstam conjecture for $\mathcal{P}$ and $\mathcal{E}_2$. Then we have*

$$\liminf_{n\to\infty}(q_{n+2}-q_n) \le 12. \tag{1.17}$$

2. Notation and preparations for the proofs.

Let $\mathcal{H} = \{h_1, \dots, h_k\}$ be an admissible set. Throughout this paper, we assume that the elements of $\mathcal{H}$ are bounded, that is, there exists a positive constant $C = C_k$ depending only on k such that $h_i \le C$ holds for $i = 1, \dots, k$. We denote by $\chi_{\mathcal{P}}$ the characteristic function of $\mathcal{P}$. We put

$$D_0 = \log\log\log N, \quad W = \prod_{p\le D_0} p \ll (\log\log N)^2.$$

We assume that both prime numbers and E_2-numbers have level of distribution θ. By the Chinese remainder theorem, we can choose $\nu_0 \in \mathbf{N}$ so that all of $\nu_0 + h_i$ $(i = 1, \dots, k)$ are coprime to W. For a smooth function $F : \mathbf{R}^k \to \mathbf{R}$ supported in

$$\mathcal{R}_k := \left\{ (x_1, \dots, x_k) \;\middle|\; x_1, \dots, x_k \ge 0, \sum_{i=1}^k x_i \le 1 \right\},$$

we put

$$\lambda_{d_1,\dots,d_k} = \left(\prod_{i=1}^k \mu(d_i)d_i\right) \sum_{\substack{r_1,\dots,r_k\\ d_i|r_i(\forall i)\\ (r_i,W)=1(\forall i)}} \frac{\mu(\prod_{i=1}^k r_i)^2}{\prod_{i=1}^k \varphi(r_i)} F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_k}{\log R}\right) \tag{2.1}$$

if $(d_1, \dots, d_k)$ satisfies the conditions that $\prod_{i=1}^k d_i$ is square-free, $\prod_{i=1}^k d_i < R$, and $(d_i, W) = 1$ for $i = 1, \dots, k$, where $R = N^{\theta/2-\delta}$ and δ is a sufficiently small positive constant. If $(d_1, \dots, d_k)$ does not satisfy at least one of these conditions, put $\lambda_{d_1,\dots,d_k} := 0$. We define the weight w_n by

$$w_n = \left(\sum_{d_i|n+h_i(\forall i)} \lambda_{d_1,\dots,d_k}\right)^2. \tag{2.2}$$

To find small gaps between E_2-numbers, for a natural number ρ, we consider the sum

$$S(N,\rho) = \sum_{\substack{N\le n<2N\\ n\equiv\nu_0 (\mathrm{mod}\ W)}} \left(\sum_{m=1}^k \beta(n+h_m) - \rho\right) w_n. \tag{2.3}$$

If $S(N, \rho)$ becomes positive for any sufficiently large N, there exists $n \in [N, 2N)$ such that at least $\rho + 1$ of $n + h_1, \dots, n + h_k$ are E_2-numbers. Hence one has

$$\liminf_{n\to\infty}(q_{n+\rho}-q_n) \le \max_{1\le i<j\le k} |h_j - h_i|.$$

Similarly, to find small gaps between the set of primes and E_2-numbers, for a natural number ρ, we consider the sum

$$S^{'}(N,\rho)=\sum_{\substack{N\le n<2N\\ n\equiv\nu_0(\mathrm{mod}\ W)}}\left(\sum_{m=1}^{k}(\beta(n+h_m)+\chi_{\mathcal{P}}(n+h_m))-\rho\right)w_n. \tag{2.4}$$

If $S^{'}(N,\rho)$ becomes positive for any sufficiently large N, there exists $n\in[N,2N)$ such that at least $\rho+1$ of $n+h_1,\dots,n+h_k$ are in $\mathcal{P}\cup\mathcal{E}_2$. Hence our problem is to evaluate the sums

$$S_0=\sum_{\substack{N\le n<2N\\ n\equiv\nu_0(\mathrm{mod}\ W)}}w_n,\quad S_1^{(m)}=\sum_{\substack{N\le n<2N\\ n\equiv\nu_0(\mathrm{mod}\ W)}}\chi_{\mathcal{P}}(n+h_m)w_n, \tag{2.5}$$

and

$$S_2^{(m)}=\sum_{\substack{N\le n<2N\\ n\equiv\nu_0(\mathrm{mod}\ W)}}\beta(n+h_m)w_n \tag{2.6}$$

for $m=1,\dots,k$. Maynard ([**10**, Proposition 4.1]) computed the sums in (2.5). The results are as follows:

Proposition 2.1. *We put*

$$I_k(F)=\int_0^1\cdots\int_0^1 F(t_1,\dots,t_k)^2dt_1\cdots dt_k,$$

$$J_k^{(m)}(F)=\int_0^1\cdots\int_0^1\left(\int_0^1 F(t_1,\dots,t_k)dt_m\right)^2dt_1\cdots dt_{m-1}dt_{m+1}\cdots dt_k$$

for $m=1,\dots,k$. Then, if $I_k(F)\ne 0$, we have

$$S_0=\frac{(1+o(1))\varphi(W)^kN(\log R)^k}{W^{k+1}}I_k(F), \tag{2.7}$$

and if $J_k^{(m)}(F)\ne 0$, we have

$$S_1^{(m)}=\frac{(1+o(1))\varphi(W)^kN(\log R)^{k+1}}{W^{k+1}\log N}J_k^{(m)}(F)\quad (m=1,\dots,k) \tag{2.8}$$

as $N\to\infty$.

Hence the main problem of this paper is the computation of $S_2^{(m)}$. By substituting (2.2) into (2.6) and interchanging the summations, we have

$$S_2^{(m)} = \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} \lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} \sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 (\mathrm{mod}\ W) \\ [d_i,e_i] | n+h_i (\forall i)}} \beta(n+h_m). \tag{2.9}$$

The integers d_m, e_m must satisfy

$$d_m, e_m | n + h_m. \tag{2.10}$$

Since $\beta(n+h_m) = 0$ unless $n + h_m = p_1 p_2$, $Y < p_1 \le N^{1/2} < p_2$, and since $\lambda_{d_1,\dots,d_k} = 0$ unless $\prod_{i=1}^k d_i < R < N^{1/2}$, only the following four types contribute to the sum above:
1) $d_m = p, e_m = 1 \quad (Y < p < R)$,
2) $d_m = 1, e_m = p \quad (Y < p < R)$,
3) $d_m = e_m = 1$,
4) $d_m = e_m = p \quad (Y < p < R)$.
Correspondingly we decompose

$$S_2^{(m)} = S_{2,I}^{(m)} + S_{2,II}^{(m)} + S_{2,III}^{(m)} + S_{2,IV}^{(m)}. \tag{2.11}$$

The following three sections will be devoted to compute these terms.

3. The computation of $S_{2,I}^{(m)}, S_{2,II}^{(m)}$.

We first compute $S_{2,I}^{(m)} (= S_{2,II}^{(m)})$. By interchanging the summations, we have

$$S_{2,I}^{(m)} = \sum_{Y<p<R} \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m = p, e_m = 1}} \lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} \sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 (\mathrm{mod}\ W) \\ [d_i,e_i] | n+h_i (\forall i)}} \beta(n+h_m). \tag{3.1}$$

By our choice of ν_0 and the assumption that the elements of $\mathcal{H}$ are bounded, the inner sum is empty if $W, [d_1,e_1], \dots, [d_k,e_k]$ are not pairwise coprime. We put

$$q = W \prod_{i=1}^k [d_i, e_i].$$

When $W, [d_1,e_1], \dots, [d_k,e_k]$ are pairwise coprime, the sum over n in (3.1) is rewritten as a sum over a single residue class modulo q. That is, there exists a unique $\nu \pmod{q}$ such that $\nu \equiv \nu_0 \pmod{W}$, $\nu + h_i \equiv 0 \pmod{[d_i, e_i]}$ and

$$\sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 (\mathrm{mod}\ W) \\ [d_i,e_i] | n+h_i (\forall i)}} \beta(n+h_m) = \sum_{\substack{N \le n < 2N \\ n \equiv \nu (\mathrm{mod}\ q)}} \beta(n+h_m) \tag{3.2}$$

holds. We put $\nu_m = \nu + h_m$. Then,

$$(\nu_m, q) = p. \tag{3.3}$$

We will check this briefly. Since $[d_m, e_m] = p$, p divides q, and since $\nu_m = \nu + h_m \equiv 0 \pmod{[d_m, e_m]}$, p divides ν_m. Clearly, p^2 does not divide q. Let p' be another prime and assume that $p' | (\nu_m, W)$. Since $p' | \nu_m$, we have $\nu + h_m \equiv 0 \pmod{p'}$. Since $p' | W$, we have $\nu \equiv \nu_0 \pmod{p'}$. Therefore, we find that $\nu_0 + h_m \equiv 0 \pmod{p'}$, hence $p' | (\nu_0 + h_m, W)$. This fact contradicts our assumption that $\nu_0 + h_m$ is coprime to W. If $p' | (\nu_m, d_j)$ for some $j \neq m$, then $\nu + h_m \equiv 0 \pmod{p'}$ and $\nu \equiv -h_j \pmod{p'}$. Hence $h_m \equiv h_j \pmod{p'}$. However, since d_j is coprime to W, the condition $p' | d_j$ implies $p' \geq \log\log\log N$. Therefore, the conclusion that $h_m \equiv h_j \pmod{p'}$ $(j \neq m)$ contradicts our assumption that the elements of $\mathcal{H}$ are bounded. Hence p' does not divide (ν_m, d_j). In a similar way, we find that p' does not divide (ν_m, e_j). Thus we obtain (3.3).

Therefore, there exists a unique $\nu'_m \pmod{q/p}$ such that $p\nu'_m \equiv \nu_m \pmod q$ and the right hand side of (3.2) becomes

$$\begin{aligned}\sum_{\substack{N \leq n < 2N \\ n \equiv \nu \pmod q}} \beta(n + h_m) &= \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_m \pmod q}} \beta(n) + O(1) \\ &= \sum_{\substack{N/p \leq n' < 2N/p \\ n' \equiv \nu'_m \pmod{q/p}}} \beta(pn') + O(1) \\ &= \sum_{\substack{N/p \leq n' < 2N/p \\ n' \equiv \nu'_m \pmod{q/p}}} \chi_{\mathcal{P}}(n') + O(1). \end{aligned} \tag{3.4}$$

We note that $(\nu'_m, q/p) = 1$, by (3.3). The sum in the right hand side of (3.4) becomes

$$\begin{aligned}\sum_{\substack{N/p \leq n' < 2N/p \\ n' \equiv \nu'_m \pmod{q/p}}} \chi_{\mathcal{P}}(n') &= \frac{1}{\varphi(q/p)} \sum_{N/p \leq n' < 2N/p} \chi_{\mathcal{P}}(n') + \Delta\left(\frac{N}{p}; \frac{q}{p}, \nu'_m\right) \\ &= \frac{1}{\varphi(q/p)} \pi^\flat\left(\frac{N}{p}\right) + \Delta\left(\frac{N}{p}; \frac{q}{p}, \nu'_m\right), \end{aligned} \tag{3.5}$$

where

$$\Delta(N; q, a) = \sum_{\substack{N \leq n < 2N \\ n \equiv a \pmod q}} \chi_{\mathcal{P}}(n) - \frac{\pi^\flat(N)}{\varphi(q)}.$$

By combining (3.4), (3.5), we have

$$\sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod W \\ [d_i, e_i] | n + h_i (\forall i)}} \beta(n + h_m) = \frac{\pi^\flat(N/p)}{\varphi(q/p)} + \Delta\left(\frac{N}{p}; \frac{q}{p}, \nu'_m\right) + O(1). \tag{3.6}$$

By substituting (3.6) into (3.1), we obtain

$$S_{2,I}^{(m)} = \frac{1}{\varphi(W)} \sum_{Y<p<R} \pi^\flat\left(\frac{N}{p}\right) \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ d_m=p,e_m=1}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i\neq m}\varphi([d_i,e_i])}$$

$$+ O\left(\sum_{Y<p<R} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ d_m=p,e_m=1}} |\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}| \left(\left|\Delta\left(\frac{N}{p};\frac{q}{p},\nu'_m\right)\right| + 1\right)\right), \tag{3.7}$$

where the sum $\sum'$ indicates that $d_1,\ldots,d_k,e_1,\ldots,e_k$ are restricted to those satisfying the condition that $W,[d_1,e_1],\ldots,[d_k,e_k]$ are pairwise coprime. We now evaluate the error term of (3.7). The conductor q is square-free, and satisfies $q < R^2W$. Moreover, p divides q. The number of pairs $(d_1,\ldots,d_k,e_1,\ldots,e_k)$ satisfying

$$q = W\prod_{i=1}^{k}[d_i,e_i]$$

is at most $\tau_{3k}(q)$. Therefore, by the Cauchy–Schwarz inequality, we have

$$\sum_{Y<p<R} \sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ d_m=p,e_m=1}} |\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}| \left(\left|\Delta\left(\frac{N}{p};\frac{q}{p},\nu'_m\right)\right| + 1\right)$$

$$\ll \lambda_{\max}^2 \sum_{Y<p<R} \sum_{\substack{q<R^2W\\ p\mid q}} \mu^2(q)\tau_{3k}(q)\left(\left|\Delta\left(\frac{N}{p};\frac{q}{p},\nu'_m\right)\right| + 1\right)$$

$$\ll \lambda_{\max}^2 \sum_{Y<p<R}\left(\sum_{q'<R^2W/p} \mu^2(pq')\tau_{3k}(pq')^2\frac{N}{p\varphi(q')}\right)^{1/2}\left(\sum_{q'<R^2W/p}\mu^2(q')\Delta^*\left(\frac{N}{p};q'\right)\right)^{1/2},$$

where

$$\lambda_{\max} = \sup_{d_1,\ldots,d_k}|\lambda_{d_1,\ldots,d_k}|, \quad \Delta^*(N;q) = \max_{(a,q)=1}|\Delta(N;q,a)|.$$

