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L^p boundedness of some rough operators with different weights

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Abstract. In this paper we prove that the maximal operator M_Ω, the singular integral operator T_Ω, and the maximal singular integral operator T_Ω^* with rough kernels are all bounded operators from $L^p(v)$ to $L^p(u)$ for the weight functions pair (u,v). Here the kernel function Ω satisfies a size condition only; that is, $\Omega \in L^q(S^{n-1})$, $q>1$, but has no smoothness on S^{n-1}.

§1. Introduction.

Suppose that S^{n-1} is the unit sphere of $\boldsymbol{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma(x')$. If $\Omega(x)$ is a homogeneous function of degree zero on $\boldsymbol{R}^n$, then the maximal operator M_Ω, the singular integral operator T_Ω, and the maximal singular integral operator T_Ω^* are defined respectively by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)|\,|f(y)|\,dy,$$

$$T_\Omega f(x) = p.v. \int_{\boldsymbol{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy,$$

and

$$T_\Omega^* f(x) = \sup_{\varepsilon>0} |T_\Omega^\varepsilon f(x)| = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy \right|.$$

In 1990, Watson [**W**] proved that if $\Omega \in L^q(S^{n-1})$, $q>1$, and Ω has average zero on S^{n-1}, then the operators T_Ω and T_Ω^* are both bounded on the weighted spaces $L^p(\omega)$ for $1<p<\infty$, where the weight function $\omega(x)$ is in the Muckenhoupt weights class (see below for the definition). In 1993, using a method different from the one in [**W**], Duoandikoetxea [**D**] obtained independently the same weighted results of the operators T_Ω and T_Ω^* as in [**W**]. Moreover, the weighted L^p boundedness of the maximal operator M_Ω was also given in [**D**].

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In this paper we shall consider the weighted norm inequalities for the operators M_Ω, T_Ω, and T_Ω^* with different weight functions. To state our results, let us recall the definitions of A_p weight class, A_p^* weights pair, and S_p^* weights pair for $1 < p < \infty$.

Definition 1. A locally integrable nonnegative function ω is said to belong to A_p if there is a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C,$$

where and below, Q denotes a cube in $\boldsymbol{R}^n$ with its sides parallel to the coordinate axes and the supremun is taken over all cubes.

Definition 2. A locally integrable nonnegative functions pair (u,v) is said to belong to A_p^* if there is a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C.$$

Definition 3. A locally integrable nonnegative functions pair (u,v) is said to belong to S_p^* if there is a constant $C > 0$ such that for any cube $Q \in \boldsymbol{R}^n$,

$$\int_Q [M(v^{-1/(p-1)}\chi_Q)]^p u(x)\,dx \le C\int_Q v(x)^{-1/(p-1)}\,dx,$$

where $\chi_Q(x)$ denotes the characteristic function of Q and M is the Hardy-Littlewood maximal operator.

Now let us state precisely our results as follows. In this paper, we always denote $p' = p/(p-1)$ for $1 < p < \infty$.

Theorem 1. *Suppose that $\Omega(x') \in L^q(S^{n-1})$, $q > 1$, is homogeneous of degree zero on $\boldsymbol{R}^n$. If p, q and the weights pair (u,v) satisfy one of the following conditions*:

(a) $1 \le q' < p < \infty$, $(u,v) \in S_{p/q'}^*$;

(b) $1 < p < q$, $(v^{1-p'}, u^{1-p'}) \in A_{p'/q'}^*$, *in addition* $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$,

then M_Ω is bounded from $L^p(v)$ to $L^p(u)$; that is, there is a constant $C > 0$ independent of f such that

$$\left(\int_{\boldsymbol{R}^n}[M_\Omega f(x)]^p u(x)\,dx\right)^{1/p} \le C\left(\int_{\boldsymbol{R}^n}|f(x)|^p v(x)\,dx\right)^{1/p}. \tag{1.1}$$

Theorem 2. *Suppose that $\Omega(x') \in L^q(S^{n-1})$, $q > 1$, is homogeneous of degree zero on $\boldsymbol{R}^n$ and has average zero on S^{n-1}. If p, q and the weights pair (u,v) satisfy one of the following conditions*:

(a) $1 \le q' < p < \infty$, $(u, v) \in A^*_{p/q'}$, *in addition* $u(x), v(x) \in A_{p/q'}$;

(b) $1 < p < q$, $(v^{1-p'}, u^{1-p'}) \in A^*_{p'/q'}$, *in addition* $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$,

then T_Ω is bounded operator from $L^p(v)$ to $L^p(u)$; that is, there is a constant $C > 0$ independent of f such that

$$\left(\int_{\boldsymbol{R}^n} |T_\Omega f(x)|^p u(x)\, dx\right)^{1/p} \le C\left(\int_{\boldsymbol{R}^n} |f(x)|^p v(x)\, dx\right)^{1/p}. \tag{1.2}$$

Theorem 3. *Under the same conditions as in Theorem 2, the operator T^*_Ω is bounded operator from $L^p(v)$ to $L^p(u)$; that is, there is a constant $C > 0$ independent of f such that*

$$\left(\int_{\boldsymbol{R}^n} [T^*_\Omega f(x)]^p u(x)\, dx\right)^{1/p} \le C\left(\int_{\boldsymbol{R}^n} |f(x)|^p v(x)\, dx\right)^{1/p}. \tag{1.3}$$

To prove the above theorems, we need to use the weighted inequality for the vector-valued Hardy-Littlewood maximal operator M with a weight function pair. However, the result itself is very interesting, since it is an extension of known results obtained by Andersen and John [**AJ**] in 1980.

Suppose that $\vec{f}(x) = \{f_k(x)\}_1^\infty$, is a sequence of locally integrable functions on $\boldsymbol{R}^n$, $M(\vec{f})(x) = \{Mf_k(x)\}_1^\infty$ and for $1 < r < \infty$, $\|\vec{f}(x)\|_{l^r} = (\sum_{k \in \boldsymbol{Z}} |f_k(x)|^r)^{1/r}$. We have the following result.

Theorem 4. *Let $1 < r < \infty$. If $1 < p < \infty$, $(u, v) \in A^*_p$ and $u(x), v(x) \in A_p$, then there is a constant $C_{r,p}$ independent of $\vec{f}$ such that*

$$\left(\int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|^p_{l^r} u(x)\, dx\right)^{1/p} \le C_{r,p}\left(\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|^p_{l^r} v(x)\, dx\right)^{1/p}. \tag{1.4}$$

Remark 1. Notice that if $u = v$, then the results of Theorems 1 through 4 are identical with the related conclusions in [**W**], [**D**] and [**AJ**].

The paper is organized as follows. Section 2 contains some elementary properties of the weight classes A_p, A^*_p, and S^*_p. The proof of Theorem 1 is given in Section 3. The proof outlines of Theorems 2 and 3 can be found in Section 4. Finally, in Section 5 we give the proof of Theorem 4.

§2. Some elementary facts.

Let us begin by giving some properties of the weight classes A_p, A^*_p, and S^*_p.

The elementary properties of A_p $(1 < p < \infty)$.

(2.1) $A_{p_1} \subset A_{p_2}$ *if* $1 < p_1 \le p_2 < \infty$.

(2.2) $\omega(x) \in A_p$ *if and only if* $\omega(x)^{1-p'} \in A_{p'}$.

(2.3) *If* $\omega(x) \in A_p$, *then there is an* $\varepsilon > 0$ *such that* $p - \varepsilon > 1$ *and* $\omega(x) \in A_{p-\varepsilon}$.

(2.4) *If* $\omega(x) \in A_p$, *then there is an* $\varepsilon > 0$ *such that* $\omega(x)^{1+\varepsilon} \in A_p$.

(2.5) *If* $\omega(x) \in A_p$, *then there are* $C > 0$ *and* $\varepsilon > 0$ *such that for any cube* $Q \in \boldsymbol{R}^n$

$$\frac{1}{|Q|}\int_Q \omega(x)^{1+\varepsilon}\,dx \le C\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)^{1+\varepsilon}.$$

See [**GR**, Chapter IV] for the proofs of (2.1)–(2.5).

The elementary properties of A_p^* $(1 < p < \infty)$**.**

(2.6) $(u, v) \in A_p^*$ *if and only if* $(v^{1-p'}, u^{1-p'}) \in A_{p'}^*$.

(2.7) $S_p^* \subset A_p^*$ *for* $1 < p < \infty$.

(2.8) *If* $(u, v) \in A_p^*$, *then for any* $0 < \varepsilon < 1$, $(u^\varepsilon, v^\varepsilon) \in S_p^*$.

(2.9) *If* $u(x), v(x) \in A_p$ *and* $(u, v) \in A_p^*$, *then there is an* $\varepsilon > 0$ *such that* $(u^{1+\varepsilon}, v^{1+\varepsilon}) \in A_p^*$ *and* $(v^{(1-p')(1+\varepsilon)}, u^{(1-p')(1+\varepsilon)}) \in A_{p'}^*$.

