J. Math. Soc. Japan
Vol. 55, No. 1, 2003

# Models of rationally connected manifolds

By Paltin Ionescu and Cristian Voica

(Received Jan. 5, 2001)
(Revised Jul. 23, 2001)

**Abstract.** We study rationally connected (projective) manifolds $X$ via the concept of a *model* $(X, Y)$, where $Y$ is a smooth rational curve on $X$ having ample normal bundle. Models are regarded from the view point of Zariski equivalence, birational on $X$ and biregular around $Y$. Several numerical invariants of these objects are introduced and a notion of minimality is proposed for them. The important special case of models Zariski equivalent to $(\boldsymbol{P}^n, \text{line})$ is investigated more thoroughly. When the (ample) normal bundle of $Y$ in $X$ has minimal degree, new such models are constructed via special vector bundles on $X$. Moreover, the formal geometry of the embedding of $Y$ in $X$ is analysed for some non-trivial examples.

## 0. Introduction.

A complex projective manifold is *rationally connected* if there is a rational curve passing through two general points of it. Rationally connected manifolds were introduced by Kollár, Miyaoka and Mori in [**18**] and turned out to be a very useful generalization to higher dimensions of the classical notion of rational surface. For instance, rational connectedness is both birationally invariant and deformation invariant; moreover, unirational manifolds and Fano manifolds are rationally connected.

The main purpose of this paper is to propose a point of view in the study of rationally connected manifolds by introducing the concept of a *model*, defined to be a pair $(X, Y)$, where $X$ is a projective manifold and $Y$ is a smooth rational curve with ample normal bundle in $X$. The existence of such a curve $Y$ on $X$ is actually equivalent to the fact that $X$ is rationally connected in the sense of [**18**]. There is an obvious notion of isomorphism of models. More importantly, we want to study them from the point of view of *Zariski equivalence*: two models $(X, Y)$ and $(X', Y')$ are said to be Zariski equivalent if there are open subsets $U \subset X$, $Y \subset U$ and $U' \subset X'$, $Y' \subset U'$, and an isomorphism $\varphi : U \to U'$ with $\varphi(Y) = Y'$. The advantage of this notion is that it provides a convenient link between the birational and the biregular point of view (e.g. see Remark 1.5). We

2000 *Mathematics Subject Classification*. Primary 14M99, 14E25; Secondary 14M20, 14J45, 14B20.

*Key Words and Phrases*. Rationally connected, model, quasi-line, formal geometry.

also propose (see Definition 1.15) a notion of *minimal Zariski model*, which is a normal (in general singular) model $(X, Y)$ such that $Y$ meets every effective divisor of $X$. Proposition 1.21 shows that this definition generalizes the usual (relatively) minimal models of rational surfaces. However, at present, neither existence, nor uniqueness of a minimal Zariski model in a Zariski equivalence class of models, is proved.

In the first section we examine systematically natural local and global numerical invariants of models. Several nontrivial examples show that these invariants are in general independent. We pay special attention to models $(X, Y)$ for which there is some divisor $D$ on $X$ with $(D \cdot Y) = 1$ and $s =: \dim|D| \geq 1$. This particular class of models is studied in Theorem 1.12; the result states that, modulo a deformation of $Y$, we may find a Zariski equivalent model $(X', Y')$ admitting a surjective morphism with connected fibres $\varphi : X' \to \boldsymbol{P}^s$ and mapping $Y'$ to a line. Moreover, $\varphi$ is smooth along $Y'$ and smooth fibres of $\varphi$ are rationally connected. In case $s = \dim(X)$ we get that $(X', Y')$ is Zariski equivalent to the simplest possible model, $(\boldsymbol{P}^s, \text{line})$; in particular, it follows that $X$ is rational. This improves Theorem 4.4 from [**1**].

In section two we propose a conjecture (see 2.3) in which models $(X, Y)$ Zariski equivalent to $(\boldsymbol{P}^n, \text{line})$ are characterized by the presence of a linear system $|D|$ on $X$ having the highest possible dimension with respect to $(D \cdot Y)$ (see also Proposition 2.1 for the precise statement). In Proposition 2.10 we show (using Mori's solution of Hartshorne's conjecture, see [**23**]) that our conjecture follows from the existence of a smooth minimal Zariski model equivalent to the given one.