Under the assumption of $BV[\theta,\mathcal{P}]$, the above is at most

$$\begin{aligned} &\ll_A \lambda_{\max}^2 \sum_{Y<p<R} p^{-1/2}N^{1/2}(\log N)^{a_k}\cdot\frac{(N/p)^{1/2}}{(\log N)^A} \\ &\ll_A \lambda_{\max}^2 N(\log N)^{a_k-A}\sum_{Y<p<R}\frac{1}{p} \ll_B \frac{\lambda_{\max}^2 N}{(\log N)^B}, \end{aligned} \tag{3.8}$$

where a_k is some positive integer depending only on k, and A is an arbitrary positive number, and $B = A - a_k - 1$. Here, we evaluated the first q'-sum by a standard method. Hence we may regard B as an arbitrary positive number, once k is fixed. Combining

(3.7), (3.8), we have

$$S_{2,I}^{(m)} = \frac{1}{\varphi(W)} \sum_{Y<p<R} (p-1)\pi^{\flat}\left(\frac{N}{p}\right) \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=p, e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} + O_B\left(\frac{\lambda_{\max}^2 N}{(\log N)^B}\right). \tag{3.9}$$

Next we compute the sum

$$\sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=p, e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])}.$$

Let g be the totally multiplicative function defined by $g(q) = q-2$ for $q \in \mathcal{P}$. Then, when d_i, e_i are square-free, we have

$$\frac{1}{\varphi([d_i,e_i])} = \frac{1}{\varphi(d_i)\varphi(e_i)} \sum_{u_i \mid d_i, e_i} g(u_i).$$

Moreover, the condition that $(d_i, e_j) = 1$ $(i \neq j)$ is replaced by multiplying $\sum_{s_{i,j} \mid d_i, e_j} \mu(s_{i,j})$. Since $\lambda_{d_1,\dots,d_k} = 0$ unless $(d_i, d_j) = 1$ $(\forall i \neq j)$, we may add the condition that $s_{i,j}$ is coprime to $u_i, u_j, s_{i,a} (a \neq j), s_{b,j} (b \neq i)$. We denote by $\sum^*$ the sum over $s_{1,2}, \dots, s_{k,k-1}$ restricted to those satisfying this condition. Then we have

$$\begin{aligned} \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=p, e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} &= \sum_{u_1,\dots,u_k} \prod_{i=1}^k g(u_i) \sideset{}{^*}\sum_{s_{1,2},\dots,s_{k,k-1}} \left(\prod_{1 \le i \neq j \le k} \mu(s_{i,j})\right) \\ &\quad \times \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ u_i \mid d_i, e_i (\forall i) \\ s_{i,j} \mid d_i, e_j (i \neq j) \\ d_m=p, e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi(d_i)\varphi(e_i)}. \end{aligned} \tag{3.10}$$

We put

$$y_{r_1,\dots,r_k}^{(m)}(p) = \left(\prod_{i=1}^k \mu(r_i) g(r_i)\right) \sum_{\substack{d_1,\dots,d_k \\ r_i \mid d_i (\forall i) \\ d_m = p}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^k \varphi(d_i)}, \tag{3.11}$$

$$y_{r_1,\dots,r_k}^{(m)} = \left(\prod_{i=1}^k \mu(r_i) g(r_i)\right) \sum_{\substack{d_1,\dots,d_k \\ r_i \mid d_i (\forall i) \\ d_m = 1}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^k \varphi(d_i)}, \tag{3.12}$$

and $r := \prod_{i=1}^k r_i$. Then, $y_{r_1,\dots,r_k}^{(m)}(p) = 0$ unless r is square-free, $(r, W) = 1$, $r < R$, and

$r_m = 1$ or p. Similarly, $y^{(m)}_{r_1,\ldots,r_k} = 0$ unless r is square-free, $(r, W) = 1$, $r < R$, and $r_m = 1$. Then the right hand side of (3.10) is expressed by

$$\sum_{u_1,\ldots,u_k} \left(\prod_{i=1}^{k} \frac{\mu^2(u_i)}{g(u_i)} \right) \sideset{}{^*}\sum_{s_{1,2},\ldots,s_{k,k-1}} \left(\prod_{1\le i\ne j\le k} \frac{\mu(s_{i,j})}{g(s_{i,j})^2} \right) y^{(m)}_{a_1,\ldots,a_k}(p) y^{(m)}_{b_1,\ldots,b_k}, \tag{3.13}$$

where $a_i = u_i \prod_{j\ne i} s_{i,j}$, $b_i = u_i \prod_{j\ne i} s_{j,i}$. To obtain (3.13), we used $\mu(a_i) = \mu(u_i)\prod_{j\ne i}\mu(s_{i,j})$, $\mu(b_i) = \mu(u_i)\prod_{j\ne i}\mu(s_{j,i})$, $g(a_i) = g(u_i)\prod_{j\ne i} g(s_{i,j})$ and $g(b_i) = g(u_i)\prod_{j\ne i} g(s_{j,i})$. Since $s_{i,j}$ is coprime to $u_i, u_j, s_{i,a}(a \ne j), s_{b,j}(b \ne i)$, these identities hold. We consider the contribution of the terms with $s_{i,j} \ne 1$ to (3.13). By the condition of the support of $y^{(m)}_{r_1,\ldots,r_k}$, only the terms with $s_{i,j} = 1$ or $s_{i,j} > D_0$ contribute to the sum above. Hence the contribution of the terms with $s_{i,j} \ne 1, a_m = 1$ is at most

$$\begin{aligned}
&y^{(m)}_{\max}(p)|_{r_m=1}\ y^{(m)}_{\max} \left(\sum_{\substack{u<R \\ (u,W)=1}} \frac{\mu^2(u)}{g(u)} \right)^{k-1} \left(\sum_{s} \frac{\mu^2(s)}{g(s)^2} \right)^{k^2-k-1} \left(\sum_{s_{i,j}>D_0} \frac{\mu^2(s_{i,j})}{g(s_{i,j})^2} \right) \\
&\ll y^{(m)}_{\max}(p)|_{r_m=1}\ y^{(m)}_{\max} \left(\frac{\varphi(W)}{W} \log R \right)^{k-1} \cdot 1 \cdot D_0^{-1} \\
&\ll \frac{\varphi(W)^{k-1}(\log R)^{k-1}}{W^{k-1}D_0} y^{(m)}_{\max}(p)|_{r_m=1}\ y^{(m)}_{\max},
\end{aligned} \tag{3.14}$$

where

$$y^{(m)}_{\max}(p)|_{r_m=\mathfrak{p}} := \sup_{\substack{r_1,\ldots,r_k \\ r_m=\mathfrak{p}}} |y^{(m)}_{r_1,\ldots,r_k}(p)| \quad (\mathfrak{p} = 1 \text{ or } p), \quad y^{(m)}_{\max} := \sup_{r_1,\ldots,r_k} |y^{(m)}_{r_1,\ldots,r_k}|.$$

Similarly, the contribution of the terms with $s_{i,j} \ne 1$, $a_m = p$ is, since in this case u_m or some $s_{m,j}$ is equal to p, at most

$$\frac{\varphi(W)^{k-1}(\log R)^{k-1}}{pW^{k-1}D_0} y^{(m)}_{\max}(p)|_{r_m=p}\ y^{(m)}_{\max}. \tag{3.15}$$

Combining (3.10), (3.13), (3.14) and (3.15), we have

$$\begin{aligned}
&\sideset{}{'}\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ d_m=p, e_m=1}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^{k}\varphi([d_i,e_i])} \\
&= \sum_{u_1,\ldots,u_k} \frac{y^{(m)}_{u_1,\ldots,u_k}(p) y^{(m)}_{u_1,\ldots,u_k}}{\prod_{i=1}^{k} g(u_i)} \\
&\quad + O\left(\frac{\varphi(W)^{k-1}(\log R)^{k-1}}{W^{k-1}D_0} \left(y^{(m)}_{\max}(p)|_{r_m=1} + \frac{y^{(m)}_{\max}(p)|_{r_m=p}}{p} \right) y^{(m)}_{\max} \right).
\end{aligned} \tag{3.16}$$

(We note that we may remove $\mu^2(u_i)$ in (3.13), since $y^{(m)}_{u_1,\ldots,u_k} = 0$ unless $u_1, \ldots, u_k$ are all square-free.) We put

$$y_{r_1,\dots,r_k} = \left(\prod_{i=1}^{k} \mu(r_i)\varphi(r_i)\right) \sum_{\substack{d_1,\dots,d_k \\ r_i \mid d_i (\forall i)}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^{k} d_i}. \tag{3.17}$$

Using the test function F, this is expressed by

$$y_{r_1,\dots,r_k} = F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_k}{\log R}\right) \tag{3.18}$$

(see [**10**, p. 400]). It is proved in [**10**] that

$$\lambda_{\max} \ll y_{\max}(\log R)^k,$$

where

$$y_{\max} = \sup_{r_1,\dots,r_k} |y_{r_1,\dots,r_k}|.$$

Hence the error term in (3.9) is replaced by $y_{\max}^2 N/(\log N)^B$. We substitute (3.16) into (3.9). Since

$$(p-1)\pi^\flat\left(\frac{N}{p}\right) = \frac{N}{\log N/p} + O_\eta\left(\frac{N}{(\log N)^2}\right)$$

for $Y = N^\eta < p < R = N^{(\theta/2)-\delta}$, we obtain the following result:

LEMMA 3.1. *Assume $BV[\theta, \mathcal{P}]$ for $0 < \theta \le 1$. Then*

$$\begin{aligned} S_{2,I}^{(m)} &= \frac{N}{\varphi(W)}\left(1 + O\left(\frac{1}{\log N}\right)\right) \sum_{Y<p<R} \frac{1}{\log N/p} \sum_{u_1,\dots,u_k} \frac{y_{u_1,\dots,u_k}^{(m)}(p) y_{u_1,\dots,u_k}^{(m)}}{\prod_{i=1}^{k} g(u_i)} \\ &\quad + O\left(\frac{N\varphi(W)^{k-2}(\log N)^{k-2}}{W^{k-1}D_0} y_{\max}^{(m)} \sum_{Y<p<R} \left(y_{\max}^{(m)}(p)|_{r_m=1} + \frac{y_{\max}^{(m)}(p)|_{r_m=p}}{p}\right)\right) \\ &\quad + O_B\left(\frac{N y_{\max}^2}{(\log N)^B}\right). \end{aligned} \tag{3.19}$$

By symmetry, the same result also holds for $S_{2,II}^{(m)}$. Next, we compute the inner sum in the main term of (3.19). The following result is obtained by Maynard ([**10**, Lemma 5.3]).

LEMMA 3.2. *If $r_m = 1$, we have*

$$y_{r_1,\dots,r_k}^{(m)} = \sum_{a_m} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\left(\frac{y_{\max}\varphi(W)\log R}{WD_0}\right). \tag{3.20}$$

Next, if $\prod_{i=1}^{k} d_i$ is square-free, we have

$$\lambda_{d_1,\dots,d_k} = \left(\prod_{i=1}^{k} \mu(d_i)d_i\right) \sum_{\substack{a_1,\dots,a_k \\ d_i \mid a_i (\forall i)}} \frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^{k} \varphi(a_i)}$$

(see [**10**, p. 393, (5.8)]). By substituting this into (3.11) and interchanging the order of summation, we have

$$y^{(m)}_{r_1,\dots,r_k}(p) = \left(\prod_{i=1}^{k} \mu(r_i)g(r_i)\right) \sum_{\substack{a_1,\dots,a_k \\ r_i \mid a_i (\forall i) \\ p \mid a_m}} \frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^{k} \varphi(a_i)} \sum_{\substack{d_1,\dots,d_k \\ d_m = p \\ r_i \mid d_i, d_i \mid a_i (\forall i)}} \frac{\prod_{i=1}^{k} \mu(d_i)d_i}{\prod_{i=1}^{k} \varphi(d_i)}.$$

If $r_m = 1$ or p, we find that

$$\begin{aligned} \sum_{\substack{d_1,\dots,d_k \\ d_m = p \\ r_i \mid d_i, d_i \mid a_i (\forall i)}} \frac{\prod_{i=1}^{k} \mu(d_i)d_i}{\prod_{i=1}^{k} \varphi(d_i)} &= \frac{\mu(p)p}{\varphi(p)} \sum_{\substack{d_1,\dots,d_{m-1},d_{m+1},\dots,d_k \\ r_i \mid d_i, d_i \mid a_i (\forall i \neq m)}} \frac{\prod_{i \neq m} \mu(d_i)d_i}{\prod_{i \neq m} \varphi(d_i)} \\ &= -\frac{p}{p-1} \prod_{i \neq m} \frac{\mu(a_i)r_i}{\varphi(a_i)}. \end{aligned}$$