(2.10) *If* $u(x), v(x) \in A_p$ *and* $(u, v) \in A_p^*$, *then* $(u, v) \in S_p^*$ *and* $(v^{1-p'}, u^{1-p'}) \in S_{p'}^*$.

(2.11) *If* $(u, v) \in A_p^*$, *then for any* $r > p$, $(u, v) \in A_r^*$.

(2.12) *If* $u(x), v(x) \in A_p$ *and* $(u, v) \in A_p^*$, *then there is an* $\varepsilon > 0$ *such that* $p - \varepsilon > 1$ *and* $(u, v) \in A_{p-\varepsilon}^*$.

Proof. The result (2.6) can be deduced from the definition of A_p^*, and (2.7) can be found in [**GR**, p. 433]. On the other hand, (2.8) is just a Corollary in [**N**, p. 644]. Now let us prove (2.9). Since $u(x) \in A_p$, by (2.5) there are $C_1, \varepsilon_1 > 0$ such that for any $Q \in \boldsymbol{R}^n$

$$\frac{1}{|Q|}\int_Q u(x)^{1+\varepsilon_1}\,dx \le C_1\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)^{1+\varepsilon_1}. \tag{2.13}$$

It follows from $v(x) \in A_p$ and (2.2) that $v(x)^{1-p'} \in A_{p'}$. Using (2.5) again, there are $C_2, \varepsilon_2 > 0$ such that for any $Q \in \boldsymbol{R}^n$

$$\frac{1}{|Q|}\int_Q v(x)^{(1-p')(1+\varepsilon_2)}\,dx \le C_2\left(\frac{1}{|Q|}\int_Q v(x)^{1-p'}\,dx\right)^{1+\varepsilon_2};$$

that is,

$$\frac{1}{|Q|}\int_Q v(x)^{-(1+\varepsilon_2)/(p-1)}\,dx \le C_2\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{1+\varepsilon_2}. \tag{2.14}$$

Thus, there are $C = \max\{C_1, C_2\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ such that (2.13) and (2.14) hold at the same time. Hence by $(u, v) \in A_p^*$ we have

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)^{1+\varepsilon}\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-(1+\varepsilon)/(p-1)}\,dx\right)^{p-1}$$

$$\le C\left(\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1}\right)^{1+\varepsilon} < \infty;$$

that is,

$$(u^{1+\varepsilon}, v^{1+\varepsilon}) \in A_p^*. \tag{2.15}$$

From (2.15) and (2.6), we can get $(v^{(1-p')(1+\varepsilon)}, u^{(1-p')(1+\varepsilon)}) \in A_{p'}^*$. Thus we proved (2.9). In order to prove (2.10) we take $\delta = 1/(1+\varepsilon)$. Then $0 < \delta < 1$. By (2.15) and (2.8) we get $(u, v) = (u^{(1+\varepsilon)\delta}, v^{(1+\varepsilon)\delta}) \in S_p^*$. On the other hand, by (2.15) and (2.6) we have $(v^{(1+\varepsilon)(1-p')}, u^{(1+\varepsilon)(1-p')}) \in A_{p'}^*$. As above, if we take $\delta = 1/(1+\varepsilon) < 1$, then $(v^{1-p'}, u^{1-p'}) \in S_{p'}^*$ which proves (2.10).

Using Hölder's inequality, we may easily obtain (2.11).

Now let us give the proof of (2.12). By (2.9) we know that there is an $\eta > 0$ such that $(v^{(1-p')(1+\eta)}, u^{(1-p')(1+\eta)}) \in A_{p'}^*$. Taking $\varepsilon = \eta(p-1)/(1+\eta)$, then we can easily see that $\varepsilon > 0$ and $1 < p - \varepsilon < p$. Hence we have $p' < (p-\varepsilon)'$ and $(v^{(1-p')(1+\eta)}, u^{(1-p')(1+\eta)}) \in A_{(p-\varepsilon)'}^*$ by (2.11). From (2.6) we get $(u^{(1-p')(1+\eta)[1-(p-\varepsilon)]}, v^{(1-p')(1+\eta)[1-(p-\varepsilon)]}) \in A_{p-\varepsilon}^*$. However, $(1-p')(1+\eta)\cdot[1-(p-\varepsilon)] = 1$. Thus we have $(u, v) \in A_{p-\varepsilon}^*$. □

In the proof of theorems in this paper, we need still the following conclusion.

Lemma 1. *Let $p, q > 1$. If the weights pair (u, v) satisfies one of the following conditions*:

(i) *for $p' > q'$, $(v^{1-p'}, u^{1-p'}) \in A_{p'/q'}^*$ and $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$*;

(ii) *for $p > q'$ $(u, v) \in A_{p/q'}^*$ and $u(x), v(x) \in A_{p/q'}$*,

then $u(x), v(x) \in A_p$ and $(u, v) \in S_p^*$.

Proof. We show the case (i) only; the proof of case (ii) is similar. Since $p'/q' < p'$, we have $u(x)^{1-p'}, v(x)^{1-p'} \in A_{p'/q'} \subset A_{p'}$ by (2.1). From (2.2) we get $u(x), v(x) \in A_p$. On the other hand, since $(v^{1-p'}, u^{1-p'}) \in A_{p'/q'}^*$ and $p'/q' < p'$, using (2.11) we obtain $(v^{1-p'}, u^{1-p'}) \in A_{p'}^*$. From this and (2.6) we get $(u, v) \in A_p^*$. Note that $u(x), v(x) \in A_p$ proved above, using (2.10) we obtain $(u, v) \in S_p^*$. □

Finally, let us recall the well known Sawyer's result about weighted norm inequality of the Hardy-Littlewood maximal operator M for a weights pair, which will be used in the proof of our results.

Theorem A ([**Sa**, Theorem B]). *For $1 < p < \infty$, M is bounded from $L^p(v)$ to $L^p(u)$ if and only if the weights pair $(u, v) \in S_p^*$.*

§3. Proof of Theorem 1.

The case for the condition (a) is simple. In fact by $M_\Omega f(x) \le C[M(|f|^{q'})(x)]^{1/q'}$, $1 < p/q' < \infty$, and Theorem A, we obtain the conclusion of Theorem 1 for the condition (a) immediately.

We now turn to the proof of Theorem 1 for the condition (b). First let us introduce some notations. For $j \in \boldsymbol{Z}$ we denote

$$K_{\Omega,j}(x) = \Omega(x)\chi_{[2^j,2^{j+1})}(x)/|x|^n,$$

$$T_{\Omega,j}f(x) = K_{\Omega,j} * f(x) = \int_{2^j \le |x-y| < 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy,$$

and

$$g_\Omega(f)(x) = \left(\sum_{j\in\boldsymbol{Z}} |T_{\Omega,j}f(x)|^2\right)^{1/2}. \tag{3.1}$$

It is easy to see that there is a $C > 0$ such that

$$M_\Omega f(x) \le C \sup_j T_{|\Omega|,j}(|f|)(x). \tag{3.2}$$

We denote $\Omega_0(x) = |\Omega(x)| - \|\Omega\|_1/|S^{n-1}|$, where $\|\Omega\|_1 = \int_{S^{n-1}} |\Omega(x')|\,d\sigma(x')$. Then it is easy to check that Ω_0 is also homogeneous of degree zero on $\boldsymbol{R}^n$, $\Omega_0 \in L^q(S^{n-1})$, and $\int_{S^{n-1}} \Omega_0(x')\,d\sigma(x') = 0$. By (3.2), we thus have the following pointwise inequality

$$\begin{aligned} M_\Omega f(x) &\le C \sup_j \int_{2^j \le |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|\,dy \\ &= C \sup_j \left(\int_{2^j \le |x-y| < 2^{j+1}} \frac{\Omega_0(x-y)}{|x-y|^n} |f(y)|\,dy \right. \\ &\qquad \left. + \frac{\|\Omega\|_1}{|S^{n-1}|} \int_{2^j \le |x-y| < 2^{j+1}} \frac{|f(y)|}{|x-y|^n}\,dy \right) \\ &\le C g_{\Omega_0}(|f|)(x) + CMf(x). \end{aligned} \tag{3.3}$$

The proof of Theorem 1 for the case (b) will be completed by a bootstrapping argument. According to the range of q, let us establish several propositions.