In the third section we study polarizations of models and their deformations. The main result here is Theorem 3.2 which shows that, modulo deformations, there are only finitely many isomorphism classes of polarized models $(X, Y, H)$ having fixed dimension and "degree" $=: (Y \cdot H)$ (here $H$ is an ample divisor on $X$). The proof uses Matsusaka's theorem ([**22**]) together with its refinement in [**17**] and several facts from Mori theory.

In the last section we investigate models $(X, Y)$ where $Y$ is a *quasi-line* in the sense of [**1**], i.e. the (ample) normal bundle of $Y$ in $X$ has minimal degree, $N_{Y|X} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$. We first give a useful construction which produces new such models starting with a given one and with a suitable vector bundle on $X$ (see Proposition 4.2). Next we answer in the negative Questions 5.2, 5.3 and 5.7 from [**1**]. First we show by an example that a smooth minimal Zariski model $(X, Y)$, where $Y$ is a quasi-line, need not be Fano (see Proposition 4.6). Then we construct a model $(X, Y)$, where $X \simeq \boldsymbol{P}(T_{\boldsymbol{P}^2})$, such that the formal completions $\hat{X}_{|Y}$ and $\hat{\boldsymbol{P}}^3_{|\text{line}}$ are not isomorphic, although the local numerical invariants of $(X, Y)$ and $(\boldsymbol{P}^3, \text{line})$ are the same (see Corollary 4.9 and Proposition 4.10). Here we use standard local cohomology techniques (see [**8**]) and results of Hartshorne ([**9**] and [**10**]) relating cohomological dimension and the properties G2, G3 introduced by Hironaka and Matsumura ([**12**]).

In writing down this paper we were largely motivated by [**1**]. For instance, all results in the fourth section of [**1**] are generalized to the case of models; we also quoted several nontrivial examples from [**1**]. Moreover, as already explained, we answered (in the negative) Questions 5.2, 5.3 and 5.7 from [**1**].

We work over $\boldsymbol{C}$ and we follow the standard notation in algebraic geometry (see e.g. [**11**]).

## 1. Models and their Zariski equivalence.

Let $X$ be a complex projective manifold of dimension $n \geq 2$.

Definition 1.1. (cf. [**18**]) $X$ is said to be *rationally connected* if it contains some smooth rational curve having ample normal bundle.

This definition, which is better suited for our purposes, is apparently stronger than the original one in [**18**], but it turns out to be equivalent to it. Indeed, if $n \geq 3$ and $X$ is rationally connected in the sense of [**18**], Theorem 3.9 from [**16**] Chapter IV yields the existence of a smooth rational curve with ample normal bundle on $X$. In case $n = 2$, being rationally connected in the sense of [**18**] is equivalent to being rational, and every rational surface contains a smooth rational curve of positive self-intersection.

The main properties of rationally connected manifolds are summarized in the following theorem (see [**18**], [**19**], [**2**] or [**16**], Chapter IV).

Theorem 1.2. (Kollár-Miyaoka-Mori, Campana)

(i) *If $Y$ is a smooth rational curve with ample normal bundle on $X$, then the deformations of $Y$ with a fixed point fill up an open subset of $X$*;

(ii) *unirational manifolds and Fano manifolds are rationally connected*;

(iii) *being rationally connected is a birationally invariant property*;

(iv) *smooth deformations of rationally connected manifolds are rationally connected*;

(v) *rationally connected manifolds are simply connected and they satisfy*: $H^0(X, \Omega_X^{\otimes m}) = 0$ *for* $m > 0$ *and* $H^i(X, \mathcal{O}_X) = 0$ *for* $i > 0$.

Definition 1.3. By a *model* (of rationally connected manifolds) we mean in what follows a pair $(X, Y)$, where $X$ is a projective manifold and $Y \subset X$ is a smooth rational curve with ample normal bundle in $X$.

Note that the simplest example is $(\boldsymbol{P}^n, \text{line})$.

Two models $(X, Y)$ and $(X', Y')$ are isomorphic if there is an isomorphism $\varphi : X \to X'$ such that $\varphi(Y) = Y'$. The basic definition for us is the following.