Therefore,

$$y^{(m)}_{r_1,\dots,r_k}(p) = -\frac{p}{p-1} \left(\prod_{i=1}^{k} \mu(r_i)g(r_i)\right) \sum_{\substack{a_1,\dots,a_k \\ r_i \mid a_i (\forall i) \\ p \mid a_m}} \frac{y_{a_1,\dots,a_k}}{\prod_{i=1}^{k} \varphi(a_i)} \prod_{i \neq m} \frac{\mu(a_i)r_i}{\varphi(a_i)}. \tag{3.21}$$

By the condition of the support of $y_{a_1,\dots,a_k}$, we may restrict the sum to $(a_i, W) = 1$ $(\forall i)$. Then, if $a_j \neq r_j$, it follows that $a_j > D_0 r_j$. For $j \neq m$, the contribution of such terms is at most

$$\begin{aligned} &y_{\max} r_m^{-1} \left(\prod_{i=1}^{k} g(r_i)r_i\right) \left(\sum_{\substack{a_j > D_0 r_j \\ r_j \mid a_j}} \frac{\mu^2(a_j)}{\varphi(a_j)^2}\right) \prod_{i \neq j,m} \left(\sum_{r_j \mid a_j} \frac{\mu^2(a_j)}{\varphi(a_j)^2}\right) \sum_{\substack{p \mid a_m \\ a_m < R \\ (a_m, W) = 1}} \frac{\mu^2(a_m)}{\varphi(a_m)} \\ &\ll y_{\max} \left(\prod_{i \neq m} \frac{g(r_i)r_i}{\varphi(r_i)^2}\right) \cdot g(r_m) \cdot D_0^{-1} \cdot 1 \cdot \sum_{\substack{a'_m < R/p \\ (a'_m, W) = 1}} \frac{\mu^2(a'_m)}{\varphi(p)\varphi(a'_m)} \\ &\ll \frac{y_{\max} g(r_m) \varphi(W) \log R/p}{W D_0 \varphi(p)}. \end{aligned}$$

Hence we have

$$y^{(m)}_{r_1,\dots,r_k}(p) = -\frac{p}{p-1}\left(\prod_{i\neq m}\frac{\mu^2(r_i)g(r_i)r_i}{\varphi(r_i)^2}\right)\mu(r_m)g(r_m)\sum_{p|a_m}\frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\left(\frac{y_{\max}g(r_m)\varphi(W)\log R/p}{WD_0\varphi(p)}\right). \tag{3.22}$$

Since $y_{r_1,\dots,r_k} = 0$ unless $r_1,\dots,r_k$ are square-free, we may remove the factors $\mu^2(r_i)$ $(i\neq m)$. Finally, by applying

$$\frac{p}{p-1} = 1 + O(N^{-\eta}), \quad \prod_{i\neq m}\frac{g(r_i)r_i}{\varphi(r_i)^2} = 1 + O(D_0^{-1}) \quad \left(\text{if } \left(\prod_{i\neq m} r_i, W\right) = 1\right),$$

we obtain the following result:

Lemma 3.3. *If $r_m = 1$ or p, we have*

$$y^{(m)}_{r_1,\dots,r_k}(p) = -\mu(r_m)g(r_m)\sum_{\substack{a_m\\ p|a_m}}\frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} + O\left(\frac{y_{\max}g(r_m)\varphi(W)\log R/p}{WD_0\varphi(p)}\right). \tag{3.23}$$

By (3.23), we have

$$y^{(m)}_{\max}(p)|_{r_m=p} \ll \frac{y_{\max}\varphi(W)\log R/p}{W}, \tag{3.24}$$

and

$$y^{(m)}_{\max}(p)|_{r_m=1} \ll \frac{y_{\max}\varphi(W)\log R/p}{pW}. \tag{3.25}$$

Next we compute the sum over a_m. For this purpose, we use the following lemma, proved in [**8**] (see Lemma 6.1 of [**10**]).

Lemma 3.4. *Let $A_1, A_2, L > 0$ and γ be a multiplicative function satisfying*

$$0 \le \frac{\gamma(q)}{q} \le 1 - A_1,$$

$$-L \le \sum_{\substack{w\le q\le z\\ q\in\mathcal{P}}}\frac{\gamma(q)\log q}{q} - \log\frac{z}{w} \le A_2$$

for any $2 \le w \le z$. Let h be the totally multiplicative function defined by

$$h(q) = \frac{\gamma(q)}{q-\gamma(q)}$$

for primes q. For a smooth function $G : [0,1] \to \mathbf{R}$, put

$$G_{\max} = \sup_{t\in[0,1]} (|G(t)| + |G'(t)|).$$

Then, we have

$$\sum_{d<z} \mu^2(d)h(d)G\left(\frac{\log d}{\log z}\right) = \mathfrak{S}\log z \int_0^1 G(x)dx + O(\mathfrak{S}LG_{\max}),$$

where

$$\mathfrak{S} = \prod_{q\in\mathcal{P}}\left(1-\frac{\gamma(q)}{q}\right)^{-1}\left(1-\frac{1}{q}\right).$$

The implied constant is dependent on A_1, A_2.

The following lemma is a direct consequence of Lemma 3.4:

Lemma 3.5. *Under the same situation as in Lemma 3.4, put*

$$G_p(x) = G\left(\frac{\log R/p}{\log R}\left(\frac{\log p}{\log R/p}+x\right)\right).$$

Then, we have

$$\sum_{d<R/p} \mu^2(d)h(d)G\left(\frac{\log pd}{\log R}\right) = \mathfrak{S}\log\frac{R}{p}\int_0^1 G_p(x)dx + O(\mathfrak{S}LG_{\max}).$$

Proof. Since

$$G\left(\frac{\log pd}{\log R}\right) = G_p\left(\frac{\log d}{\log R/p}\right), \quad (G_p)_{\max} \ll G_{\max},$$

by applying Lemma 3.4 with $z = R/p$, we obtain the result. □

We compute the sum in (3.23). Using the conditions of the support of $y_{r_1,\dots,r_k}$, we have

$$\begin{aligned}
&\sum_{\substack{a_m\\ p|a_m}} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)}\\
&\quad= \frac{1}{\varphi(p)} \sum_{\substack{a'_m\\ (p,a'_m)=1}} \frac{y_{r_1,\dots,r_{m-1},pa'_m,r_{m+1},\dots,r_k}}{\varphi(a'_m)}\\
&\quad= \frac{1}{\varphi(p)} \sum_{\substack{a'_m<R/p\\ (a'_m,pW\prod_{i\neq m} r_i)=1}} \frac{\mu^2(a'_m)}{\varphi(a'_m)}\\
&\qquad\times F\left(\frac{\log r_1}{\log R},\dots,\frac{\log r_{m-1}}{\log R},\frac{\log pa'_m}{\log R},\frac{\log r_{m+1}}{\log R},\dots,\frac{\log r_k}{\log R}\right). \qquad (3.26)
\end{aligned}$$

We apply Lemma 3.5 with

$$\gamma(q) = \begin{cases} 1 & (q \nmid pW \prod_{i \neq m} r_i) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.27}$$

The prime number theorem assures that the conditions in Lemma 3.4 are satisfied with $A_1 = 1/2$ and sufficiently large $A_2 > 1$. In this case, we have

$$\begin{aligned} L &\ll 1 + \sum_{q \mid pW \prod_{i \neq m} r_i} \frac{\log q}{q} \\ &\ll \sum_{q < \log R} \frac{\log q}{q} + \sum_{\substack{q \mid pW \prod_{i \neq m} r_i \\ q \geq \log R}} \frac{\log \log R}{\log R} \\ &\ll \log \log N. \end{aligned}$$

Moreover, since $(r_i, p) = 1$ $(\forall i \neq m)$, we have

$$\mathfrak{S} = \prod_{q \mid pW \prod_{i \neq m} r_i} \left(1 - \frac{1}{q}\right) = \frac{\varphi(p)\varphi(W)}{pW} \prod_{i \neq m} \frac{\varphi(r_i)}{r_i}.$$

Therefore, by applying Lemma 3.5 to the right hand side of (3.26), we obtain

$$\begin{aligned} &\sum_{\substack{a_m \\ p \mid a_m}} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{\varphi(a_m)} \\ &\quad = \frac{\varphi(W)}{pW} \log \frac{R}{p} \prod_{i \neq m} \frac{\varphi(r_i)}{r_i} \int_0^1 F_p^{[m]}\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, u, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R}\right) du \\ &\qquad + O\left(\frac{\varphi(W)}{pW} F_{\max} \log \log N\right), \end{aligned} \tag{3.28}$$

where the function $F_p^{[m]}(\cdots)$ is obtained by replacing the m-th component x of $F(\cdots)$ with $((\log R/p)/\log R)((\log p/\log R/p) + x)$, and

$$F_{\max} := \sup_{(t_1,\dots,t_k) \in [0,1]^k} \left(|F(t_1,\dots,t_k)| + \sum_{i=1}^k \left| \frac{\partial F}{\partial t_i}(t_1,\dots,t_k) \right| \right).$$

We put

$$F_{r_1,\dots,r_k}^{(p;m)} = \int_0^1 F_p^{[m]}\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, u, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R}\right) du. \tag{3.29}$$

By substituting (3.28) into (3.23) and using $y_{\max} \ll F_{\max}$, $\log R/p \ll \log R$, we obtain the following result:

LEMMA 3.6. *If $r_m = 1$, we have*

$$y^{(m)}_{r_1,\dots,r_k}(p) = -\frac{\varphi(W)}{pW}\log\frac{R}{p}\left(\prod_{i\neq m}\frac{\varphi(r_i)}{r_i}\right)F^{(p;m)}_{r_1,\dots,r_k} + O\left(\frac{F_{\max}\varphi(W)\log R}{pWD_0}\right) \tag{3.30}$$

for $Y < p < R$, where $F^{(p;m)}_{r_1,\dots,r_k}$ is defined by (3.29)*.*

The summand in the main term of (3.19) is zero unless $u_1,\dots,u_k$ satisfy $u_m = 1$, $(u_i,u_j) = 1$ $(i \neq j)$, $\prod_{i=1}^k u_i < R$, $\prod_{i=1}^k u_i$ is square-free and $(u_i,pW) = 1$ $(\forall i)$. It is proved in [**10**, p. 403, (6.13)] that if $u_1,\dots,u_k$ satisfy these conditions, we have

$$y^{(m)}_{u_1,\dots,u_k} = (\log R)\frac{\varphi(W)}{W}\left(\prod_{i=1}^k\frac{\varphi(u_i)}{u_i}\right)F^{(m)}_{u_1,\dots,u_k} + O\left(\frac{F_{\max}\varphi(W)\log R}{WD_0}\right), \tag{3.31}$$

where

$$F^{[m]}_{u_1,\dots,u_k} = \int_0^1 F\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_{m-1}}{\log R},v,\frac{\log u_{m+1}}{\log R},\dots,\frac{\log u_k}{\log R}\right)dv. \tag{3.32}$$

Combining (3.30), (3.31), we obtain

$$\begin{aligned} y^{(m)}_{u_1,\dots,u_k}(p)y^{(m)}_{u_1,\dots,u_k} &= -\frac{\varphi(W)^2}{pW^2}(\log R)\left(\log\frac{R}{p}\right)\left(\prod_{i\neq m}\frac{\varphi(u_i)^2}{u_i^2}\right)F^{(p;m)}_{u_1,\dots,u_k}\ F^{[m]}_{u_1,\dots,u_k} \\ &\quad + O\left(\frac{\varphi(W)^2F_{\max}^2\log^2 R}{pW^2D_0}\right) \end{aligned} \tag{3.33}$$

if $u_m = 1$. In the above computation, we used the trivial estimates

$$\sup_{u_1,\dots,u_k}|F^{[m]}_{u_1,\dots,u_k}| \ll F_{\max}, \qquad \sup_{u_1,\dots,u_k}|F^{(p;m)}_{u_1,\dots,u_k}| \ll F_{\max}.$$

We substitute this into the sum over $u_1,\dots,u_k$ in (3.19). The contribution of the error term is

$$\ll \frac{\varphi(W)^2F_{\max}^2\log^2 R}{pW^2D_0}\sum_{\substack{u_1,\dots,u_{m-1},u_{m+1},\dots,u_k<R\\(u_i,W)=1(\forall i)}}\frac{1}{\prod_{i=1}^k g(u_i)} \ll \frac{\varphi(W)^{k+1}F_{\max}^2\log^{k+1}N}{pW^{k+1}D_0}.$$

Therefore,

$$\begin{aligned} &\sum_{u_1,\dots,u_k}\frac{y^{(m)}_{u_1,\dots,u_k}(p)y^{(m)}_{u_1,\dots,u_k}}{\prod_{i=1}^k g(u_i)} \\ &\quad = -\frac{\varphi(W)^2}{pW^2}(\log R)\left(\log\frac{R}{p}\right)\sum_{\substack{u_1,\dots,u_k\\u_m=1\\(u_i,u_j)=1(\forall i\neq j)\\(u_i,pW)=1(\forall i)}}\left(\prod_{i\neq m}\frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2g(u_i)}\right)F^{(p;m)}_{u_1,\dots,u_k}\ F^{[m]}_{u_1,\dots,u_k} \end{aligned}$$

$$+\,O\left(\frac{\varphi(W)^{k+1}F_{\max}^2\log^{k+1}N}{pW^{k+1}D_0}\right). \tag{3.34}$$