Proposition 1. *Suppose that $\Omega(x') \in L^q(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$, and $q > \max\{p, 2\}$ and $(v^{1-p'}, u^{1-p'}) \in A^*_{p'/q'}$, in addition $v^{1-p'}, u^{1-p'} \in A_{p'/q'}$. Then there is a constant $C > 0$ independent of f such that $\|M_\Omega f\|_{p,u} \le C\|f\|_{p,v}$.*

In order to prove Proposition 1, by (3.3) it is sufficient to show that, under the conditions of Proposition 1,

$$\|g_{\Omega_0}(|f|)\|_{p,u} \le C\|f\|_{p,v} \tag{3.4}$$

and

$$\|Mf\|_{p,u} \le C\|f\|_{p,v}, \tag{3.5}$$

where $g_{\Omega_0}(|f|)(x) = (\sum_{j \in \boldsymbol{Z}} |T_{\Omega_0,j}(|f|)(x)|^2)^{1/2}$ by (3.1). Obviously, by Lemma 1 and Theorem A we can obtain (3.5) immediately. Hence it remains to verify (3.4) to finish the proof of Proposition 1. For any a sequence $\varepsilon = \{\varepsilon_j\}$ with $\varepsilon_j = +1$ or -1, we define a linear operator by

$$T_{\varepsilon,\Omega_0} f(x) = \sum_{j \in \boldsymbol{Z}} \varepsilon_j (K_{\Omega_0,j} * f)(x).$$

Thus by using the argument related to Rademacher functions [**K**, Theorem 4.2], the proof of (3.4) is reduced to verify that, under the condition of Proposition 1, there is a constant $C > 0$, independent of f and $\{\varepsilon_j\}$, such that

$$\left(\int_{\boldsymbol{R}^n} |T_{\varepsilon,\Omega_0} f(x)|^p u(x)\,dx\right)^{1/p} \le C\left(\int_{\boldsymbol{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p}. \tag{3.6}$$

However, (3.6) may be obtained from the following Lemma 2 and Lemma 3.

Lemma 2. *Suppose that $\Omega(x') \in L^q(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$ and has average zero on S^{n-1}. If $q' < p$, $q > 2$ and $(u,v) \in A^*_{p/q'}$, in addition $u(x), v(x) \in A_{p/q'}$, then there is a constant $C > 0$, independent of f and $\{\varepsilon_j\}$, such that $\|T_{\varepsilon,\Omega} f\|_{p,u} \le C\|f\|_{p,v}$.*

Proof. We make a new decomposition of $T_{\varepsilon,\Omega}$. Choose a radial real function $\psi \in C_0^\infty(\boldsymbol{R}^n)$ satisfying $0 \le \psi \le 1$, $\operatorname{supp}(\psi) \subset \{x \in \boldsymbol{R}^n : 1/2 \le |x| \le 2\}$, and $\sum_{k \in \boldsymbol{Z}} \psi^2(2^k x) = 1$ for any $x \ne 0$. Define S_k by $(S_k f)\hat{}(\xi) = \psi(2^k \xi)\hat{f}(\xi)$, then for any $f \in \mathscr{S}(\boldsymbol{R}^n)$, $\sum_{k \in \boldsymbol{Z}} S_k^2 f(x) = f(x)$. Hence for $f \in \mathscr{S}(\boldsymbol{R}^n)$ we have

$$\begin{aligned} T_{\varepsilon,\Omega} f(x) &= \sum_{j \in \boldsymbol{Z}} \varepsilon_j (K_{\Omega,j} * f)(x) = \sum_{j \in \boldsymbol{Z}} \varepsilon_j K_{\Omega,j} * \left(\sum_{k \in \boldsymbol{Z}} (S_{j+k}^2 f)(x)\right) \\ &= \sum_{k \in \boldsymbol{Z}} \sum_{j \in \boldsymbol{Z}} \varepsilon_j S_{j+k}(K_{\Omega,j} * S_{j+k} f)(x) \\ &:= \sum_{k \in \boldsymbol{Z}} T^k_{\varepsilon,\Omega} f(x), \end{aligned} \tag{3.7}$$

where $T^k_{\varepsilon,\Omega}f(x) = \sum_{j\in\boldsymbol{Z}} \varepsilon_j S_{j+k}(K_{\Omega,j} * S_{j+k}f)(x)$. By Plancherel's Theorem we get

$$\begin{aligned}
\|T^k_{\varepsilon,\Omega}f\|_2^2 &= \int_{\boldsymbol{R}^n} \left| \sum_{j\in\boldsymbol{Z}} \varepsilon_j S_{j+k}(K_{\Omega,j} * S_{j+k}f)(x) \right|^2 dx \\
&\le C\|\{\varepsilon_j\}\|_{l^\infty} \sum_{j\in\boldsymbol{Z}} \int_{\boldsymbol{R}^n} |S_{j+k}(K_{\Omega,j} * S_{j+k}f)(x)|^2\, dx \\
&\le C \sum_{j\in\boldsymbol{Z}} \int_{2^{-j-k-1}\le|\xi|\le 2^{-j-k+1}} |\hat{K}_{\Omega,j}(\xi)|^2 |\hat{f}(\xi)|^2\, d\xi.
\end{aligned}$$

On the other hand, by [**DR**, pp. 551–552] we know that there are $C > 0$ and $\theta > 0$ such that for any $j \in \boldsymbol{Z}$, $|\hat{K}_{\Omega,j}(\xi)| \le C \min\{|2^j\xi|^\theta, |2^j\xi|^{-\theta}\}$. Hence for $k > 0$, we have

$$\|T^k_{\varepsilon,\Omega}f\|_2^2 \le \sum_{j\in\boldsymbol{Z}} \int_{2^{-j-k-1}\le|\xi|\le 2^{-j-k+1}} |2^j\xi|^{2\theta} |\hat{f}(\xi)|^2\, d\xi \le C2^{-2\theta k}\|f\|_2^2.$$

If $k \le 0$, the estimate $|\hat{K}_{\Omega,j}(\xi)| \le C|2^j\xi|^{-\theta}$, implies $\|T^k_{\varepsilon,\Omega}f\|_2^2 \le C2^{-2\theta(-k)}\|f\|_2^2$. Thus there are $C, \theta > 0$, independent of f and $\{\varepsilon_j\}$, such that

$$\|T^k_{\varepsilon,\Omega}f\|_2 \le C2^{-\theta|k|}\|f\|_2 \quad \text{for any } k \in \boldsymbol{Z}. \tag{3.8}$$

Below we prove that, under the conditions of Lemma 2, there is $C > 0$, independent of f and $\{\varepsilon_j\}$, such that

$$\|T^k_{\varepsilon,\Omega}f\|_{p,u} \le C\|f\|_{p,v} \quad \text{for any } k \in \boldsymbol{Z}. \tag{3.9}$$

In fact, by Lemma 1 (ii) we know that $u(x) \in A_p$. Using the weighted Littlewood-Paley theory [**K**, Theorem 2.1], there is a $C > 0$, independent of f and $\{\varepsilon_j\}$, such that

$$\begin{aligned}
\|T^k_{\varepsilon,\Omega}f\|_{p,u} &= \left(\int_{\boldsymbol{R}^n} \left| \sum_{j\in\boldsymbol{Z}} \varepsilon_j S_{j+k}(K_{\Omega,j} * S_{j+k}f)(x) \right|^p u(x)\, dx \right)^{1/p} \\
&\le C\|\{\varepsilon_j\}\|_{l^\infty} \cdot \left\| \left(\sum_{j\in\boldsymbol{Z}} |S_{j+k}(K_{\Omega,j} * S_{j+k}f)(\cdot)|^2 \right)^{1/2} \right\|_{p,u} \\
&\le C \left(\int_{\boldsymbol{R}^n} \left\{ \left(\sum_{j\in\boldsymbol{Z}} |K_{\Omega,j} * S_{j+k}f(x)|^2 \right)^{1/2} \right\}^p u(x)\, dx \right)^{1/p}.
\end{aligned}$$

For fixed k, denote $h_j(x) = S_{j+k}f(x)$. Then we have

$$|K_{\Omega,j} * S_{j+k}f(x)| \le \left(\int_{2^j \le |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|^q}{|x-y|^n}\,dy\right)^{1/q} \left(\int_{2^j \le |x-y| < 2^{j+1}} \frac{|h_j(y)|^{q'}}{|x-y|^n}\,dy\right)^{1/q'}$$

$$\le C[M(|h_j|^{q'})(x)]^{1/q'}.$$

Thus

$$\|T^k_{\varepsilon,\Omega}f\|_{p,u} \le C\left\{\int_{\boldsymbol{R}^n}\left[\left(\sum_{j\in\boldsymbol{Z}}[M(|h_j|^{q'})(x)]^{2/q'}\right)^{q'/2}\right]^{p/q'} u(x)\,dx\right\}^{1/p}$$

$$= C\big\|\|M(|h_j|^{q'})(\cdot)\|_{l^{2/q'}}\big\|^{1/q'}_{p/q',u}.$$

Since $(u,v) \in A^*_{p/q'}$ and $u(x), v(x) \in A_{p/q'}$, by Theorem 4 we have

$$\|T^k_{\varepsilon,\Omega}f\|_{p,u} \le C\big\|\|M(|h_j|^{q'})(\cdot)\|_{l^{2/q'}}\big\|^{1/q'}_{p/q',u} \le C\big\|\|\,|h_j|^{q'}(\cdot)\|_{l^{2/q'}}\big\|^{1/q'}_{p/q',v}$$

$$= C\left\{\int_{\boldsymbol{R}^n}\left[\left(\sum_{j\in\boldsymbol{Z}}|S_{j+k}f(x)|^2\right)^{1/2}\right]^{p} v(x)\,dx\right\}^{1/p}.$$

By Lemma 1 we know $v(x) \in A_p$. Using the weighted Littlewood-Paley theory again, we get

$$\|T^k_{\varepsilon,\Omega}f\|_{p,u} \le C\left\{\int_{\boldsymbol{R}^n}\left[\left(\sum_{j\in\boldsymbol{Z}}|S_{j+k}f(x)|^2\right)^{1/2}\right]^{p} v(x)\,dx\right\}^{1/p}$$

$$\le C\left(\int_{\boldsymbol{R}^n}|f(x)|^p v(x)\,dx\right)^{1/p},$$

and (3.9) follows.