Definition 1.4. Two models $(X, Y)$ and $(X', Y')$ are *Zariski equivalent*, denoted $(X, Y) \sim_Z (X', Y')$, if there are open subsets $U \subset X$, $Y \subset U$ and $U' \subset X'$, $Y' \subset U'$, and an isomorphism $\varphi : U \to U'$ such that $\varphi(Y) = Y'$.

Note that $X$ and $X'$ are in this case birationally equivalent.

Remark 1.5. We stress at this point that we are not merely studying rationally connected manifolds, the fixed curve $Y$ being part of the structure. On the other hand, given a rationally connected manifold $X$, we can use models $(X, Y)$, for suitably chosen curves $Y$, in order to understand the birational geometry of $X$. We illustrate this by the following example.

Example 1.6. (see [**14**]) Let $X$ be a smooth quartic threefold in $\boldsymbol{P}^4$. Being Fano, $X$ is rationally connected by Theorem 1.2 (ii). If one could prove that any birational automorphism $\varphi$ of $X$ induces a Zariski equivalence between two models $(X, Y)$ and $(X, Y')$ (for some well-chosen curves $Y$ and $Y'$), it would follow easily (see e.g. Corollary 1.19 below) that $\varphi$ has to be a biregular automorphism of $X$. Actually, it was proved in [**14**], using delicate techniques, that indeed, every birational automorphism of $X$ is a biregular one. As the group of biregular automorphisms of $X$ is finite (see [**21**]) it follows that for any choice of the curve $Y$, if $Y'$ is a general deformation of $Y$, the models $(X, Y)$ and $(X, Y')$ are not Zariski-equivalent.

We first consider *local* properties of a given model $(X, Y)$, which depend only on a neighbourhood of $Y$ in $X$ and thus are invariant with respect to Zariski equivalence. Fix such a model $(X, Y)$. By Grothendieck's theorem (see [**7**]), $N_{Y|X} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}_{\boldsymbol{P}^1}(a_i)$, $a_i > 0$ for every $i$. A first set of local numerical invariants of $(X, Y)$ is:

$$\underline{a}(X, Y) =: (a_1, \dots, a_{n-1}).$$

We let $a(X, Y) = \sum_{i=1}^{n-1} a_i = \deg N_{Y|X}$, so we have by adjunction formula: $-(K \cdot Y) = a(X, Y) + 2$, where $K$ is the canonical divisor of $X$. Recall from [**1**] that $Y$ is called a *quasi-line if* $\underline{a}(X, Y) = (1, \dots, 1)$ or, equivalently, $a(X, Y) = n - 1$.

Consider now $\hat{X}_{|Y}$, the formal completion of $X$ along $Y$, and $K(\hat{X}_{|Y})$ the ring of formal-rational functions of $X$ along $Y$. Recall the following definitions, which make sense for any subvariety $Y \subset X$ (see [**12**]).

Definition 1.7. (i) $Y$ is G2 in $X$ if $K(\hat{X}_{|Y})$ is a field and the extension of fields $K(X) \hookrightarrow K(\hat{X}_{|Y})$ is finite (here $K(X)$ is the field of rational functions on $X$). Denote by $b(X, Y)$ the degree of this extension.

(ii) $Y$ is G3 in $X$ if $b(X, Y) = 1$.

For any model $(X, Y)$, since $N_{Y|X}$ is ample, a basic result due to Hartshorne ([**9**]) says that $Y$ is G2 in $X$. Note that, obviously, if $(X, Y) \sim_Z (X', Y')$ then the formal completions $\hat{X}_{|Y}$ and $\hat{X}'_{|Y'}$ are isomorphic. Thus $b(X, Y)$ is a local numerical invariant of the model $(X, Y)$. For us, the importance of the G3 condition comes from the following fact, which is a consequence of results due to Gieseker ([**6**]).

Proposition 1.8. *If we have two models $(X, Y)$ and $(X', Y')$ such that $Y$ is* G3 *in $X$, $Y'$ is* G3 *in $X'$, and moreover the formal completions $\hat{X}_{|Y}$ and $\hat{X}'_{|Y'}$ are isomorphic, then $(X, Y) \sim_Z (X', Y')$.*

We recall also the following useful notion, due to Hartshorne ([**9**]).