We compute the sum over $u_1,\dots,u_k$ in (3.34). First, we remove the condition that u_i and u_j are coprime if $i\neq j$. Since $(u_i,W)=(u_j,W)=1$, if $(u_i,u_j)>1$, there exists a prime $q>D_0$ such that $q|u_i,u_j$. Therefore, the possible error is at most

$$F_{\max}^2\left(\sum_{q>D_0}\frac{\varphi(q)^4}{g(q)^2q^4}\right)\left(\sum_{\substack{u<R\\(u,W)=1}}\frac{\varphi(u)^2}{u^2g(u)}\right)^{k-1}\ll\frac{F_{\max}^2\varphi(W)^{k-1}(\log R)^{k-1}}{D_0W^{k-1}}.$$

Hence we have

$$\begin{aligned}&\sum_{\substack{u_1,\dots,u_k\\u_m=1\\(u_i,u_j)=1(\forall i\neq j)\\(u_i,pW)=1(\forall i)}}\left(\prod_{i\neq m}\frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2g(u_i)}\right)F^{(p;m)}_{u_1,\dots,u_k}\ F^{[m]}_{u_1,\dots,u_k}\\&=\sum_{\substack{u_1,\dots,u_k\\u_m=1\\(u_i,pW)=1(\forall i)}}\left(\prod_{i\neq m}\frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2g(u_i)}\right)F^{(p;m)}_{u_1,\dots,u_k}\ F^{[m]}_{u_1,\dots,u_k}\\&\quad+O\left(\frac{F_{\max}^2\varphi(W)^{k-1}(\log R)^{k-1}}{D_0W^{k-1}}\right).\end{aligned} \tag{3.35}$$

Now we apply Lemma 3.4 with

$$\gamma(q)=\begin{cases}1-\dfrac{q^2-3q+1}{q^3-q^2-2q+1} & (q\nmid pW)\\ 0 & (\text{otherwise})\end{cases} \tag{3.36}$$

to the sum over $u_1,\dots,u_{m-1},u_{m+1},\dots,u_k$. Then we have

$$\begin{aligned}&\sum_{\substack{u_1,\dots,u_k\\u_m=1\\(u_i,pW)=1(\forall i)}}\left(\prod_{i\neq m}\frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2g(u_i)}\right)F^{(p;m)}_{u_1,\dots,u_k}\ F^{[m]}_{u_1,\dots,u_k}\\&=\mathfrak{S}^{k-1}(\log R)^{k-1}\\&\quad\times\int_0^1\cdots\int_0^1\left\{\int_0^1F_p^{[m]}(u_1,\dots,u_{m-1},u,u_{m+1},\dots,u_k)du\right\}\\&\quad\times\left\{\int_0^1F(u_1,\dots,u_{m-1},u,u_{m+1},\dots,u_k)du\right\}du_1\cdots du_{m-1}du_{m+1}\cdots du_k\\&\quad+O(\mathfrak{S}^{k-1}L(\log R)^{k-2}F_{\max}^2).\end{aligned}$$

In this case,

$$L \ll 1 + \sum_{q|pW} \frac{\log q}{q} \ll \log D_0,$$

$$\begin{aligned}\mathfrak{S} &= \left\{ \prod_{q \nmid pW} \left(1 - \frac{1}{q} + O\left(\frac{1}{q^2}\right)\right)^{-1} \left(1 - \frac{1}{q}\right) \right\} \prod_{q|pW} \left(1 - \frac{1}{q}\right) \\ &= \frac{\varphi(p)\varphi(W)}{pW} \prod_{\substack{q \geq D_0 \\ q \neq p}} \left(1 + O\left(\frac{1}{q^2}\right)\right) \\ &= \frac{\varphi(p)\varphi(W)}{pW}(1 + O(D_0^{-1})).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{\substack{u_1,\ldots,u_k \\ u_m=1 \\ (u_i,pW)=1(\forall i)}} \left(\prod_{i\neq m} \frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2 g(u_i)}\right) F^{(p;m)}_{u_1,\ldots,u_k} F^{[m]}_{u_1,\ldots,u_k} \\ &= \frac{\varphi(p)^{k-1}\varphi(W)^{k-1}}{p^{k-1}W^{k-1}}(\log R)^{k-1} \\ &\times \int_0^1 \cdots \int_0^1 \left\{ \int_0^1 F_p^{[m]}(u_1,\ldots,u_{m-1},u,u_{m+1},\ldots,u_k)du \right\} \\ &\times \left\{ \int_0^1 F(u_1,\ldots,u_{m-1},u,u_{m+1},\ldots,u_k)du \right\} du_1 \cdots du_{m-1} du_{m+1} \cdots du_k \\ &+ O\left(\frac{\varphi(p)^{k-1}\varphi(W)^{k-1}}{D_0 p^{k-1}W^{k-1}}(\log R)^{k-1} F_{\max}^2 \right). \qquad (3.37)\end{aligned}$$

We put

$$\begin{aligned}J_k^{(m)}[p] = &\int_0^1 \cdots \int_0^1 \left\{ \int_0^1 F_p^{[m]}(u_1,\ldots,u_{m-1},u,u_{m+1},\ldots,u_k)du \right\} \\ &\times \left\{ \int_0^1 F(u_1,\ldots,u_{m-1},u,u_{m+1},\ldots,u_k)du \right\} du_1 \cdots du_{m-1} du_{m+1} \cdots du_k. \qquad (3.38)\end{aligned}$$

By substituting (3.37), (3.38) into (3.35), we have

$$\begin{aligned}&\sum_{\substack{u_1,\ldots,u_k \\ u_m=1 \\ (u_i,u_j)=1(\forall i\neq j) \\ (u_i,pW)=1(\forall i)}} \left(\prod_{i\neq m} \frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2 g(u_i)}\right) F^{(p;m)}_{u_1,\ldots,u_k}\ F^{[m]}_{u_1,\ldots,u_k} \\ &= \frac{\varphi(p)^{k-1}\varphi(W)^{k-1}}{p^{k-1}W^{k-1}}(\log R)^{k-1} J_k^{(m)}[p] + O\left(\frac{F_{\max}^2 \varphi(W)^{k-1}(\log R)^{k-1}}{D_0 W^{k-1}} \right). \qquad (3.39)\end{aligned}$$

We substitute (3.39) into (3.34). Since $\log R \ll \log N$, we obtain

$$\begin{aligned}
&\sum_{u_1,\dots,u_k} \frac{y^{(m)}_{u_1,\dots,u_k}(p) y^{(m)}_{u_1,\dots,u_k}}{\prod_{i=1}^k g(u_i)} \\
&= -\frac{\varphi(p)^{k-1}\varphi(W)^{k+1}}{p^k W^{k+1}} (\log R)^k \left(\log \frac{R}{p}\right) J^{(m)}_k[p] + O\left(\frac{F^2_{\max}\varphi(W)^{k+1}(\log N)^{k+1}}{pD_0 W^{k+1}}\right) \\
&= -\frac{\varphi(W)^{k+1}}{pW^{k+1}} (\log R)^k \left(\log \frac{R}{p}\right) J^{(m)}_k[p] + O\left(\frac{F^2_{\max}\varphi(W)^{k+1}(\log N)^{k+1}}{pD_0 W^{k+1}}\right). \qquad (3.40)
\end{aligned}$$

We substitute this into (3.19). We compute the sum over p. Recall that $J^{(m)}_k[p]$ is defined by (3.38), and the function $F^{[m]}_p$ is obtained by replacing the m-th component u of F with $((\log R/p)/\log R)(\log p/(\log R/p) + u)$. To see how the sum over p becomes, let us compute the sum

$$\sum_{Y<p<R} \frac{1}{p} \left(\log \frac{R}{p}\right) \left(\log \frac{N}{p}\right)^{-1} f\left(\frac{\log R/p}{\log R}\left(\frac{\log p}{\log R/p} + u\right)\right) \qquad (3.41)$$

for any smooth function f, where $Y = N^\eta, R = N^{(\theta/2)-\delta}$. We denote by $\pi(v)$ the number of primes equal or less than v. Then, the sum (3.41) is expressed by

$$\int_{N^\eta}^{N^{(\theta/2)-\delta}} \frac{1}{v} \left(\log \frac{R}{v}\right) \left(\log \frac{N}{v}\right)^{-1} f\left(\frac{\log R/v}{\log R}\left(\frac{\log v}{\log R/v} + u\right)\right) d\pi(v).$$

Using the Prime Number Theorem, this is asymptotically

$$\int_{N^\eta}^{N^{(\theta/2)-\delta}} \frac{1}{v} \left(\log \frac{R}{v}\right) \left(\log \frac{N}{v}\right)^{-1} f\left(\frac{\log R/v}{\log R}\left(\frac{\log v}{\log R/v} + u\right)\right) \frac{dv}{\log v}.$$

By putting $\log v/\log N = \xi$, this becomes

$$\int_\eta^{(\theta/2)-\delta} \frac{(\theta/2)-\delta-\xi}{1-\xi} f\left(\frac{\xi}{(\theta/2)-\delta} + \frac{(\theta/2)-\delta-\xi}{(\theta/2)-\delta} u\right) \frac{d\xi}{\xi}.$$

We put

$$\begin{aligned}
&F_{m,\delta}(u_1,\dots,u_k;\xi) \\
&\quad := F\left(u_1,\dots,u_{m-1}, \frac{\xi}{(\theta/2)-\delta} + \frac{(\theta/2)-\delta-\xi}{(\theta/2)-\delta} u_m, u_{m+1},\dots,u_k\right), \qquad (3.42)
\end{aligned}$$

$$\begin{aligned}
L^{[m]}_{k,\delta}(\xi) := &\int_0^1 \cdots \int_0^1 \left\{\int_0^1 F_{m,\delta}(u_1,\dots,u_k;\xi) du_m\right\} \\
&\times \left\{\int_0^1 F(u_1,\dots,u_k) du_m\right\} du_1 \cdots du_{m-1} du_{m+1} \cdots du_k. \qquad (3.43)
\end{aligned}$$

Then, by the above argument and simple estimate

$$\sum_{Y<p<R}\frac{1}{p\log N/p}\ll_\eta \frac{1}{\log N}\quad (Y=N^\eta, R=N^{(\theta/2)-\delta}),$$

we have

$$\begin{aligned}
\sum_{Y<p<R}\frac{1}{\log N/p}\sum_{u_1,\dots,u_k}\frac{y^{(m)}_{u_1,\dots,u_k}(p)y^{(m)}_{u_1,\dots,u_k}}{\prod_{i=1}^k g(u_i)}
&= -\frac{\varphi(W)^{k+1}}{W^{k+1}}(\log R)^k(1+o(1))\int_\eta^{(\theta/2)-\delta}\frac{(\theta/2)-\delta-\xi}{1-\xi}L^{[m]}_{k,\delta}(\xi)\frac{d\xi}{\xi}\\
&\quad + O\left(\frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^k}{D_0W^{k+1}}\right),
\end{aligned}\tag{3.44}$$

if the integral of the main term is not zero. Finally, by substituting this into (3.19), and combining (3.24), (3.25) and

$$y^{(m)}_{\max}\ll\frac{\varphi(W)}{W}F_{\max}\log N,\quad y_{\max}\ll F_{\max}$$

(see [**10**, p. 403]), we obtain the following result:

Proposition 3.7. *Assume $BV[\theta,\mathcal{P}]$. Then, if*

$$L^{(m)}_{k,\delta}(F):=\int_\eta^{(\theta/2)-\delta}\frac{(\theta/2)-\delta-\xi}{1-\xi}L^{[m]}_{k,\delta}(\xi)\frac{d\xi}{\xi}\neq 0,$$

we have

$$\begin{aligned}
S^{(m)}_{2,I} &= -\frac{\varphi(W)^kN}{W^{k+1}}(\log R)^k(1+o(1))L^{(m)}_{k,\delta}(F)\\
&\quad + O\left(\frac{F_{\max}^2\varphi(W)^kN(\log N)^k}{D_0W^{k+1}}\right)+O_B\left(\frac{NF_{\max}^2}{(\log N)^B}\right)
\end{aligned}\tag{3.45}$$

as $N\to\infty$, where $L^{[m]}_{k,\delta}(F)$ is defined by (3.43).

Notice that the same result holds for $S^{(m)}_{2,II}$. If $L^{(m)}_{k,\delta}(F)=0$, the leading term vanishes and hence

$$S^{(m)}_{2,I}=S^{(m)}_{2,II}=o\left(\frac{F_{\max}^2\varphi(W)^kN(\log N)^k}{W^{k+1}}\right)+O_B\left(\frac{NF_{\max}^2}{(\log N)^B}\right).$$

4. The computation of $S^{(m)}_{2,III}$.

To compute

$$S^{(m)}_{2,III}:=\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ d_m=e_m=1}}\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}\sum_{\substack{N\le n<2N\\ n\equiv\nu_0(\mathrm{mod}\ W)\\ [d_i,e_i]\mid n+h_i(\forall i)}}\beta(n+h_m),\tag{4.1}$$

we use the following lemma:

Lemma 4.1. *Let $\beta(n)$ be the function defined by* (1.10) *with $Y = N^{\eta}$, $1 \ll \eta < 1/4$. Then, we have*

$$\sum_{\substack{N \le n < 2N \\ (n,q)=1}} \beta(n) = \frac{N}{\log N} \log \frac{1-\eta}{\eta} + O\left(\frac{N \log\log N}{(\log N)^2}\right) \tag{4.2}$$

uniformly for $q \le N$. Here, the implicit constant might be dependent on η.