To complete the proof of Lemma 2, we still need to use Stein-Weiss's interpolation theorem with change of measures [**SW**, Theorem 2.11]. Let us discuss by dividing into the following three cases.

(i) *The case for* $p > 2$.

Since $u(x), v(x) \in A_{p/q'}$ and $(u,v) \in A^*_{p/q'}$, by (2.9) and (2.4), there is a $\sigma > 0$ such that $(u^{1+\sigma}, v^{1+\sigma}) \in A^*_{p/q'}$ and $u(x)^{1+\sigma}, v(x)^{1+\sigma} \in A_{p/q'}$ hold simultaneously time. Choose p_1 satisfying $(p_1 - p)/(p-2) = \sigma$. We then have $p_1 > p$, $(u^{1+\sigma}, v^{1+\sigma}) \in A^*_{p_1/q'}$, and $u(x)^{1+\sigma}, v(x)^{1+\sigma} \in A_{p_1/q'}$ by (2.11) and (2.1), respectively. From the proof of (3.9) we can get

$$\|T^k_{\varepsilon,\Omega}f\|_{p_1,u^{1+\sigma}} \le C_1\|f\|_{p_1,v^{1+\sigma}}, \tag{3.10}$$

where $C_1 > 0$ independent of any $k \in \boldsymbol{Z}$ and f. Now we let $t = p_1/(1+\sigma)p$. It is easy to check that $0 < t < 1$ and $1/p = (1-t)/2 + t/p_1$. Therefore, using the interpolation theorem with change of measures between (3.8) and (3.10), we obtain

$$\|T^k_{\varepsilon,\Omega} f\|_{p,u} \le C_1 2^{-\theta\gamma|k|} \|f\|_{p,v}, \tag{3.11}$$

where $C_1, \theta > 0$ and $\gamma > 0$ are independent of any $k \in \boldsymbol{Z}$ and f.

(ii) *The case for* $p < 2$.

Since $u(x), v(x) \in A_{p/q'}$ and $(u,v) \in A^*_{p/q'}$, by (2.9) and (2.4), there is an $\varepsilon > 0$ such that $(u^{1+\varepsilon}, v^{1+\varepsilon}) \in A^*_{p/q'}$ and $u(x)^{1+\varepsilon}, v(x)^{1+\varepsilon} \in A_{p/q'}$ hold simultaneously. By (2.12) and (2.3), we can choose an ℓ satisfying $q' < \ell < p$ such that $(u^{1+\varepsilon}, v^{1+\varepsilon}) \in A^*_{\ell/q'}$ and $u(x)^{1+\varepsilon}, v(x)^{1+\varepsilon} \in A_{\ell/q'}$ hold at the same time. By the choice of ε and ℓ, we may obtain σ and p_0 satisfying $0 < \sigma \le \varepsilon$ and $q' < \ell \le p_0 < p$ such that $\sigma = (p-p_0)/(2-p)$, $(u^{1+\sigma}, v^{1+\sigma}) \in A^*_{p_0/q'}$, and $u(x)^{1+\sigma}, v(x)^{1+\sigma} \in A_{p_0/q'}$.

In fact, if $\varepsilon = (p-\ell)/(2-p)$, then let $\sigma = \varepsilon$ and $p_0 = \ell$. Thus σ and p_0 are just ones we need. If $\varepsilon < (p-\ell)/(2-p)$, then take $\sigma = \varepsilon$ and $\ell < p_0 < p$ such that $\sigma = (p-p_0)/(2-p)$. Thus, we still have $(u^{1+\sigma}, v^{1+\sigma}) \in A^*_{p_0/q'}$ and $u(x)^{1+\sigma}, v(x)^{1+\sigma} \in A_{p_0/q'}$ by (2.11) and (2.1). If $\varepsilon > (p-\ell)/(2-p)$, we may take $0 < \sigma < \varepsilon$ and $p_0 = \ell$ such that $\sigma = (p-p_0)/(2-p)$. Thus, by Hölder's inequality, it is easy to see that $(u^{1+\sigma}, v^{1+\sigma}) \in A^*_{p_0/q'}$ and $u(x)^{1+\sigma}, v(x)^{1+\sigma} \in A_{p_0/q'}$.

As the proof of (3.9), we have

$$\|T^k_{\varepsilon,\Omega} f\|_{p_0,u^{1+\sigma}} \le C_2 \|f\|_{p_0,v^{1+\sigma}}, \tag{3.12}$$

where $C_2 > 0$ is independent of any $k \in \boldsymbol{Z}$ and f. Let $t = p_0/(1+\sigma)p$, then $0 < t < 1$ and $1/p = (1-t)/2 + t/p_0$. Using the interpolation theorem with change of measures between (3.8) and (3.12), we obtain

$$\|T^k_{\varepsilon,\Omega} f\|_{p,u} \le C_2 2^{-\theta\gamma'|k|} \|f\|_{p,v}, \tag{3.13}$$

where $C_2, \theta > 0$ and $\gamma' > 0$ are independent of any $k \in \boldsymbol{Z}$ and f.

(iii) *The case for* $p = 2$.

Since $u(x), v(x) \in A_{2/q'}$ and $(u,v) \in A^*_{2/q'}$, by (2.9) and (2.4) there is a $\sigma > 0$ such that $(u^{1+\sigma}, v^{1+\sigma}) \in A^*_{2/q'}$ and $u(x)^{1+\sigma}, v(x)^{1+\sigma} \in A_{2/q'}$ hold simultaneously. It follows from the process of proving (3.9), we can get

$$\|T^k_{\varepsilon,\Omega} f\|_{2,u^{1+\sigma}} \le C_3 \|f\|_{2,v^{1+\sigma}}. \tag{3.14}$$

Let $t = 1/(1+\sigma)$. Using the interpolation theorem with change of measures between (3.8) and (3.14), we obtain

$$\|T^k_{\varepsilon,\Omega} f\|_{2,u} \le C_3 2^{-\theta\gamma''|k|} \|f\|_{2,v}, \tag{3.15}$$

where $C_3, \theta > 0$ and $\gamma'' > 0$ are independent of any $k \in \boldsymbol{Z}$ and f.

We set $C = \max\{C_1, C_2, C_3\}$, $\eta = \min\{\gamma, \gamma', \gamma''\}$. Then, for $p > q'$ and $q > 2$,

$$\|T^k_{\varepsilon,\Omega} f\|_{p,u} \le C 2^{-\theta\eta|k|} \|f\|_{p,v} \tag{3.16}$$

by (3.11), (3.13) and (3.15). Thus Lemma 2 follows from (3.7) and (3.16). □

LEMMA 3. *Suppose that $\Omega(x') \in L^q(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$ and has average zero on S^{n-1}. If $q > \max\{p, 2\}$ and $(v^{1-p'}, u^{1-p'}) \in A^*_{p'/q'}$, in addition $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$, then there is a constant $C > 0$, independent of f and $\{\varepsilon_j\}$, such that $\|T_{\varepsilon,\Omega} f\|_{p,u} \le C\|f\|_{p,v}$.*

PROOF. Clearly we have $\|T_{\varepsilon,\Omega} f\|_{p,u} = \sup_g |\int_{\boldsymbol{R}^n} T_{\varepsilon,\Omega} f(x) g(x)\, dx|$, where the supremum is taken over all $g(x)$ with $\|g\|_{p',u^{1-p'}} \le 1$. On the other hand, let $(T_{\varepsilon,\Omega})^*$ be the adjoint operator of $T_{\varepsilon,\Omega}$, which means $(T_{\varepsilon,\Omega})^* = T_{\varepsilon,\Omega^*}$ with $\Omega^*(x) = \Omega(-x)$. We thus have

$$\left| \int_{\boldsymbol{R}^n} T_{\varepsilon,\Omega} f(x) g(x)\, dx \right| = \left| \int_{\boldsymbol{R}^n} f(x) (T_{\varepsilon,\Omega})^* g(x)\, dx \right| \le \|f\|_{p,v} \cdot \|(T_{\varepsilon,\Omega})^* g\|_{p',v^{1-p'}}.$$

Obviously Ω^* has also the same properties as Ω. Since $(v^{1-p'}, u^{1-p'}) \in A^*_{p'/q'}$ and $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$, Lemma 2 with the choice of g yields

$$\|T_{\varepsilon,\Omega} f\|_{p,u} \le \|f\|_{p,v} \cdot \sup_g \|(T_{\varepsilon,\Omega})^* g\|_{p',v^{1-p'}} \le C\|f\|_{p,v}. \qquad \square$$

It follows from Lemma 3 that, under the assumptions of Proposition 1, (3.6) holds and hence Proposition 1 follows. We now are going to extend the range of q to the case of $q > \max\{p, 4/3\}$.