Definition 1.9. The *cohomological dimension* of a scheme $W$, denoted $\mathrm{cd}(W)$, is defined by

$$\mathrm{cd}(W) = \min\{j \geq 0 \mid H^i(W, \mathscr{F}) = 0, \text{ for all } i > j \text{ and for all coherent sheaves } \mathscr{F} \text{ on } W\}.$$

The following proposition (to be used in the last section) is a special case of results proved in [**10**], Chapter V, Corollary 2.2.

Proposition 1.10. *Let $(X, Y)$ be a model and let $n =: \dim(X)$.*

(i) *If* $\mathrm{cd}(X \setminus Y) < n - 1$, *then $Y$ is* G3 *in $X$;*

(ii) *if $Y$ meets every effective divisor of $X$ and $Y$ is* G3 *in $X$, then* $\mathrm{cd}(X \setminus Y) < n - 1$.

We consider now another local numerical invariant of a model $(X, Y)$:

$$c(X, Y) =: \min\{(D \cdot Y) > 0 \mid D \in \mathrm{Div}(X)\}.$$

Note that $c(X, Y)$ is indeed a local invariant, since for any open subset $U \subset X$, the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective. Remark that, by Theorem 1.2 (v), the exponential sequence and the GAGA principle, we have $\mathrm{Pic}(X) \simeq H^2(X, \boldsymbol{Z})$. Thus $c(X, Y)$ is nothing but the order of the cokernel of the natural restriction maps $\mathrm{Pic}(X) \to \mathrm{Pic}(Y) \simeq \boldsymbol{Z}$ or $H^2(X, \boldsymbol{Z}) \to H^2(Y, \boldsymbol{Z}) \simeq \boldsymbol{Z}$. Recall from [**1**] that $Y$ is called an *almost-line* if it is a quasi-line and moreover $c(X, Y) = 1$.

Examples 1.11. (i) (cf. [**1**], (2.7)). For each $n \geq 3$, there is a model $(X, Y)$ with $\dim(X) = n$ such that:

(a) $Y$ is an almost-line;

(b) $b(X, Y) = n + 1$;

(c) $\hat{X}_{|Y} \simeq \hat{\boldsymbol{P}}^n_{|\mathrm{line}}$.

(ii) Let $X$ be a Fano threefold of index two with Picard group generated by the class of the hyperplane section. It was first proved in [**26**] (see also [**1**], Theorem 3.2 for a more conceptual argument) that $X$ contains a quasi-line $Y$ with $c(X, Y) = 2$.

Let $(X, Y)$ be a model such that $c(X, Y) = 1$; fix some $D \in \mathrm{Div}(X)$ with $(D \cdot Y) = 1$ and let $s =: \dim|D|$. We remark (e.g. by applying Proposition 2.1 below) that we must have $s \leq n$; actually, in our special case, we can say more. The following result is a generalization of Theorem 4.4 in [**1**], where $Y$ was assumed to be an almost-line and, moreover, $s$ was supposed to take on its maximal value, $s = \dim(X)$.

Theorem 1.12. *Keeping the above notation and assumptions, suppose moreover that $s \geq 1$. Then, for a general deformation of $Y$, say $Y_1$, there is a model $(X', Y')$ Zariski equivalent to $(X, Y_1)$ such that the following hold:*

(i) *there is a surjective morphism with connected fibres $\varphi : X' \to \boldsymbol{P}^s$, which is smooth along $Y'$;*

(ii) *there is a line $l$ in $\boldsymbol{P}^s$ such that $Z =: \varphi^{-1}(l)$ is smooth and $Y'$ is contained in $Z$ such that $N_{Y'|Z}$ is ample; moreover, $Y'$ is a section for $\varphi_{|Z}$;*