Proof. We denote by $\omega(q)$ the number of distinct prime factors of q. Then,

$$\begin{aligned}
\sum_{\substack{N \le n < 2N \\ (n,q)=1}} \beta(n) &= \sum_{\substack{Y < p_1 \le N^{1/2} \\ (p_1,q)=1}} \sum_{\substack{N/p_1 \le p_2 < 2N/p_1 \\ (p_2,q)=1}} 1 \\
&= \sum_{\substack{Y < p_1 \le N^{1/2} \\ (p_1,q)=1}} \left\{ \pi^{\flat}\left(\frac{N}{p_1}\right) + O(\omega(q)) \right\} \\
&= \sum_{\substack{Y < p_1 \le N^{1/2} \\ (p_1,q)=1}} \pi^{\flat}\left(\frac{N}{p_1}\right) + O(N^{1/2}\omega(q)) \\
&= \sum_{Y < p_1 \le N^{1/2}} \pi^{\flat}\left(\frac{N}{p_1}\right) + O\left(\pi^{\flat}\left(\frac{N}{Y}\right)\omega(q)\right) + O(N^{1/2}\omega(q)).
\end{aligned}$$

By applying the Prime Number Theorem to the final line, we obtain

$$\sum_{\substack{N \le n < 2N \\ (n,q)=1}} \beta(n) = \sum_{Y < p_1 \le N^{1/2}} \left\{ \frac{N}{p_1 \log N/p_1} + O\left(\frac{N}{p_1 (\log N)^2}\right) \right\} + O\left(\omega(q)\frac{N^{1-\eta}}{\log N}\right). \tag{4.3}$$

The contribution of the first error term is

$$\sum_{Y < p_1 \le N^{1/2}} \frac{N}{p_1 (\log N)^2} \ll \frac{N \log\log N}{(\log N)^2}.$$

On the other hand, the main term is

$$\begin{aligned}
\sum_{Y < p_1 \le N^{1/2}} \frac{N}{p_1 \log N/p_1} &= N \int_Y^{N^{1/2}} \frac{d\pi(u)}{u(\log N - \log u)} \\
&= \frac{N}{\log N} \sum_{k=0}^{\infty} \frac{1}{\log^k N} \int_Y^{N^{1/2}} \frac{\log^k u}{u} d\pi(u).
\end{aligned} \tag{4.4}$$

By partial integration, we find that

$$\begin{aligned}\int_Y^{N^{1/2}} \frac{d\pi(u)}{u} &= \left[\frac{1}{\log u} + O\left(\frac{1}{\log^2 u}\right)\right]_{N^\eta}^{N^{1/2}} + \int_{N^\eta}^{N^{1/2}} \frac{1}{u^2}\left(\frac{u}{\log u} + O\left(\frac{u}{\log^2 u}\right)\right) du \\ &= \log\left(\frac{1}{2\eta}\right) + O\left(\frac{1}{\log N}\right),\end{aligned}$$

$$\begin{aligned}\int_Y^{N^{1/2}} \frac{\log u}{u} d\pi(u) &= \left[1 + O\left(\frac{1}{\log u}\right)\right]_{N^\eta}^{N^{1/2}} \\ &\quad - \int_{N^\eta}^{N^{1/2}} \frac{1-\log u}{u^2}\left(\frac{u}{\log u} + O\left(\frac{u}{\log^2 u}\right)\right) du \\ &= \left(\frac{1}{2} - \eta\right)\log N + O(1),\end{aligned}$$

and for $k \geq 2$, we have

$$\begin{aligned}\int_Y^{N^{1/2}} \frac{\log^k u}{u} d\pi(u) &= \left[\frac{\log^k u}{u}\left(\frac{u}{\log u} + O\left(\frac{u}{\log^2 u}\right)\right)\right]_{N^\eta}^{N^{1/2}} \\ &\quad - \int_{N^\eta}^{N^{1/2}} \frac{k\log^{k-1} u - \log^k u}{u^2}\left(\frac{u}{\log u} + O\left(\frac{u}{\log^2 u}\right)\right) du \\ &= \frac{1}{k}\left(\frac{1}{2^k} - \eta^k\right)\log^k N + O(2^{-k}\log^{k-1} N).\end{aligned}$$

(The implied constant might be dependent on η, but independent of k.) Combining these, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{\log^k N}\int_Y^{N^{1/2}} \frac{\log^k u}{u} d\pi(u) &= \log\left(\frac{1}{2\eta}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{2^k} - \eta^k\right) + O\left(\sum_{k=1}^{\infty}\frac{2^{-k}}{\log N}\right) \\ &= \log\frac{1-\eta}{\eta} + O\left(\frac{1}{\log N}\right).\end{aligned} \tag{4.5}$$

By substituting (4.5) into (4.4), we have

$$\sum_{Y<p_1\leq N^{1/2}} \frac{N}{p_1 \log N/p_1} = \frac{N}{\log N}\log\frac{1-\eta}{\eta} + O\left(\frac{N}{\log^2 N}\right).$$

Consequently,

$$\sum_{\substack{N\leq n<2N \\ (n,q)=1}} \beta(n) = \frac{N}{\log N}\log\frac{1-\eta}{\eta} + O\left(\frac{N\log\log N}{\log^2 N}\right) + O\left(\frac{\omega(q)N^{1-\eta}}{\log N}\right). \tag{4.6}$$

Since $\omega(q) \ll N^\epsilon$ ($\forall\epsilon > 0$) holds uniformly for $q \leq N$, the second error term is dominated by the first one. Hence we obtain the result. □

We return to the computation of $S_{2,III}^{(m)}$, defined by (4.1). Only those $(d_1, \dots, d_k, e_1, \dots, e_k)$ for which $W, [d_1, e_1], \dots, [d_k, e_k]$ are pairwise coprime and $\prod_{i=1}^k d_i$, $\prod_{i=1}^k e_i < R$ contribute to the sum. We denote the restricted sum by $\sum'$. We put

$$q = W \prod_{i=1}^{k} [d_i, e_i].$$

Then, there exists a unique $\nu \pmod q$ such that $\nu \equiv \nu_0 \pmod W$, $h_i + \nu \equiv 0 \pmod{[d_i, e_i]}$ $(i = 1, \dots, k)$ and the sum over n is rewritten as the sum over integers congruent to ν modulo q. Therefore,

$$\begin{aligned} \sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 (\mathrm{mod}\ W) \\ [d_i, e_i] | n + h_i (\forall i)}} \beta(n + h_m) &= \sum_{\substack{N \le n < 2N \\ n \equiv \nu \ (\mathrm{mod}\ q)}} \beta(n + h_m) \\ &= \sum_{\substack{N < n \le 2N \\ n \equiv \nu' \ (\mathrm{mod}\ q)}} \beta(n) + O(1), \end{aligned} \tag{4.7}$$

where $\nu' = \nu + h_m$. This ν' satisfies $(\nu', q) = 1$. This fact follows from our choice of ν_0 and the condition that the elements of $\mathcal{H}$ are bounded. We have treated the similar situation in Section 3, hence we omit to prove this. Hence by (4.7), we have

$$\sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 \ (\mathrm{mod}\ W) \\ [d_i, e_i] | n + h_i (\forall i)}} \beta(n + h_m) = \frac{1}{\varphi(q)} \sum_{\substack{N \le n < 2N \\ (n, q) = 1}} \beta(n) + \Delta_\beta(N; q, \nu') + O(1), \tag{4.8}$$

where

$$\Delta_\beta(N; q, \nu') = \sum_{\substack{N \le n < 2N \\ n \equiv \nu' \ (\mathrm{mod}\ q)}} \beta(n) - \frac{1}{\varphi(q)} \sum_{\substack{N \le n < 2N \\ (n, q) = 1}} \beta(n).$$

Now we apply Lemma 4.1 to the sum in (4.8). Then we have

$$\sum_{\substack{N \le n < 2N \\ n \equiv \nu_0 \ (\mathrm{mod}\ W) \\ [d_i, e_i] | n + h_i (\forall i)}} \beta(n + h_m) = \frac{X_{N,\eta}}{\varphi(q)} + \Delta_\beta(N; q, \nu') + O(1),$$

where

$$X_{N,\eta} = \frac{N}{\log N} \log \frac{1 - \eta}{\eta} + O\left(\frac{N \log \log N}{(\log N)^2} \right). \tag{4.9}$$

By substituting this into (4.1), we obtain

$$S_{2,III}^{(m)} = \frac{X_{N,\eta}}{\varphi(W)} \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} + O\left(\sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} |\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}|(|\Delta_\beta(N;q,\nu')|+1) \right). \tag{4.10}$$

Under the assumption of the estimation $BV[\theta,\mathcal{E}_2]$, the error term above is evaluated by $O_B(Ny_{\max}^2/(\log N)^B)$. The proof of this statement is essentially the same as that in the argument around (5.19)–(5.20) of [**10**], hence we omit it. Moreover, the sum in the main term is also computed in [**10**] (see the proof of Lemma 5.2 of [**10**]). The result is

$$\sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} = \sum_{u_1,\dots,u_k} \frac{(y^{(m)}_{u_1,\dots,u_k})^2}{\prod_{i=1}^k g(u_i)} + O\left(\frac{(y^{(m)}_{\max})^2 \varphi(W)^{k-1}(\log N)^{k-1}}{D_0 W^{k-1}} \right).$$

Consequently, we obtain

$$S_{2,III}^{(m)} = \frac{X_{N,\eta}}{\varphi(W)} \sum_{u_1,\dots,u_k} \frac{(y^{(m)}_{u_1,\dots,u_k})^2}{\prod_{i=1}^k g(u_i)} + O\left(\frac{(y^{(m)}_{\max})^2 \varphi(W)^{k-2} N(\log N)^{k-2}}{D_0 W^{k-1}} \right) + O_B\left(\frac{Ny_{\max}^2}{(\log N)^B} \right). \tag{4.11}$$

We put

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \left(\int_0^1 F(t_1,\dots,t_k)dt_m \right)^2 dt_1 \cdots dt_{m-1}dt_{m+1}\cdots dt_k. \tag{4.12}$$

By Lemma 6.3 of [**10**], we have

$$\frac{N}{\varphi(W)\log N} \sum_{u_1,\dots,u_k} \frac{(y^{(m)}_{u_1,\dots,u_k})^2}{\prod_{i=1}^k g(u_i)} = \frac{\varphi(W)^k N(\log R)^{k+1}}{W^{k+1}\log N} J_k^{(m)}(F) + O\left(\frac{F_{\max}^2 \varphi(W)^k N(\log N)^k}{W^{k+1}D_0} \right).$$

Moreover, we have

$$y_{\max} \ll F_{\max}, \quad y^{(m)}_{\max} \ll \frac{F_{\max}\varphi(W)\log N}{W}$$

(see [**10**, p. 403]). By substituting the definition of $X_{N,\eta}$ (4.9) into (4.10) and combining these, we obtain the following result:

Proposition 4.2. *Under the assumption of $BV[\theta,\mathcal{E}_2]$, we have*

$$S_{2,III}^{(m)} = \frac{\varphi(W)^k N(\log R)^{k+1}}{W^{k+1}\log N}\left(\log\frac{1-\eta}{\eta}\right)(1+o(1))J_k^{(m)}(F) + O\left(\frac{F_{\max}^2\varphi(W)^k N(\log N)^k}{W^{k+1}D_0}\right) + O_B\left(\frac{F_{\max}^2 N}{(\log N)^B}\right) \tag{4.13}$$

as $N\to\infty$, where $J_k^{(m)}(F)$ is defined by (4.12).