PROPOSITION 2. *Suppose that $\Omega(x') \in L^q(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$, and $q > \max\{p, 4/3\}$ and $(v^{1-p'}, u^{1-p'}) \in A^*_{p'/q'}$, in addition $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$. Then there is a constant $C > 0$ independent of f such that $\|M_\Omega f\|_{p,u} \le C\|f\|_{p,v}$.*

Note that if we divide the region of q into $q > 2$ and $\max\{p, 4/3\} < q \le 2$, then the case $q > 2$ is covered by Proposition 1. Thus, to show Proposition 2 it suffices to consider the case $\max\{p, 4/3\} < q \le 2$. Following the proof of Proposition 1, we see that the key of proving Proposition 2 for this case is to establish the following lemma.

LEMMA 4. *Suppose that $\Omega(x') \in L^q(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$ and has average zero on S^{n-1}. For $q' < p$ and $4/3 < q \le 2$, if $(u, v) \in A^*_{p/q'}$ and $u(x), v(x) \in A_{p/q'}$, then there is a constant $C > 0$, independent of f and $\{\varepsilon_j\}$, such that $\|T_{\varepsilon,\Omega} f\|_{p,u} \le C\|f\|_{p,v}$.*

However, the proof of Lemma 4 depends heavily on the following lemma.

Lemma 5. *Under the conditions of Lemma* 4, *we have* $\|T^k_{\varepsilon,\Omega} f\|_{p,u} \le C\|f\|_{p,v}$, *where the constant* $C > 0$ *is independent of* f, k, *and* $\{\varepsilon_j\}$.

Proof. Let us first consider the case $4/3 < q < 2$. In this case we have $2 < q' < p$. Denote $\overline{K}_{\Omega,j}(x) = |K_{\Omega,j}(x)|^{2-q}$. Then Hölder's inequality implies

$$
\begin{aligned}
(3.17)\qquad |K_{\Omega,j} * g(x)|^2 &\le \left(\int_{\boldsymbol{R}^n} |K_{\Omega,j}(x-y)|^q \, dy\right) \cdot \left(\int_{\boldsymbol{R}^n} |K_{\Omega,j}(x-y)|^{2-q} |g(y)|^2 \, dy\right) \\
&\le C 2^{j(n-nq)} \overline{K}_{\Omega,j} * (|g|^2)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\qquad \overline{K}_{\Omega,j} * |h|(x) &\le \int_{2^j \le |x-y| < 2^{j+1}} \left(\frac{|\Omega(x-y)|}{|x-y|^n}\right)^{2-q} |h(y)| \, dy \\
&\le C 2^{-jn(2-q)} \int_{|x-y|<2^{j+1}} |\Omega(x-y)|^{2-q} |h(y)| \, dy \\
&\le C 2^{-j(n-nq)} M_{\Omega^{2-q}} h(x).
\end{aligned}
$$

By Lemma 1 we know that $u(x), v(x) \in A_p$ and $(u, v) \in S_p^*$. Using the weighted Littlewood-Paley theory and (3.17), we get

$$
\begin{aligned}
\|T^k_{\varepsilon,\Omega} f\|^2_{p,u} &= \left(\int_{\boldsymbol{R}^n} \left|\sum_{j\in\boldsymbol{Z}} \varepsilon_j S_{j+k}(K_{\Omega,j} * S_{j+k} f)(x)\right|^p u(x)\, dx\right)^{2/p} \\
&\le C\|\{\varepsilon_j\}\|^2_{l^\infty} \cdot \left\|\left(\sum_{j\in\boldsymbol{Z}} |S_{j+k}(K_{\Omega,j} * S_{j+k} f)(\cdot)|^2\right)^{1/2}\right\|^2_{p,u} \\
&\le C\left\|\left(\sum_{j\in\boldsymbol{Z}} |(K_{\Omega,j} * S_{j+k} f)(\cdot)|^2\right)^{1/2}\right\|^2_{p,u} \\
&\le C\left\|\left(\sum_{j\in\boldsymbol{Z}} 2^{j(n-nq)} \overline{K}_{\Omega,j} * |S_{j+k} f|^2(\cdot)\right)^{1/2}\right\|^2_{p,u} \\
&= C\left(\int_{\boldsymbol{R}^n} \left|\sum_{j\in\boldsymbol{Z}} 2^{j(n-nq)} \overline{K}_{\Omega,j} * |S_{j+k} f|^2(x)\right|^{p/2} u(x)\, dx\right)^{2/p} \\
&= C \sup_h \left|\int_{\boldsymbol{R}^n} \left(\sum_{j\in\boldsymbol{Z}} 2^{j(n-nq)} \overline{K}_{\Omega,j} * |S_{j+k} f|^2(x)\right) h(x)\, dx\right|,
\end{aligned}
$$

where the supremum is taken over all $h(x)$ with $\|h\|_{(p/2)',u^{1-(p/2)'}} \le 1$. By (3.18) we get

$$\begin{aligned}
&\int_{\boldsymbol{R}^n}\left(\sum_{j\in\boldsymbol{Z}} 2^{j(n-nq)}\bar{K}_{\Omega,j} * |S_{j+k}f|^2(x)\right)h(x)\,dx \\
&\quad = \int_{\boldsymbol{R}^n}\sum_{j\in\boldsymbol{Z}} 2^{j(n-nq)}|S_{j+k}f(x)|^2(\bar{K}_{\Omega,j} * h)(x)\,dx \\
&\quad \le C\int_{\boldsymbol{R}^n}\sum_{j\in\boldsymbol{Z}} |S_{j+k}f(x)|^2 M_{\Omega^{2-q}}h(x)\,dx \\
&\quad \le C\left(\int_{\boldsymbol{R}^n}\left(\sum_{j\in\boldsymbol{Z}}|S_{j+k}f(x)|^2\right)^{p/2} v(x)\,dx\right)^{2/p} \\
&\qquad \times\left(\int_{\boldsymbol{R}^n}[M_{\Omega^{2-q}}h(x)]^{(p/2)'}v(x)^{1-(p/2)'}\,dx\right)^{1/(p/2)'}.
\end{aligned}$$

We claim that the following weighted norm inequality holds:

$$\|M_{\Omega^{2-q}}h\|_{(p/2)',v^{1-(p/2)'}} \le C\|h\|_{(p/2)',u^{1-(p/2)'}}. \tag{3.19}$$

Since $4/3 < q < 2$, if denote $r = q/(2-q)$ (thus $r' = q'/2$), it is easy to see that $\Omega^{2-q} \in L^r(S^{n-1})$, $r > \max\{(p/2)', 2\}$, and $(p/2)/r' = p/q'$. It follows from $(u,v) \in A^*_{p/q'}$ and $u(x), v(x) \in A_{p/q'}$ that

$$([u^{1-(p/2)'}]^{1-(p/2)}, [v^{1-(p/2)'}]^{1-(p/2)}) \in A^*_{(p/2)/r'},$$

$$[u(x)^{1-(p/2)'}]^{1-(p/2)} \in A_{(p/2)/r'} \quad\text{and}\quad [v(x)^{1-(p/2)'}]^{1-(p/2)} \in A_{(p/2)/r'}.$$

Pluging $r, (p/2)'$ and weight pair $(u^{1-(p/2)'}, v^{1-(p/2)'})$ in Proposition 1, we get (3.19).

By $v(x) \in A_{p/q'} \subset A_p$ and using the weighted Littlewood-Paley theory again and (3.19), we get

$$\begin{aligned}
\|T^k_{\varepsilon,\Omega}f\|^2_{p,u} &\le C\sup_h \|h\|_{(p/2)',u^{1-(p/2)'}}\left(\int_{\boldsymbol{R}^n}\left(\sum_{j\in\boldsymbol{Z}}|S_{j+k}f(x)|^2\right)^{p/2}v(x)\,dx\right)^{2/p} \\
&\le C\left\|\left(\sum_{j\in\boldsymbol{Z}}|S_{j+k}f(\cdot)|^2\right)^{1/2}\right\|^2_{p,v} \le C\|f\|^2_{p,v}.
\end{aligned}$$

Thus we prove Lemma 5 for the case $4/3 < q < 2$.

We now consider the case of $q = 2$. In this case $2 = q' < p$, $(u, v) \in A^*_{p/2}$ and $u(x), v(x) \in A_{p/2}$. Hence by (2.12) and (2.3) we can choose a $\sigma > 0$ such that

$$
(3.20) \quad
\begin{aligned}
&\text{(i)} \quad (2-\sigma)' < p;\\
&\text{(ii)} \quad (u, v) \in A^*_{p/(2-\sigma)'} \quad \text{and} \quad u(x), v(x) \in A_{p/(2-\sigma)'};\\
&\text{(iii)} \quad 4/3 < (2-\sigma) < 2;\\
&\text{(iv)} \quad \Omega \in L^2(S^{n-1}) \subset L^{2-\sigma}(S^{n-1}).
\end{aligned}
$$

The above (3.20) shows that, for $(2-\sigma), p$ and the weights pair (u, v), we may apply the conclusion proved in the case $4/3 < q < 2$. We thus still have $\|T^k_{\varepsilon,\Omega} f\|_{p,u} \le C\|f\|_{p,v}$ for $q = 2$. This completes the proof of Lemma 5. □

As in the proof of Lemma 2, using Stein-Weiss's interpolation theorem with change measures between (3.8) and the conclusion of Lemma 5, we can prove Lemma 4. Then by Lemma 4 and using the method of proving Lemma 3, we may obtain the conclusion of Proposition 2. We omit the details here.