(iii) *any smooth fibre of $\varphi$ is rationally connected.*

Proof. The proof runs parallel to that of Theorem 4.4 in [**1**]. Note, however, that here we shall proceed by induction on $s$, while in [**1**] induction was made with respect to $n$. We include the details, for reader's convenience. Write $|D| = E + |M|$, $E$ being the fixed part and $|M|$ the moving part of $|D|$, respectively. We have $(E \cdot Y) \geq 0$ and $(M \cdot Y) > 0$ since both $Y$ and $M$ move. The relation $(E \cdot Y) + (M \cdot Y) = (D \cdot Y) = 1$ implies $(M \cdot Y) = 1$; so, by replacing $|D|$ by $|M|$, we may assume $|D|$ to be free from fixed components. Next, using Hironaka's desingularization theory, we may find a composition of blowing-ups along smooth centers, say $\sigma : X' \to X$ such that $\sigma^*(|D|) = E' + |D'|$, where $|D'|$ is base points free, $E'$ is the fixed part of $|E' + D'|$ and $\sigma(\mathrm{Supp}(E'))$ is contained in the base locus of $|D|$. We let $\varphi = \varphi_{|D'|} : X' \to \boldsymbol{P}^s$. Using [**16**], Chapter II, Proposition 3.7, we can find a deformation of $Y$, say $Y_1$, having ample normal bundle and not meeting the base locus of $|D|$. We let $Y' =: \sigma^{-1}(Y_1)$. The proof of (i) proceeds by induction on $s$. In case $s = 1$, we only have to remark that the fibres of $\varphi$ are connected, since $(D \cdot Y) = (D' \cdot Y') = 1$. Assume now that $s \geq 2$. Using again that $(D' \cdot Y') = 1$, it follows that $\varphi$ is not composed with a pencil. As $s \geq 2$, by Bertini's theorem we may find a smooth, connected, member $\Delta \in |D'|$ passing through two general points $x, y \in X'$. Using Theorem 1.2 (i) we may replace $Y'$ by a deformation of it (still having ample normal bundle) that passes through $x$ and $y$. As $(\Delta \cdot Y') = 1$, it follows that $Y' \subset \Delta$. Consider the exact sequence of conormal bundles:

$$0 \to \mathcal{O}_{\boldsymbol{P}^1}(-1) \to N^*_{Y'|X'} \to N^*_{Y'|\Delta} \to 0.$$

Since $H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-1)) = 0$ and $H^0(Y', N^*_{Y'|X'}) = 0$, it follows that $H^0(Y', N^*_{Y'|\Delta}) = 0$. Therefore $N_{Y'|\Delta}$ is ample. We also have the standard exact sequence:

$$0 \to \mathcal{O}_{X'} \to \mathcal{O}_{X'}(D') \to \mathcal{O}_{\Delta}(D') \to 0$$

and we know by Theorem 1.2 (v) that $H^1(X', \mathcal{O}_{X'}) = 0$. It follows that $\dim|D'_{|\Delta}| = s - 1$, so we may apply the induction hypothesis to the model $(\Delta, Y')$. In particular, as we have $\Delta = \varphi^{-1}(H)$ for some hyperplane $H \simeq \boldsymbol{P}^{s-1} \subset \boldsymbol{P}^s$, we get that the restriction of $\varphi$ to $\Delta$ maps $\Delta$ onto $H$. As the image of $\varphi$ is non-degenerate in $\boldsymbol{P}^s$, $\varphi$ must be surjective, too. It also follows that the general fibre of $\varphi$ is connected. This ensures (using Stein factorisation) that all fibres of $\varphi$ are connected, since $\varphi$ is surjective and $\boldsymbol{P}^s$ is normal (in fact smooth). This proves (i). (ii) follows by the preceding argument noting that, if $x, y \in X'$ are two general points, the line $l$ in $\boldsymbol{P}^s$ determined by $\varphi(x)$, $\varphi(y)$ is also general, so $Z =: \varphi^{-1}(l)$ is smooth, by Bertini's theorem. To show (iii), by what we already proved and by Theorem 1.2 (iv), it is enough to see that a general fibre of $\varphi|_Z$ is rationally connected. To this end we shall use the model $(Z, Y')$. Let $z$ be a general point of $Z$ and let $F$ be the fibre of $\varphi|_Z$ passing through $z$. Take some other point $z' \in F$. It follows that there is some deformation of $Y'$ inside $Z$ which is a tree $T$ of rational curves joining $z$ and $z'$. Since we have $(F \cdot Y') = (F \cdot T) = 1$, it follows that the intersection number of $F$ with an irreducible component of $T$ can take on only the values zero or one. Moreover, there is exactly one component for which this intersection number is one and the components of $T$ contained in $F$ give a chain of rational curves joining $z$ and $z'$. So $F$ is rationally connected (see [**18**]). □

Keeping the notations and assumptions of Theorem 1.12, let $\underline{a}(X, Y) = (a_1, \ldots, a_{n-1})$.