5. The computation of $S_{2,IV}^{(m)}$.

The next problem is to compute

$$S_{2,IV}^{(m)} := \sum_{Y<p<R}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ d_m=e_m=p}}\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}\sum_{\substack{N\le n<2N\\ n\equiv\nu_0 (\mathrm{mod}\ W)\\ [d_i,e_i]\mid n+h_i(\forall i)}}\beta(n+h_m). \tag{5.1}$$

Only those $(d_1,\dots,d_k,\ e_1,\dots,e_k)$ for which $W,[d_1,e_1],\dots,[d_k,e_k]$ are pairwise coprime and $\prod_{i=1}^k d_i$, $\prod_{i=1}^k e_i<R$ contribute to the sum. We put

$$q = W\prod_{i=1}^k [d_i,e_i].$$

Then,

$$\sum_{\substack{N\le n<2N\\ n\equiv\nu_0 (\mathrm{mod}\ W)\\ [d_i,e_i]\mid n+h_i(\forall i)}}\beta(n+h_m) = \sum_{\substack{N\le n<2N\\ n\equiv\nu\ (\mathrm{mod}\ q)}}\beta(n+h_m) \tag{5.2}$$

for some ν (mod q), given in Section 3. The right hand side of (5.2) is given by (3.6). Combining this and

$$\frac{1}{\varphi(q/p)} = \frac{\varphi(p)}{\varphi(q)} = \frac{p-1}{\varphi(q)},$$

we obtain

$$S_{2,IV}^{(m)} = \frac{1}{\varphi(W)}\sum_{Y<p<R}(p-1)\pi^\flat\left(\frac{N}{p}\right)\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ d_m=e_m=p}}\frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k\varphi([d_i,e_i])} + O\left(\sum_{Y<p<R}\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ d_m=e_m=p}}|\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}|\left(\left|\Delta\left(\frac{N}{p};\frac{q}{p},\nu_m'\right)\right|+1\right)\right), \tag{5.3}$$

where the sum $\sum'$ implies that $d_1, \ldots, d_k, e_1, \ldots, e_k$ are restricted to those satisfying the condition that $W, [d_1, e_1], \ldots, [d_k, e_k]$ are pairwise coprime. The error term is, under the assumption of $BV[\theta, \mathcal{P}]$, evaluated by

$$\ll_B \frac{Ny_{\max}^2}{(\log N)^B} \tag{5.4}$$

for any $B > 0$. The proof is almost the same as that in Section 3, hence we omit this. Next, we compute the sum over $d_1, \ldots, d_k, e_1, \ldots, e_k$. By the similar way as (3.10), we obtain

$$\begin{aligned}\sideset{}{'}\sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ d_m=e_m=p}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k \varphi([d_i, e_i])} &= \sum_{u_1,\ldots,u_k} \prod_{i=1}^k g(u_i) \sideset{}{^*}\sum_{s_{1,2},\ldots,s_{k,k-1}} \left(\prod_{1\le i\ne j\le k} \mu(s_{i,j})\right)\\ &\quad\times \sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k\\ u_i\mid d_i,e_i(\forall i)\\ s_{i,j}\mid d_i,e_j(i\ne j)\\ d_m=e_m=p}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^k \varphi(d_i)\varphi(e_i)}.\end{aligned} \tag{5.5}$$

Using the function $y_{r_1,\ldots,r_k}^{(m)}(p)$ defined by (3.11), the right hand side of (5.5) is expressed by

$$\sum_{u_1,\ldots,u_k} \left(\prod_{i=1}^k \frac{\mu^2(u_i)}{g(u_i)}\right) \sideset{}{^*}\sum_{s_{1,2},\ldots,s_{k,k-1}} \left(\prod_{1\le i\ne j\le k} \frac{\mu(s_{i,j})}{g(s_{i,j})^2}\right) y_{a_1,\ldots,a_k}^{(m)}(p) y_{b_1,\ldots,b_k}^{(m)}(p), \tag{5.6}$$

where $a_i = u_i \prod_{j\ne i} s_{i,j}$, $b_i = u_i \prod_{j\ne i} s_{j,i}$. Since $a_m = p$ implies $p = u_m$ or $s_{m,j}$ $(\exists j)$ and $b_m = p$ implies $p = u_m$ or $s_{j,m}$ $(\exists j)$, the contribution of the terms with $s_{i,j} \ne 1$ (hence $s_{i,j} > D_0$) is at most

$$\begin{aligned}\sum_{\mathfrak{p}_1,\mathfrak{p}_2=1 \text{ or } p} \quad \sum_{u_1,\ldots,u_k} \prod_{i=1}^k \frac{\mu^2(u_i)}{g(u_i)} \sum_{\substack{s_{i',j'}\\ (i',j')\ne(i,j)}} \prod_{\substack{1\le i'\ne j'\le k\\ (i',j')\ne(i,j)}} \frac{\mu^2(s_{i',j'})}{g(s_{i',j'})^2}\\ \times \left(\sum_{s_{i,j}>D_0} \frac{\mu^2(s_{i,j})}{g(s_{i,j})^2}\right) y_{a_1,\ldots,a_k}^{(m)}(p)|_{a_m=\mathfrak{p}_1} y_{b_1,\ldots,b_k}^{(m)}(p)|_{b_m=\mathfrak{p}_2},\end{aligned}$$

which is bounded by

$$\begin{aligned}&\ll y_{\max}^{(m)}(p)_{r_m=1}^2 \left(\frac{\varphi(W)}{W}\log R\right)^{k-1} \cdot 1 \cdot D_0^{-1}\\ &\quad + \left(y_{\max}^{(m)}(p)_{r_m=1} y_{\max}^{(m)}(p)_{r_m=p} + y_{\max}^{(m)}(p)_{r_m=p}^2\right) \cdot \frac{1}{p} \left(\frac{\varphi(W)}{W}\log R\right)^{k-1} \cdot 1 \cdot D_0^{-1}\end{aligned}$$

$$\ll \frac{\varphi(W)^{k-1}(\log R)^{k-1}}{W^{k-1}D_0} y_{\max}^{(m)}(p)_{r_m=1}^2$$
$$+ \frac{\varphi(W)^{k-1}(\log R)^{k-1}}{pW^{k-1}D_0} \max\left\{y_{\max}^{(m)}(p)_{r_m=1},\ y_{\max}^{(m)}(p)_{r_m=p}\right\}^2. \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we have

$$\begin{aligned}\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ d_m=e_m=p}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k \varphi([d_i,e_i])} &= \sum_{u_1,\dots,u_k} \frac{y_{u_1,\dots,u_k}^{(m)}(p)^2}{\prod_{i=1}^k g(u_i)} \\ &\quad + O\left(\frac{\varphi(W)^{k-1}(\log R)^{k-1}}{W^{k-1}D_0} y_{\max}^{(m)}(p)_{r_m=1}^2\right) \\ &\quad + O\left(\frac{\varphi(W)^{k-1}(\log R)^{k-1}}{pW^{k-1}D_0} Y^{(m)}(p)^2\right),\end{aligned} \tag{5.8}$$

where

$$Y^{(m)}(p) = \max\{y_{\max}^{(m)}(p)_{r_m=1},\ y_{\max}^{(m)}(p)_{r_m=p}\}.$$

By substituting (5.8) into (5.3) and combining (5.4), we obtain

$$\begin{aligned}S_{2,IV}^{(m)} &= \frac{1}{\varphi(W)} \sum_{Y<p<R} (p-1)\pi^\flat\left(\frac{N}{p}\right) \sum_{u_1,\dots,u_k} \frac{y_{u_1,\dots,u_k}^{(m)}(p)^2}{\prod_{i=1}^k g(u_i)} \\ &\quad + O\left(\frac{N\varphi(W)^{k-2}(\log N)^{k-2}}{W^{k-1}D_0} \sum_{Y<p<R} \left(y_{\max}^{(m)}(p)_{r_m=1}^2 + \frac{Y^{(m)}(p)^2}{p}\right)\right) \\ &\quad + O_B\left(\frac{Ny_{\max}^2}{(\log N)^B}\right).\end{aligned} \tag{5.9}$$

Moreover, by the estimates (3.24), (3.25) and $\sum_{Y<p<R} 1/p \ll_\eta 1$, the first error term of (5.9) is at most

$$\frac{N\varphi(W)^{k-2}(\log N)^{k-2}}{W^{k-1}D_0} \cdot \frac{y_{\max}^2\varphi(W)^2(\log N)^2}{W^2} = \frac{y_{\max}^2 N\varphi(W)^k(\log N)^k}{W^{k+1}D_0}.$$

Since

$$(p-1)\pi^\flat\left(\frac{N}{p}\right) = \frac{N}{\log N/p} + O\left(\frac{N}{\log^2 N}\right),$$

if we replace the factor $(p-1)\pi^\flat(N/p)$ in (5.9) with $N/(\log N/p)$, the possible error is at most

$$\frac{1}{\varphi(W)} \cdot \frac{N}{(\log N)^2} \sum_{Y<p<R} \sum_{u_1,\dots,u_k} \frac{y_{u_1,\dots,u_k}^{(m)}(p)^2}{\prod_{i=1}^k g(u_i)}$$

$$\ll \frac{N}{\varphi(W)(\log N)^2} \sum_{Y<p<R} \left\{ \frac{y_{\max}^{(m)}(p)|_{r_m=p}^2}{p} + y_{\max}^{(m)}(p)|_{r_m=1}^2 \right\} \left(\sum_{\substack{u<R \\ (u,W)=1}} \frac{1}{g(u)} \right)^{k-1}$$

$$\ll \frac{N}{\varphi(W)(\log N)^2} \sum_{Y<p<R} \left(\frac{y_{\max}^2 \varphi(W)^2 \log^2 N}{pW^2} + \frac{y_{\max}^2 \varphi(W)^2 \log^2 N}{p^2 W^2} \right) \left(\frac{\varphi(W) \log N}{W} \right)^{k-1}$$

$$\ll \frac{y_{\max}^2 N (\log N)^{k-1} \varphi(W)^k}{W^{k+1}},$$

which is dominated by the error term above. Therefore,

$$\begin{aligned} S_{2,IV}^{(m)} &= \frac{N}{\varphi(W)} \sum_{Y<p<R} \frac{1}{\log N/p} \sum_{u_1,\ldots,u_k} \frac{y_{u_1,\ldots,u_k}^{(m)}(p)^2}{\prod_{i=1}^k g(u_i)} \\ &\quad + O\left(\frac{y_{\max}^2 N \varphi(W)^k (\log N)^k}{W^{k+1} D_0} \right) + O_B\left(\frac{N y_{\max}^2}{(\log N)^B} \right). \end{aligned} \tag{5.10}$$

Let us compute the sum over $u_1, \ldots, u_k$ in the main term of (5.10) by using Lemma 3.3. Let u_m be 1 or p. Then, by (3.23), we have

$$\begin{aligned} &y_{u_1,\ldots,u_k}^{(m)}(p)^2 \\ &= \left\{ -\mu(u_m) g(u_m) \sum_{\substack{a_m \\ p|a_m}} \frac{y_{u_1,\ldots,u_{m-1},a_m,u_{m+1},\ldots,u_k}}{\varphi(a_m)} + O\left(\frac{y_{\max} g(u_m) \varphi(W) \log R/p}{W D_0 \varphi(p)} \right) \right\}^2 \\ &= g(u_m)^2 \left(\sum_{\substack{a_m \\ p|a_m}} \frac{y_{u_1,\ldots,u_{m-1},a_m,u_{m+1},\ldots,u_k}}{\varphi(a_m)} \right)^2 + O\left(\frac{y_{\max}^2 g(u_m)^2 \varphi(W)^2 (\log R/p)^2}{W^2 D_0 \varphi(p)^2} \right). \end{aligned}$$

Therefore, by taking the sum over $u_1, \ldots, u_k$, we obtain

$$\begin{aligned} &\sum_{u_1,\ldots,u_k} \frac{y_{u_1,\ldots,u_k}^{(m)}(p)^2}{\prod_{i=1}^k g(u_i)} \\ &= \sum_{u_m=1,p} g(u_m) \sum_{u_1,\ldots,u_{m-1},u_{m+1},\ldots,u_k} \frac{1}{\prod_{i\neq m} g(u_i)} \left(\sum_{\substack{a_m \\ p|a_m}} \frac{y_{u_1,\ldots,u_{m-1},a_m,u_{m+1},\ldots,u_k}}{\varphi(a_m)} \right)^2 \\ &\quad + O\left(\frac{y_{\max}^2 \varphi(W)^{k+1} (\log N)^{k+1}}{W^{k+1} D_0 p} \right). \end{aligned} \tag{5.11}$$

By (3.28), if $u_1, \ldots, u_k$ satisfy the conditions that $(u_i, u_j) = 1$ $(i \neq j)$, $(u_i, pW) = 1$ $(\forall i \neq m)$, $u_1, \ldots, u_k$ are square-free, we have

$$
\begin{aligned}
&\left(\sum_{\substack{a_m\\ p|a_m}} \frac{y_{u_1,\dots,u_{m-1},a_m,u_{m+1},\dots,u_k}}{\varphi(a_m)}\right)^2 \\
&\quad = \frac{\varphi(W)^2}{p^2W^2}\left(\log\frac{R}{p}\right)^2 \prod_{i\neq m}\frac{\varphi(u_i)^2}{u_i^2} \\
&\quad\quad \times \left(\int_0^1 F_p^{[m]}\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_{m-1}}{\log R},u,\frac{\log u_{m+1}}{\log R},\dots,\frac{\log u_k}{\log R}\right)du\right)^2 \\
&\quad\quad + O\left(\frac{\varphi(W)^2}{p^2W^2}\left(F_{\max}^2(\log\log N)^2 + F_{\max}\log\frac{R}{p}\log\log N\right)\right). \qquad (5.12)
\end{aligned}
$$

We substitute (5.12) into (5.11). Then, the contribution of the error term is at most

$$
\begin{aligned}
&\sum_{u_m=1,p} g(u_m)\left(\sum_{u_1,\dots,u_{m-1},u_{m+1},\dots,u_k}\frac{1}{\prod_{i\neq m} g(u_i)}\right)\frac{F_{\max}^2\varphi(W)^2(\log R/p)(\log\log N)^2}{p^2W^2} \\
&\quad \ll \frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^k(\log\log N)^2}{pW^{k+1}},
\end{aligned}
$$

which is dominated by the error term of (5.11). Therefore,

$$
\begin{aligned}
&\sum_{u_1,\dots,u_k}\frac{y^{(m)}_{u_1,\dots,u_k}(p)^2}{\prod_{i=1}^k g(u_i)} \\
&\quad = \frac{\varphi(W)^2}{p^2W^2}\left(\log\frac{R}{p}\right)^2 \sum_{u_m=1,p} g(u_m) \sideset{}{'}\sum_{u_1,\dots,u_{m-1},u_{m+1},\dots,u_k}\prod_{i\neq m}\frac{\varphi(u_i)^2}{u_i^2 g(u_i)} \\
&\quad\quad \times \left(\int_0^1 F_p^{[m]}\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_{m-1}}{\log R},u,\frac{\log u_{m+1}}{\log R},\dots,\frac{\log u_k}{\log R}\right)du\right)^2 \\
&\quad\quad + O\left(\frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^{k+1}}{W^{k+1}D_0p}\right). \qquad (5.13)
\end{aligned}
$$