By a similarly inductive method, it is not difficult to see that if the conclusion of Proposition 2 holds for the $q > \max\{p, 2^{m-1}/(2^{m-1} - 1)\}$, $m \ge 2$, then it also holds for $q > \max\{p, 2^m/(2^m - 1)\}$. More precisely, we have the following general conclusion.

Proposition 3. *Suppose that $\Omega(x') \in L^q(S^{n-1})$ is homogeneous of degree zero on $\boldsymbol{R}^n$, and $q > \max\{p, 2^m/(2^m - 1)\}$, $m \in \boldsymbol{N}$, $m \ge 2$. Moreover, $(v^{1-p'}, u^{1-p'}) \in A^*_{p'/q'}$ and $v(x)^{1-p'}, u(x)^{1-p'} \in A_{p'/q'}$. Then there is a constant $C > 0$, independent of f, such that $\|M_\Omega f\|_{p,u} \le C\|f\|_{p,v}$.*

Finally, let us finish the proof of Theorem 1 for the condition (b). If $p \ge 2$, then Theorem 1 follows from Proposition 1. If $p < 2$, then there exists $m \in \boldsymbol{N}$, $m \ge 2$, such that $2^m/(2^m - 1) \le p < 2^{m-1}/(2^{m-1} - 1)$. Thus $q > p$ is equivalent to $q > \max\{p, 2^m/(2^m - 1)\}$. In this case Proposition 3 is applied to get the conclusion of Theorem 1 for the condition (b) when $q > p$.

§4. Outline of proof for Theorems 2 and 3.

The outline of proving Theorem 2. In fact the process of proving Theorem 1 for the condition (b) implies the proof of Theorem 2. Let us first consider Theorem 2 for the case (a). Using the notations and decomposition introduced in the proof of Theorem 1 for the condition (b), we have the following equality.

$$(4.1) \qquad T_\Omega f(x) = \sum_{j\in\boldsymbol{Z}} K_{\Omega,j} * \left(\sum_{k\in\boldsymbol{Z}} (S_{j+k}^2 f)(x)\right)$$
$$= \sum_{k\in\boldsymbol{Z}}\sum_{j\in\boldsymbol{Z}} S_{j+k}(K_{\Omega,j} * S_{j+k} f)(x) := \sum_{k\in\boldsymbol{Z}} T_\Omega^k f(x).$$

By Plancherel's theorem, it is easy to see that the conclusion of (3.8) still holds if we replace $T_{\varepsilon,\Omega}^k$ by T_Ω^k. That is, there are $C, \theta > 0$, independent of $k \in \boldsymbol{Z}$ and f, such that

$$(4.2) \qquad \|T_\Omega^k f\|_2 \le C 2^{-\theta|k|}\|f\|_2.$$

If we can prove that, under the condition (a) of Theorem 2, there is a $C > 0$ such that for any $k \in \boldsymbol{Z}$ and f,

$$(4.3) \qquad \|T_\Omega^k f\|_{p,u} \le C\|f\|_{p,v},$$

then, by using Stein-Weiss's interpolation theorem with change of measures between (4.2) and (4.3), we may obtain the conclusion of Theorem 2 for the case (a) by (4.1). However, from the process of proving Lemma 2, we know that if $q > 2$, then (4.3) holds. On the other hand, when $q < 2$ (thus $p > q' > 2$), using the notations and method in Lemma 5, we have

$$(4.4) \qquad \|T_\Omega^k f\|_{p,u}^2 \le C \sup_h \left| \int_{\boldsymbol{R}^n} \left(\sum_{j\in\boldsymbol{Z}} 2^{j(n-nq)} \bar{K}_{\Omega,j} * |S_{j+k} f|^2(x)\right) h(x)\,dx \right|$$
$$\le C\left(\int_{\boldsymbol{R}^n}\left(\sum_{j\in\boldsymbol{Z}} |S_{j+k} f(x)|^2\right)^{p/2} v(x)\,dx\right)^{2/p}$$
$$\times \sup_h \left(\int_{\boldsymbol{R}^n} [M_{\Omega^{2-q}} h(x)]^{(p/2)'} v(x)^{1-(p/2)'}\,dx\right)^{1/(p/2)'},$$

where the supremum is taken over all $h(x)$ with $\|h\|_{(p/2)',u^{1-(p/2)'}} \le 1$.

Since $q < 2$, we have $r = q/(2-q) > (p/2)'$, $(p/2)/r' = p/q'$, and $\Omega^{2-q} \in L^r(S^{n-1})$. As done in the proof of Lemma 5, it is easy to check that under the condition (a) of Theorem 2, $r, (p/2)'$ and weight pair $(u^{1-(p/2)'}, v^{1-(p/2)'})$ satisfy the condition (b) of Theorem 1. Hence (3.19) holds. Thus, for $q < 2$, (4.3) follows from (4.4), (3.19), and the weighted Littlewood-Paley theory [**K**, Theorem 2.1].

The treatment for the case $q = 2$ is the same as one in the proof of Lemma 5. We omit the details here. We thus prove Theorem 2 for the case (a).

By applying the duality property of T_Ω and the method of proving Lemma 3 (to remove the restriction $q > 2$), the conclusion of Theorem 2 under the condition (b) easily follows from the conclusion of Theorem 2 for the case (a).

The outline of proving Theorem 3. We follow the idea of proving Theorem E in [**DR**]. For any $\varepsilon > 0$, there is $k \in \boldsymbol{Z}$ such that $2^k \le \varepsilon < 2^{k+1}$. It is easy to see that

$$|T_\Omega^\varepsilon f(x)| \le CM_\Omega f(x) + |T_\Omega^{2^k} f(x)|.$$

Now we choose a Schwartz function satisfying $0 \le \phi(x) \le 1$, $\operatorname{supp}(\phi) \subset \{x \in \boldsymbol{R}^n : |x| < 2\}$, and $\phi(x) = 1$ when $|x| < 1$. We also write $\phi_k(x) = \phi(2^k x)$ and denote $\hat{\Phi}_k(\xi) = \phi_k(\xi)$. By [**DR**, p. 548] we know that

$$|T_\Omega^{2^k} f(x)| \le CM(T_\Omega f) + CMf + \left|(\delta - \Phi_k) * \sum_{j=k}^{\infty} T_j f\right|,$$

where $T_j f(x) = K_{\Omega,j} * f(x)$ is defined in §3 and δ is the Dirac function. Thus,

(4.5)
$$T_\Omega^* f(x) \le C(M_\Omega f(x) + M(T_\Omega f)(x) + Mf(x)) + \sup_{k \in \boldsymbol{Z}} \left|(\delta - \Phi_k) * \sum_{j=k}^{\infty} K_{\Omega,j} * f(x)\right|.$$

After we use Lemma 1 and apply the weighted boundedness with weight pair (u, v) for M_Ω (Theorem 1), M (Theorem A), and T_Ω (Theorem 2) to (4.5), the proof of Theorem 3 is reduced to verify

$$\left\| \sup_{k \in \boldsymbol{Z}} \left|(\delta - \Phi_k) * \sum_{j=k}^{\infty} K_{\Omega,j} * f\right| \right\|_{p,u} \le C\|f\|_{p,v}. \tag{4.6}$$

Obviously

$$\sup_{k \in \boldsymbol{Z}} \left|(\delta - \Phi_k) * \sum_{j=k}^{\infty} K_{\Omega,j} * f(x)\right| \le \sum_{j=0}^{\infty} \sup_{k \in \boldsymbol{Z}} |(\delta - \Phi_k) * K_{\Omega,j+k} * f(x)|. \tag{4.7}$$

By [**DR**, pp. 551–552] there are $C, \theta > 0$ such that, for any $j, k \in \boldsymbol{Z}$,

$$|\hat{K}_{\Omega,j+k}(\xi)| \le C \min\{|2^{j+k}\xi|^\theta, |2^{j+k}\xi|^{-\theta}\}.$$

Applying Plancherel's theorem to the j-th term of the summation, we may get an L^2-norm of the order $2^{-\alpha j}$ with $\alpha > 0$. On the other hand we have

$$\sup_{k \in \boldsymbol{Z}} |(\delta - \Phi_k) * K_{\Omega,j+k} * f(x)| \le C(M_\Omega f(x) + M(M_\Omega f)(x)).$$

By Theorem 1, Lemma 1, and the weighted L^p boundedness with one weigh function in A_p, we know that under the conditions of Theorem 3

$$\left\|\sup_{k\in\boldsymbol{Z}}(\delta-\Phi_k)*K_{\Omega,j+k}*f\right\|_{p,u}\le C(\|M_\Omega f\|_{p,u}+\|M(M_\Omega f)\|_{p,u})\le C\|f\|_{p,v}.$$

Using Stein-Weiss's interpolation theorem with change of measures, we obtain (4.6) from (4.7) and hence finish the proof of Theorem 3.