The sum $\sum'$ indicates that $u_1,\dots,u_{m-1},u_{m+1},\dots,u_k$ are restricted to those satisfying the conditions above. In (5.13), the contribution of the terms with $u_m=1$ are dominated by the error term. Hence only the terms with $u_m=p$ contribute to the main term, and since

$$
\frac{g(p)}{p^2} = \frac{1}{p} + O\left(\frac{1}{p^2}\right) = \frac{1}{p} + O\left(p^{-1}N^{-\eta}\right),
$$

if we replace the factor $g(p)/p^2$ in (5.13) with $1/p$, the possible error is dominated by the error term of (5.13). Hence we have

$$
\sum_{u_1,\dots,u_k}\frac{y^{(m)}_{u_1,\dots,u_k}(p)^2}{\prod_{i=1}^k g(u_i)}
$$

$$= \frac{\varphi(W)^2}{pW^2}\left(\log\frac{R}{p}\right)^2 \sideset{}{'}\sum_{u_1,\dots,u_{m-1},u_{m+1},\dots,u_k} \prod_{i\neq m} \frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2 g(u_i)}$$
$$\times\left(\int_0^1 F_p^{[m]}\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_{m-1}}{\log R},u,\frac{\log u_{m+1}}{\log R},\dots,\frac{\log u_k}{\log R}\right)du\right)^2$$
$$+O\left(\frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^{k+1}}{W^{k+1}D_0 p}\right). \tag{5.14}$$

We remove the condition that $(u_i,u_j)=1$ for $i\neq j$. If $(u_i,u_j)>1$, there exists a prime $q>D_0$ for which $q|u_i,u_j$. Therefore, the difference is at most

$$\frac{\varphi(W)^2(\log N)^2F_{\max}^2}{pW^2}\left(\sum_{q>D_0}\frac{\varphi(q)^4}{g(q)^2q^4}\right)\left(\sum_{\substack{u<R\\(u,W)=1}}\frac{\varphi(u)^2}{u^2g(u)}\right)^{k-1}$$
$$\ll \frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^{k+1}}{W^{k+1}D_0 p}.$$

Therefore,

$$\sum_{u_1,\dots,u_k}\frac{y^{(m)}_{u_1,\dots,u_k}(p)^2}{\prod_{i=1}^k g(u_i)}$$
$$= \frac{\varphi(W)^2}{pW^2}\left(\log\frac{R}{p}\right)^2 \sum_{\substack{u_1,\dots,u_{m-1},u_{m+1},\dots,u_k\\(u_i,pW)=1(\forall i\neq m)}} \prod_{i\neq m} \frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2 g(u_i)}$$
$$\times\left(\int_0^1 F_p^{[m]}\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_{m-1}}{\log R},u,\frac{\log u_{m+1}}{\log R},\dots,\frac{\log u_k}{\log R}\right)du\right)^2$$
$$+O\left(\frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^{k+1}}{W^{k+1}D_0 p}\right). \tag{5.15}$$

We apply Lemma 3.4 with

$$\gamma(q)=\begin{cases}1-\dfrac{q^2-3q+1}{q^3-q^2-2q+1} & (q\nmid pW)\\ 0 & (\text{otherwise})\end{cases} \tag{5.16}$$

to the sum over $u_1,\dots,u_{m-1},u_{m+1},\dots,u_k$. In Section 3, we proved that

$$L\ll\log D_0,\quad \mathfrak{S}=\frac{\varphi(p)\varphi(W)}{pW}(1+O(D_0^{-1})).$$

Therefore, by the similar way as (3.37), we find that

$$\sum_{\substack{u_1,\dots,u_{m-1},u_{m+1},\dots,u_k\\(u_i,pW)=1(\forall i\neq m)}} \prod_{i\neq m} \frac{\mu^2(u_i)\varphi(u_i)^2}{u_i^2 g(u_i)}$$

$$
\begin{aligned}
&\times\left(\int_0^1 F_p^{[m]}\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_{m-1}}{\log R},u,\frac{\log u_{m+1}}{\log R},\dots,\frac{\log u_k}{\log R}\right)du\right)^2\\
=\;&\frac{\varphi(p)^{k-1}\varphi(W)^{k-1}}{p^{k-1}W^{k-1}}(\log R)^{k-1}\\
&\times\int_0^1\cdots\int_0^1\left(\int_0^1 F_p^{[m]}(u_1,\dots,u_{m-1},u,u_{m+1},\dots,u_k)du\right)^2\\
&\qquad\qquad\qquad\cdot du_1\cdots du_{m-1}du_{m+1}\cdots du_k\\
&+O\left(\frac{F_{\max}^2\varphi(p)^{k-1}\varphi(W)^{k-1}(\log N)^{k-1}}{p^{k-1}W^{k-1}D_0}\right).
\end{aligned} \tag{5.17}
$$

By substituting (5.17) into (5.15) and replacing the factor $\varphi(p)^{k-1}/p^k$ with $1/p$ (the possible error is sufficiently small), we have

$$
\begin{aligned}
&\sum_{u_1,\dots,u_k}\frac{y^{(m)}_{u_1,\dots,u_k}(p)^2}{\prod_{i=1}^k g(u_i)}\\
=\;&\frac{\varphi(W)^{k+1}}{pW^{k+1}}(\log R)^{k-1}\left(\log\frac{R}{p}\right)^2\\
&\times\int_0^1\cdots\int_0^1\left(\int_0^1 F_p^{[m]}(u_1,\dots,u_{m-1},u,u_{m+1},\dots,u_k)du\right)^2\\
&\qquad\qquad\qquad\cdot du_1\cdots du_{m-1}du_{m+1}\cdots du_k\\
&+O\left(\frac{F_{\max}^2\varphi(W)^{k+1}(\log N)^{k+1}}{W^{k+1}D_0 p}\right).
\end{aligned} \tag{5.18}
$$

We substitute this into (5.10). Our next purpose is to compute the sum over p. For any smooth function f, the sum

$$
\sum_{Y<p<R}\frac{1}{p}\left(\log\frac{R}{p}\right)^2\left(\log\frac{N}{p}\right)^{-1}f\left(\frac{\log R/p}{\log R}\left(\frac{\log p}{\log R/p}+u\right)\right) \tag{5.19}
$$

is expressed by

$$
\int_{N^\eta}^{N^{(\theta/2)-\delta}}\frac{1}{v}\left(\log\frac{R}{v}\right)^2\left(\log\frac{N}{v}\right)^{-1}f\left(\frac{\log R/v}{\log R}\left(\frac{\log v}{\log R/v}+u\right)\right)d\pi(v),
$$

which is asymptotically

$$
\int_{N^\eta}^{N^{(\theta/2)-\delta}}\frac{1}{v}\left(\log\frac{R}{v}\right)^2\left(\log\frac{N}{v}\right)^{-1}f\left(\frac{\log R/v}{\log R}\left(\frac{\log v}{\log R/v}+u\right)\right)\frac{dv}{\log v}.
$$

By putting $\log v/\log N=\xi$, this becomes

$$
\log N\int_\eta^{(\theta/2)-\delta}\frac{((\theta/2)-\delta-\xi)^2}{1-\xi}f\left(\frac{\xi}{(\theta/2)-\delta}+\frac{(\theta/2)-\delta-\xi}{(\theta/2)-\delta}u\right)\frac{d\xi}{\xi}.
$$

We put

$$M_{k,\delta}^{[m]}(\xi) = \int_0^1 \cdots \int_0^1 \left(\int_0^1 F_{m,\delta}(u_1, \ldots, u_k; \xi) du_m \right)^2 du_1 \cdots du_{m-1} du_{m+1} \cdots du_k, \tag{5.20}$$

where the function $F_{m,\delta}$ is defined by (3.42). Then, by applying the consequence of the above argument to (5.18), we have

$$\begin{aligned} &\sum_{Y<p<R} \frac{1}{\log N/p} \sum_{u_1,\ldots,u_k} \frac{y_{u_1,\ldots,u_k}^{(m)}(p)^2}{\prod_{i=1}^k g(u_i)} \\ &= \frac{\varphi(W)^{k+1} (\log R)^{k-1} \log N}{W^{k+1}} (1+o(1)) \int_\eta^{(\theta/2)-\delta} \frac{((\theta/2)-\delta-\xi)^2}{1-\xi} M_{k,\delta}^{[m]}(\xi) \frac{d\xi}{\xi} \\ &\quad + O\left(\frac{F_{\max}^2 \varphi(W)^{k+1} (\log N)^k}{W^{k+1} D_0} \right), \end{aligned} \tag{5.21}$$

if the integral is not zero. We substitute this into (5.10). Consequently we obtain the following result:

Proposition 5.1. *Assuming $BV[\theta, \mathcal{P}]$, we have*

$$\begin{aligned} S_{2,IV}^{(m)} = &\frac{\varphi(W)^k N \log N (\log R)^{k-1}}{W^{k+1}} (1+o(1)) M_{k,\delta}^{(m)}(F) \\ &+ O\left(\frac{F_{\max}^2 \varphi(W)^k N (\log N)^k}{W^{k+1} D_0} \right) + O_B\left(\frac{F_{\max}^2 N}{(\log N)^B} \right) \end{aligned} \tag{5.22}$$

as $N \to \infty$ if

$$M_{k,\delta}^{(m)}(F) := \int_\eta^{(\theta/2)-\delta} \frac{((\theta/2)-\delta-\xi)^2}{1-\xi} M_{k,\delta}^{[m]}(\xi) \frac{d\xi}{\xi} \neq 0,$$

where $M_{k,\delta}^{[m]}(\xi)$ is defined by (5.20).

We note that if $M_{k,\delta}^{(m)}(F) = 0$, the leading term vanishes and hence

$$S_{2,IV}^{(m)} = o\left(\frac{F_{\max}^2 \varphi(W)^k N (\log N)^k}{W^{k+1}} \right) + O_B\left(\frac{N F_{\max}^2}{(\log N)^B} \right).$$

6. Conclusion.

To establish the small gaps between almost primes, we consider the sum

$$S(N, \rho) = \sum_{m=1}^k S_2^{(m)} - \rho S_0$$

$$= \sum_{m=1}^{k} \left(S_{2,I}^{(m)} + S_{2,II}^{(m)} + S_{2,III}^{(m)} + S_{2,IV}^{(m)} \right) - \rho S_0 \tag{6.1}$$

for $\rho \in \mathbf{N}$. To establish small gaps between the set of primes and almost primes, we consider the sum

$$\begin{aligned} S'(N,\rho) &= \sum_{m=1}^{k} (S_1^{(m)} + S_2^{(m)}) - \rho S_0 \\ &= \sum_{m=1}^{k} \left(S_1^{(m)} + S_{2,I}^{(m)} + S_{2,II}^{(m)} + S_{2,III}^{(m)} + S_{2,IV}^{(m)} \right) - \rho S_0 \end{aligned} \tag{6.2}$$

for $\rho \in \mathbf{N}$. If $S(N,\rho) \to \infty$, there exist infinitely many n for which at least $\rho + 1$ of $n + h_1, \ldots, n + h_k$ are E_2-numbers. If $S'(N,\rho) \to \infty$, there exist infinitely many n for which at least $\rho+1$ of $n+h_1, \ldots, n+h_k$ are primes or E_2-numbers. We have computed all terms to obtain the asymptotic formulas for $S(N,\rho)$ and $S'(N,\rho)$. The terms $S_{2,I}^{(m)}$ and $S_{2,II}^{(m)}$ are obtained in Proposition 3.7, the term $S_{2,III}^{(m)}$ is obtained in Proposition 4.2, and the term $S_{2,IV}^{(m)}$ is obtained in Proposition 5.1. Also, the terms S_0 and $S_1^{(m)}$ were obtained by Maynard ([**10**]), and the result is given in Proposition 2.1. By these propositions, we find that under the assumptions of $BV[\theta, \mathcal{P}]$ and $BV[\theta, \mathcal{E}_2]$, $S(N,\rho)$ is asymptotically

$$\begin{aligned} &\left\{ -\theta' \sum_{m=1}^{k} L_{k,\delta}^{(m)}(F) + \frac{\theta'^2}{4} \left(\log \frac{1-\eta}{\eta} \right) \sum_{m=1}^{k} J_k^{(m)}(F) + \sum_{m=1}^{k} M_{k,\delta}^{(m)}(F) - \frac{\rho\theta'}{2} I_k(F) \right\} \\ &\quad \times \left(\frac{\theta'}{2} \right)^{k-1} \frac{\varphi(W)^k N (\log N)^k}{W^{k+1}} \end{aligned}$$

and $S'(N,\rho)$ is asymptotically

$$\begin{aligned} &\left\{ -\theta' \sum_{m=1}^{k} L_{k,\delta}^{(m)}(F) + \frac{\theta'^2}{4} \left(1 + \log \frac{1-\eta}{\eta} \right) \sum_{m=1}^{k} J_k^{(m)}(F) + \sum_{m=1}^{k} M_{k,\delta}^{(m)}(F) - \frac{\rho\theta'}{2} I_k(F) \right\} \\ &\quad \times \left(\frac{\theta'}{2} \right)^{k-1} \frac{\varphi(W)^k N (\log N)^k}{W^{k+1}}, \end{aligned}$$

whenever the leading coefficient is not zero, where

$$\theta' = \theta - 2\delta.$$

We take the limit $\delta \to +0$ and see when the leading coefficients

$$-\theta \sum_{m=1}^{k} L_{k,0}^{(m)}(F) + \frac{\theta^2}{4} \left(\log \frac{1-\eta}{\eta} \right) \sum_{m=1}^{k} J_k^{(m)}(F) + \sum_{m=1}^{k} M_{k,0}^{(m)}(F) - \frac{\rho\theta}{2} I_k(F) \tag{6.3}$$

or

$$-\theta \sum_{m=1}^{k} L_{k,0}^{(m)}(F) + \frac{\theta^2}{4}\left(1 + \log\frac{1-\eta}{\eta}\right)\sum_{m=1}^{k} J_k^{(m)}(F) + \sum_{m=1}^{k} M_{k,0}^{(m)}(F) - \frac{\rho\theta}{2} I_k(F) \quad (6.4)$$

become positive.

7. The proof of Theorem 1.1.

Let $\rho \in \mathbf{N}$ be sufficiently large. We use the same test function as in [**10**]. That is, we define the test function F by

$$F(u_1, \ldots, u_k) = \begin{cases} \prod_{i=1}^{k} g(ku_i) & (u_1, \ldots, u_k \geq 0, u_1 + \cdots + u_k \leq 1) \\ 0 & (\text{otherwise}), \end{cases} \quad (7.1)$$

where the function $g : [0, \infty) \to \mathbf{R}$ is defined by

$$g(u) = \begin{cases} \dfrac{1}{1 + Au} & (0 \leq u \leq T) \\ 0 & (u > T) \end{cases} \quad (7.2)$$

with $A = \log k - 2 \log\log k$, $T = (e^A - 1)/A$. We choose η by

$$\eta = \frac{\theta T}{k} \sim \frac{\theta}{(\log k)^3}.$$

In this case the function $F_{m,0}$ is given by

$$F_{m,0}(u_1, \ldots, u_k; \xi) = g\left(k\left(\frac{2\xi}{\theta} + \frac{\theta - 2\xi}{\theta} u_m\right)\right) \prod_{i \neq m} g(ku_i).$$

When the pair (u_m, ξ) moves in $[0, 1] \times [\eta, \theta/2]$, we have

$$k\left(\frac{2\xi}{\theta} + \frac{\theta - 2\xi}{\theta} u_m\right) \geq \frac{2k\eta}{\theta} = 2T > T.$$

Therefore, we have $F_{m,0} \equiv 0$, so $L_k^{(m)}(F) = M_k^{(m)}(F) = 0$ for $m = 1, \ldots, k$. Hence if

$$\frac{(\theta^2/4)\,(\log(1-\eta)/\eta) \sum_{m=1}^{k} J_k^{(m)}(F)}{(\theta/2) I_k(F)} > \rho, \quad (7.3)$$

(6.3) becomes positive. Using the inequality in [**10**, p. 408], for any $\epsilon > 0$, if k is sufficiently large, the left hand side of (7.3) is

$$\begin{aligned} &= \frac{\theta}{2}\left(\log\frac{1-\eta}{\eta}\right)\frac{kJ_k^{(1)}(F)}{I_k(F)} \\ &\geq \frac{3\theta}{2}(\log\log k)(1 + o(1))(\log k - 2\log\log k - 2) \end{aligned}$$

$$\geq \frac{3\theta}{2}\left(1-\frac{\epsilon}{4}\right)(\log k \log\log k - 3(\log\log k)^2). \tag{7.4}$$

We put

$$k = \left[\exp\left(\frac{(2+\epsilon)\rho}{3\theta\log\rho}\right)+1\right],$$

where $[a]$ denotes the largest integer less than or equal to a. Then the third line of (7.4) is

$$\begin{aligned}&\geq \frac{3\theta}{2}\left(1-\frac{\epsilon}{4}\right)\left(\frac{(2+\epsilon)\rho}{3\theta\log\rho}\log\left(\frac{(2+\epsilon)\rho}{3\theta\log\rho}\right)-3\log^2\left(\frac{(2+\epsilon)\rho}{3\theta\log\rho}\right)\right)\\&\geq \left(1+\frac{\epsilon}{5}\right)\rho + O\left(\frac{\rho\log\log\rho}{\log\rho}\right).\end{aligned}$$

This is greater than ρ whenever ρ is sufficiently large. Hence (7.3) holds for $k \sim \exp((2+\epsilon)\rho/(3\theta\log\rho))$. We can choose the admissible set by $\mathcal{H} = \{p_{\pi(k)+1}, p_{\pi(k)+2}, \ldots, p_{\pi(k)+k}\}$, where p_n denotes the n-th prime. Hence there exist infinitely many n for which at least $\rho+1$ of $n+p_{\pi(k)+1}, \ldots, n+p_{\pi(k)+k}$ are E_2-numbers. Since

$$p_{\pi(k)+k} - p_{\pi(k)+1} \ll k\log k \ll \exp\left(\frac{(2+2\epsilon)\rho}{3\theta\log\rho}\right),$$

by replacing ϵ with $\epsilon/2$, the proof of Theorem 1.1 is completed. $\square$

Remark 7.1. Recently Takayuki Neshime, who belongs to the master course of Tokyo Institute of Technology, told me that by choosing the parameter A in a different way and evaluating the contribution of $L_k^{(m)}(F)$, we can obtain a better upper bound

$$\liminf_{n\to\infty}(q_{n+m}-q_n) \ll \sqrt{m}\exp\sqrt{\frac{8m}{\theta}},$$

assuming $BV[\theta,\mathcal{P}]$ and $BV[\theta,\mathcal{E}_2]$.

8. The proofs of other theorems.

The numerical computations below are accomplished by *Mathematica*. To prove Theorem 1.2, it suffices to show that the leading coefficient (6.4) with $k=6, \rho=2, \theta=1/2$ becomes positive for some test function F. We define this by

$$\begin{aligned}F(x,y,z,u,v,w) = 1 &- \frac{143577}{50000}P_1 + \frac{12337}{5000}P_1^2 + \frac{86987}{50000}P_2\\&- \frac{619873}{1000000}P_1^3 - \frac{156481}{100000}P_1P_2 - \frac{230073}{5000000}P_3\end{aligned}$$

if $x,y,z,u,v,w \geq 0, x+y+z+u+v+w \leq 1$ and otherwise $F(x,y,z,u,v,w) := 0$, where $P_i = x^i+y^i+z^i+u^i+v^i+w^i$ $(i=1,2,3)$. We take $\eta = 10^{-10}$. Then, by numerical computations, we find that

$$I_6(F) = 5.30806\cdots \times 10^{-6}, \quad J_6(F) = 1.88915\cdots \times 10^{-6},$$
$$L_{6,0}^{(m)}(F) = 9.20744\cdots \times 10^{-6}, \quad M_{6,0}^{(m)}(F) = 2.22265\cdots \times 10^{-6},$$

hence

$$-\frac{1}{2}\sum_{m=1}^{6} L_{6,0}^{(m)}(F) + \frac{1}{16}\left(1+\log\frac{1-10^{-10}}{10^{-10}}\right)\sum_{m=1}^{6} J_6^{(m)}(F) + \sum_{m=1}^{6} M_{6,0}^{(m)}(F) - \frac{1}{2}I_6(F)$$
$$= 8.02\cdots \times 10^{-8} > 0.$$

This proves Theorem 1.2. □

Next we prove Theorem 1.3. It suffices to show that the leading coefficient (6.4) with $k = 3, \rho = 2, \theta = 1$ becomes positive for some test function F. We define this by

$$F(x,y,z) = \begin{cases} (1-x)(1-y)(1-z) & (x,y,z \geq 0, x+y+z \leq 1) \\ 0 & (\text{otherwise}) \end{cases} \tag{8.1}$$

and put $\eta = 10^{-10}$. Then, by numerical computations, we find that

$$I_3(F) = 0.0287919\cdots, \quad J_3^{(m)}(F) = 0.0154828\cdots,$$
$$L_{3,0}^{(m)}(F) = 0.1606331\cdots, \quad M_{3,0}^{(m)}(F) = 0.0779163\cdots$$

for $m = 1, 2, 3$. Consequently,

$$-\sum_{m=1}^{3} L_{3,0}^{(m)}(F) + \frac{1}{4}\left(1+\log\frac{1-10^{-10}}{10^{-10}}\right)\sum_{m=1}^{3} J_3^{(m)}(F) + \sum_{m=1}^{3} M_{3,0}^{(m)}(F) - I_3(F)$$
$$= 0.00204\cdots > 0.$$

This proves Theorem 1.3. □

To prove Theorem 1.4, we see that the number (6.3) with $k = 5, \rho = 2, \theta = 1$ becomes positive for some test function F. We define this by

$$F(x,y,z,u,v) = 1 + \frac{917}{500}Q_1 - \frac{281}{50}Q_1^2 - \frac{41}{25}Q_2 + \frac{287}{100}Q_1^3 + \frac{191}{100}Q_1Q_2 - \frac{81}{250}Q_3$$

if $x,y,z,u,v \geq 0, x+y+z+u+v \leq 1$, and otherwise $F(x,y,z,u,v) := 0$, where $Q_i = x^i + y^i + z^i + u^i + v^i$ $(i = 1,2,3)$. Moreover, we take $\eta = 10^{-10}$. Then by numerical computations, we find that

$$I_5(F) = \frac{1735763}{1732500000}, \quad J_5^{(m)}(F) = \frac{722755717}{1871100000000},$$
$$L_{5,0}^{(m)}(F) = 0.00392368\cdots, \quad M_{5,0}^{(m)}(F) = 0.00190092\cdots$$

for $1 \leq m \leq 5$. Consequently,

$$
\begin{aligned}
-\sum_{m=1}^{5} L_{5,0}^{(m)}(F) + \frac{1}{4}\left(\log \frac{1-10^{-10}}{10^{-10}}\right) \sum_{m=1}^{5} J_{5}^{(m)}(F) + \sum_{m=1}^{5} M_{5,0}^{(m)}(F) - I_5(F) \\
= 2.13079\cdots \times 10^{-6} > 0.
\end{aligned}
$$

Since the set $\mathcal{H} = \{0, 2, 6, 8, 12\}$ is an admissible set with five elements, the statement of Theorem 1.4 is obtained. □

Acknowledgements. The author sincerely expresses his utmost gratitude to the referee for reading this paper carefully and giving him many valuable comments and suggestions.

References

[1] J.-R. Chen, On the representation of a larger even integer as the sum of a prime and the product of at most two primes, Sci. Sinica, **16** (1973), 157–176.

[2] D. A. Goldston, J. Pintz and C. Y. Yıldırım, Primes in tuples I, Ann. of Math. (2), **170** (2009), 819–862.

[3] D. A. Goldston, J. Pintz and C. Y. Yıldırım, Primes in tuples II, Acta. Math., **204** (2010), 1–47.

[4] D. A. Goldston, J. Pintz and C. Y. Yıldırım, Primes in tuples III: On the difference $p_{n+\nu} - p_n$, Funct. Approx. Comment. Math., **35** (2006), 79–89.

[5] D. A. Goldston, J. Pintz and C. Y. Yıldırım, Primes in tuples IV: Density of small gaps between consecutive primes, Acta Arith., **160** (2013), 37–53.

[6] D. A. Goldston, J. Pintz and C. Y. Yıldırım, Small gaps between primes, In: Proceedings of the International Congress of Mathematics, Seoul, 2014, 419–441.

[7] D. A. Goldston, S. W. Graham, J. Pintz and C. Y. Yıldırım, Small gaps between primes or almost primes, Trans. Amer. Math. Soc., **361** (2009), 5285–5330.

[8] D. A. Goldston, S. W. Graham, J. Pintz and C. Y. Yıldırım, Small gaps between products of two primes, Proc. Lond. Math. Soc. (3), **98** (2009), 741–774.

[9] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes, Acta Math., **44** (1923), 1–70.

[10] J. Maynard, Small gaps between primes, Ann. of Math. (2), **181** (2015), 383–413.

[11] J. Maynard, 3-tuples have at most 7 prime factors infinitely often, Math. Proc. Cambridge Philos. Soc., **155** (2013), 443–457.

[12] Y. Motohashi, An induction principle for the generalization of Bombieri's prime number theorem, Proc. Japan Acad., **52** (1976), 273–275.

[13] J. Pintz, Some new results on gaps between consecutive primes, In: Paul Turán Memorial Volume: Number Theory, Analysis and Combinatorics, De Gruyter Proc. Math., De Gruyter, Berlin, 2014, 261–278.

[14] D. H. J. Polymath, W. Castryck, E. Fouvry, G. Harcos, E. Kowalski, P. Michel, P. Nelson, E. Paldi, J. Pintz, A. V. Sutherland, T. Tao and X. Xiao-Feng, New equidistribution estimates of Zhang type, Algebra Number Theory, **8** (2014), 2067–2199.

[15] D. H. J. Polymath, Variants of the Selberg sieve, and bounded intervals containing many primes, Res. Math. Sci., **1** (2014), art. 12, 83 pp.

[16] F. Thorne, Bounded gaps between products of primes with applications to ideal class groups and elliptic curves, Int. Math. Res. Not. IMRN, **2008** (2008), art. ID rmn 156, 41 pp.

[17] Y. Zhang, Bounded gaps between primes, Ann. of Math. (2), **179** (2014), 1121–1174.

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