§5. Proof of Theorem 4.

We shall follow the basic idea in [**AJ**]. In the proof of Theorem 4, we need to use two weighted vector-valued interpolation theorems, which are analogues of the Marcinkiewicz interpolation theorem.

Let S denote the linear space of sequences $\vec{f}=\{f_k\}_1^\infty$ of the form: $f_k(x)$ is a simple function on $\boldsymbol{R}^n$ and $f_k(x)\equiv 0$ for all sufficiently large k. Then S is dense in $L_v^p(l^r)$, $1\le p,r<\infty$.

Lemma 6. *Let $u(x),v(x)\ge 0$ be locally integrable on $\boldsymbol{R}^n$ and $1<r<\infty$, $1\le p_i<\infty$ $(i=0,1)$. Suppose that T is a sublinear operator defined on S satisfying*

$$u(\{x\in\boldsymbol{R}^n:\|T(\vec{f})(x)\|_{l^r}>\lambda\})\le C_i\lambda^{-p_i}\int_{\boldsymbol{R}^n}\|\vec{f}(x)\|_{l^r}^{p_i}v(x)\,dx\quad \textit{for } i=0,1 \textit{ and } \vec{f}\in S,$$

where and below, $u(A)=\int_A u(x)\,dx$ for a set A. Then T can be extended to a bounded operator from $L_v^p(l^r)(\boldsymbol{R}^n)$ to $L_u^p(l^r)(\boldsymbol{R}^n)$; that is,

$$\left(\int_{\boldsymbol{R}^n}\|T(\vec{f})(x)\|_{l^r}^p u(x)\,dx\right)^{1/p}\le C_\theta\left(\int_{\boldsymbol{R}^n}\|\vec{f}(x)\|_{l^r}^p v(x)\,dx\right)^{1/p},$$

where $1/p=(1-\theta)/p_0+\theta/p_1$ and $0<\theta<1$.

Lemma 7. *Let $u(x),v(x)\ge 0$ be locally integrable on $\boldsymbol{R}^n$ and $1<r_i,p_i<\infty$ $(i=0,1)$. Suppose that the sublinear operator T satisfies*

$$\left(\int_{\boldsymbol{R}^n}\|T(\vec{f})(x)\|_{l^{r_i}}^{p_i} u(x)\,dx\right)^{1/p_i}\le C_i\left(\int_{\boldsymbol{R}^n}\|\vec{f}(x)\|_{l^{r_i}}^{p_i} v(x)\,dx\right)^{1/p_i}$$

$$\textit{for } i=0,1 \textit{ and } \vec{f}\in S.$$

Then T can be extended to a bounded operator from $L_v^p(l^r)(\boldsymbol{R}^n)$ to $L_u^p(l^r)(\boldsymbol{R}^n)$; that is,

$$\left(\int_{\boldsymbol{R}^n}\|T(\vec{f})(x)\|_{l^r}^p u(x)\,dx\right)^{1/p}\le C_0^{1-\theta}C_1^\theta\left(\int_{\boldsymbol{R}^n}\|\vec{f}(x)\|_{l^r}^p v(x)\,dx\right)^{1/p},$$

where $(1/p,1/r)=(1-\theta)(1/p_0,1/r_0)+\theta(1/p_1,1/r_1)$ and $0\le\theta\le 1$.

Using the same methods as in [**BCP**], [**BP**] and [**CZ**], we may obtain Lemma 6 and Lemma 7 (see also [**AJ**, p. 21]). Now let us turn to the proof of Theorem 4. We divide the proof of Theorem 4 into three steps.

Case 1: $p = r$. Note that under the conditions of Theorem 4, we have $(u, v) \in S_p^*$ by (2.10). Applying Theorem A we get

$$(5.1) \qquad \left(\int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|_{l^r}^r u(x)\,dx\right)^{1/r} \le C_r \left(\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^r v(x)\,dx\right)^{1/r}.$$

Case 2: $1 < p < r$. In this case we first prove that, under the conditions of Theorem 4, there is a $C > 0$ such that, for any $\lambda > 0$,

$$(5.2) \qquad u(\{x \in \boldsymbol{R}^n : \|M(\vec{f})(x)\|_{l^r} > \lambda\}) \le C\lambda^{-p} \int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx.$$

Using the Calderón-Zygmund decomposition (see [**St**, p. 17, Theorem 4]), for $\|\vec{f}(x)\|_{l^r}$ and $\lambda > 0$, we get a sequence of non-overlapping cubes $\{Q_j\}$ such that

$$(5.3) \qquad \|\vec{f}(x)\|_{l^r} \le \lambda, \quad \text{for almost everywhere } x \notin E = \bigcup_{j=1}^{\infty} Q_j,$$

$$(5.4) \qquad \lambda < \frac{1}{|Q_j|}\int_{Q_j} \|\vec{f}(x)\|_{l^r}\,dx \le 2^n\lambda, \quad j = 1, 2, 3, \ldots.$$

Since $(u, v) \in A_p^*$, by (5.4) and Hölder's inequality, we have

$$\begin{aligned} u(Q_j) &= \int_{Q_j} u(x)\,dx \le \lambda^{-p}\left(\frac{1}{|Q_j|}\int_{Q_j} \|\vec{f}(x)\|_{l^r}\,dx\right)^p \int_{Q_j} u(x)\,dx \\ &\le \lambda^{-p}\left(\int_{Q_j} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx\right)\left(\frac{1}{|Q_j|}\int_{Q_j} v(x)^{-1/(p-1)}\,dx\right)^{p-1}\left(\frac{1}{|Q_j|}\int_{Q_j} u(x)\,dx\right) \\ &\le C\lambda^{-p}\int_{Q_j} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx. \end{aligned}$$

The cubes Q_j are non-overlapping, so we get

$$(5.5) \qquad u(E) = \sum_{j=1}^{\infty} u(Q_j) \le C\lambda^{-p}\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx.$$

Denote $\vec{f}(x) = \vec{f}'(x) + \vec{f}''(x)$, where $\vec{f}'(x) = \{f_k'(x)\}_{k=1}^{\infty}$ and $f_k'(x) = f_k(x)\chi_{\{\boldsymbol{R}^n \setminus E\}}(x)$. By Minkowski's inequality we have

$$(5.6) \qquad \|M(\vec{f})(x)\|_{l^r} \le \|M(\vec{f}')(x)\|_{l^r} + \|M(\vec{f}'')(x)\|_{l^r}.$$

Since $(u,v) \in A_p^*$ and $u(x), v(x) \in A_p$, by (2.1) and (2.11) we have $(u,v) \in A_r^*$ and $u(x), v(x) \in A_r$. Thus (5.1) holds if we replace $\vec{f}$ by $\vec{f}'$, and Chebyshev's inequality yields

$$u(\{x \in \boldsymbol{R}^n : \|M(\vec{f}')(x)\|_{l^r} > \lambda\}) \leq C\lambda^{-r} \int_{\boldsymbol{R}^n} \|\vec{f}'(x)\|_{l^r}^r v(x)\,dx. \tag{5.7}$$

By (5.3) we have $\|\vec{f}'(x)\|_{l^r}^r \leq \lambda^{r-p}\|\vec{f}'(x)\|_{l^r}^p$, which combined with (5.7) implies

$$u(\{x \in \boldsymbol{R}^n : \|M(\vec{f}')(x)\|_{l^r} > \lambda\}) \leq C\lambda^{-p} \int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx. \tag{5.8}$$

Hence, by (5.6) and (5.8), the proof of (5.2) is reduced to the verification of the following inequality.

$$u(\{x \in \boldsymbol{R}^n : \|M(\vec{f}'')(x)\|_{l^r} > \lambda\}) \leq C\lambda^{-p} \int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx. \tag{5.9}$$

To prove (5.9), as done in [**FS**, p. 109] we let $\vec{\tilde{f}} = \{\tilde{f}_k\}_1^\infty$, where

$$\tilde{f}_k(x) = \begin{cases} (1/|Q_j|)\int_{Q_j} |f_k(y)|\,dy, & \text{for } x \in Q_j, j = 1,2,3,\dots, \\ 0, & \text{otherwise.} \end{cases}$$

Since $u(x) \in A_p$, there is a $C > 0$ such that, for any $Q \subset \boldsymbol{R}^n$, $u(2Q) \leq Cu(Q)$ (see [**GR**, p. 396, Lemma 2.2]). If denote $\tilde{Q}_j = 2nQ_j$ and $\tilde{E} = \bigcup_j \tilde{Q}_j$, then by (5.5) we get

$$u(\tilde{E}) \leq \sum_j u(\tilde{Q}_j) \leq C\sum_j u(Q_j) \leq C\lambda^{-p} \int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx. \tag{5.10}$$

On the other hand, for $x \in Q_j$, $j = 1,2,\dots$, Minkowski's inequality and (5.4) yield

$$\|\vec{\tilde{f}}(x)\|_{l^r} = \left(\sum_k \left(\frac{1}{|Q_j|}\int_{Q_j} |f_k(y)|\,dy\right)^r\right)^{1/r} \leq \frac{1}{|Q_j|}\int_{Q_j} \|\vec{f}(y)\|_{l^r}\,dy \leq 2^n\lambda. \tag{5.11}$$

Obviously $\|\vec{\tilde{f}}(x)\|_{l^r} = 0$ for $x \notin E$. Using (5.1) for the case $u(x) = v(x) \in A_p \subset A_r$, we have

$$\begin{aligned} u(\{x \in \boldsymbol{R}^n &: \|M(\vec{\tilde{f}})(x)\|_{l^r} > \lambda\}) \\ &\leq \lambda^{-r} \int_{\boldsymbol{R}^n} \|M(\vec{\tilde{f}})(x)\|_{l^r}^r u(x)\,dx \\ &\leq C\lambda^{-r} \int_E \|\vec{\tilde{f}}(x)\|_{l^r}^r u(x)\,dx \\ &\leq Cu(E) \leq C\lambda^{-p} \int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^r}^p v(x)\,dx. \end{aligned} \tag{5.12}$$

By [**FS**, p. 110] we know that $M(f_k'')(x) \le CM(\tilde{f}_k)(x)$ for $x \notin \tilde{E}$, where C is independent of f_k. We thus get (5.9) from (5.10) and (5.12), and hence prove (5.2).

Now let us complete the proof of Theorem 4 for *case* 2. Since $1 < p < r$, $(u, v) \in A_p^*$ and $u(x), v(x) \in A_p$, by (2.3) and (2.12) we may choose an $\varepsilon > 0$ satisfying $1 < p - \varepsilon < p$ and

$$(u, v) \in A_{p-\varepsilon}^* \quad \text{and} \quad u(x), v(x) \in A_{p-\varepsilon}. \tag{5.13}$$

We write $p_0 = p - \varepsilon$ and take p_1 such that $p < p_1 < r$. Then by (2.1) and (2.11) we have

$$(u, v) \in A_{p_1}^* \quad \text{and} \quad u(x), v(x) \in A_{p_1}. \tag{5.14}$$

By (5.13) and (5.14) we know that (5.2) still holds if we replaced p by p_0 and p_1, respectively. Hence we apply Lemma 6 to get the conclusion of Theorem 4 for *case* 2.

Case 3: $1 < r < p < \infty$. Before giving the proof of Theorem 4, let us recall the following well known Fefferman-Stein's result.

LEMMA 8 ([**FS**, Lemma 1]). *Let $g(x)$ and $\phi(x)$ be two real-valued functions on $\boldsymbol{R}^n$. If $1 < q < \infty$, then there is a constant $C_{r,p}$, independent of g and ϕ, such that*

$$\int_{\boldsymbol{R}^n} (Mg(x))^q |\phi(x)|\, dx \le C_{r,p} \int_{\boldsymbol{R}^n} |g(x)|^q M\phi(x)\, dx.$$

Now let us return to the proof of Theorem 4 for *case* 3. Since $(u, v) \in A_p^*$ and $u(x), v(x) \in A_p$, by (2.12) and (2.3) we may take a r_0 arbitrarily close to 1 satisfying $1 < r_0 < r < p/r_0 < p$ such that

$$(u, v) \in A_{p/r_0}^* \quad \text{and} \quad u(x), v(x) \in A_{p/r_0}. \tag{5.15}$$

Denote $p_0 = p/r_0$. By (5.15), (2.10), and Theorem A, we get

$$\left(\int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|_{l^{p_0}}^{p_0} u(x)\, dx \right)^{1/p_0} \le C \left(\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^{p_0}}^{p_0} v(x)\, dx \right)^{1/p_0}. \tag{5.16}$$

On the other hand, by $r_0 < r < p_0$, there is a θ satisfying $0 < \theta < 1$ such that $1/r = (1-\theta)/p_0 + \theta/r_0$. For this θ, we may take a p_1 such that $1/p = (1-\theta)/p_0 + \theta/p_1$. Since $r_0 > 1$ closed enough to 1, we have $p_1 > p$. It follows from (5.15), (2.11), and (2.1) that $(u, v) \in A_{p_1/r_0}^*$ and $u(x), v(x) \in A_{p_1/r_0}$.

Set $q = p_1/r_0$. Then

$$\left(\int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|_{l^{r_0}}^{p_1} u(x)\, dx \right)^{1/p_1} = \left(\sup_{\phi} \left| \int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|_{l^{r_0}}^{r_0} \phi(x) u(x)\, dx \right| \right)^{q/p_1}, \tag{5.17}$$

where the supremum is taken over all $\phi(x)$ with $\|\phi\|_{L_u^{q'}} \le 1$. Since $(u,v) \in A_q^*$ and $u(x), v(x) \in A_q$, by (2.2) and (2.6) we have

$$(v^{1-q'}, u^{1-q'}) \in A_{q'}^* \quad \text{and} \quad u(x)^{1-q'}, v(x)^{1-q'} \in A_{q'}. \tag{5.18}$$

Thus, by (5.18) and (2.10) we have $(v^{1-q'}, u^{1-q'}) \in S_{q'}^*$. Using Theorem A and noting the choice of ϕ, we get

$$\left(\int_{\boldsymbol{R}^n} [M(\phi u)(x)]^{q'} v(x)^{1-q'}\,dx\right)^{1/q'} \le C\left(\int_{\boldsymbol{R}^n} |\phi(x)u(x)|^{q'} u(x)^{1-q'}\,dx\right)^{1/q'} \le C. \tag{5.19}$$

Applying Lemma 8 and (5.19), we have

$$\begin{aligned}
&\left|\int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|_{l^{r_0}}^{r_0} \phi(x)u(x)\,dx\right| \\
&\quad \le C\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^{r_0}}^{r_0} [M(\phi u)(x)/v(x)]v(x)\,dx \\
&\quad \le C\left(\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^{r_0}}^{p_1} v(x)\,dx\right)^{1/q} \left(\int_{\boldsymbol{R}^n} [M(\phi u)(x)/v(x)]^{q'} v(x)\,dx\right)^{1/q'} \\
&\quad \le C\left(\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^{r_0}}^{p_1} v(x)\,dx\right)^{1/q}.
\end{aligned} \tag{5.20}$$

By (5.17) and (5.20), we get

$$\left(\int_{\boldsymbol{R}^n} \|M(\vec{f})(x)\|_{l^{r_0}}^{p_1} u(x)\,dx\right)^{1/p_1} \le C\left(\int_{\boldsymbol{R}^n} \|\vec{f}(x)\|_{l^{r_0}}^{p_1} v(x)\,dx\right)^{1/p_1}. \tag{5.21}$$

If we apply Lemma 7 to (5.16) and (5.21), then the conclusion of Theorem 4 for *case* 3 follows.

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References

[AJ] K. F. Andersen and R. T. John, Weighted inequalities for vector-valued maximal functions and singular integrals, Studia Math., **69** (1980), 19–31.

[BCP] A. Benedek, A. Calderón and R. Panzones, Convolution operators on Banach space valued functions, Proc. Natl. Acad. Sci. USA, **48** (1962), 356–365.

[BP] A. Benedek and R. Panzone, The spaces L^p with mixed norm, Duke Math. J., **28** (1961), 301–321.

[CZ] A. Calderón and A. Zygmund, A note on the interpolation of Sublinear operators, Amer. J. Math., **78** (1956), 282–288.

[D] J. Duoandikoetxea, Weighted norm inequalities for homogeneous singular integrals, Trans. Amer. Math. Soc., **336** (1993), 869–880.

[DR] J. Duoandikoetxea and J. L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math., **84** (1986), 541–561.

[FS] C. Fefferman and E. M. Stein, Some maximal inequalities, Amer. J. Math., **93** (1971), 107–115.

[GR] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North-Holland, Amsterdam, 1985.

[K] D. Kurtz, Littlewood-Paley and multiplies theorems on weighted L^p spaces, Trans. Amer. Math. Soc., **259** (1980), 235–254.

[M] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc., **165** (1972), 207–226.

[N] C. Neugebauer, Inserting A_p-weights, Proc. Amer. Math. Soc., **87** (1983), 644–648.

[Sa] E. Sawyer, A characterization of a two-weight norm inequality for maximal operators, Studia Math., **75** (1982), 1–11.

[St] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, N.J., 1970.

[SW] E. M. Stein and G. Weiss, Interpolation of operators with change of measures, Trans. Amer. Math. Soc., **87** (1958), 159–172.

[W] D. Watson, Weighted estimates for singular integrals via Fourier transform estimates, Duke Math. J., **60** (1990), 389–399.

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