Corollary 1.13. *We have*:
(i) $\mathrm{card}\{i \mid a_i = 1\} \geq s - 1$;
(ii) (cf. [**1**], Theorem 4.4) *if* $s = n$, $(X', Y') \sim_Z (\boldsymbol{P}^n, \text{line})$;
(iii) *if* $s = n - 1$, *the general fibre of* $\varphi : X' \to \boldsymbol{P}^{n-1}$ *is* $\boldsymbol{P}^1$.

Proof. With the same notations as in the proof of Theorem 1.12, we have:

$$N_{Z|X'}|_{Y'} \simeq N_{l|\boldsymbol{P}^s} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus(s-1)}.$$

Moreover, the exact sequence:

$$0 \to N_{Y'|Z} \to N_{Y'|X'} \to N_{Z|X'}|_{Y'} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus(s-1)} \to 0$$

splits. If we let $\underline{a}(X, Y_1) = \underline{a}(X', Y') = (a'_1, \ldots, a'_{n-1})$, it follows that

$\mathrm{card}\{i \mid a_i' = 1\} \geq s-1$. As $Y_1$ is a general deformation of $Y$, we have moreover: $\mathrm{card}\{i \mid a_i = 1\} \geq \mathrm{card}\{i \mid a_i' = 1\}$ (see [**16**], Chapter II, Lemma 3.9.2). This proves (i). (ii) follows by induction on $s = n$, as in the proof of the theorem. Note that in [**1**], Theorem 4.4 we assumed moreover $Y$ to be an almost-line. But this follows from part (i) of the corollary. (iii) comes directly from Theorem 1.12 (iii). □

Remark 1.14. It is not clear if, given a model $(X, Y)$ with $c(X, Y) = 1$, we can always find $D \in \mathrm{Div}(X)$ such that $(D \cdot Y) = 1$ and $\dim|D| \geq 1$ (we do not know the answer even for the special case when $Y$ is an almost-line).

Next we turn to some *global* properties of models $(X, Y)$. It follows from Theorem 1.2 (v) that $\mathrm{Pic}(X) \simeq H^2(X, \boldsymbol{Z})$ is a free abelian group of finite rank; we denote by $\rho(X) =: \mathrm{rank}\,\mathrm{Pic}(X)$ the *base-number* of $X$. Clearly, $\rho$ is not an invariant for Zariski equivalence. For instance, if $\varphi : X' \to X$ is a birational morphism between projective manifolds and $Y \subset X$ is a smooth rational curve with ample normal bundle, we may apply [**16**], Chapter II, Proposition 3.7 to find a deformation $Y_1$ of $Y$ and a curve $Y'$ in $X'$ such that $\varphi$ induces a Zariski equivalence between the models $(X', Y')$ and $(X, Y_1)$. It is then natural to seek for a model in each Zariski equivalence class having a minimal base-number $\rho$.

Definition 1.15. Let $X$ be a normal projective variety such that:

(i) there is a smooth rational curve $Y$ contained in the smooth locus of $X$, such that $N_{Y|X}$ is ample;

(ii) $Y$ meets every effective (Weil) divisor of $X$.

We call the pair $(X, Y)$ a *minimal Zariski model*.

Two natural questions arise:

I. *Existence*: is every model Zariski equivalent to a minimal one?

II. *Uniqueness*: are two Zariski equivalent minimal models isomorphic?

For the moment it is not clear to us what is the "good" class of singularities that we may allow on a minimal Zariski model in order to ensure a reasonable answer to the above questions. We hope to return to this matter in the future. We shall see in Proposition 1.21 below that singularities are necessary even in the simplest case, $n = 2$. With some extra-hypothesis we can sometimes ensure uniqueness, as in the following.

Definition 1.16. A smooth model $(X, Y)$ is *anticanonical* (or Fano) if $-K_X$ is ample.

Proposition 1.17. *Two smooth anticanonical minimal Zariski models which are Zariski equivalent are isomorphic.*

Proof. Let $(X, Y)$ and $(X', Y')$ be two such models. Assume that $\varphi : X \cdots \to X'$ is a birational map yielding a Zariski equivalence between $(X, Y)$ and $(X', Y')$. By minimality $X$ and $X'$ are isomorphic in codimension one. In particular $\varphi^*(K_{X'}) = K_X$. Since $|-mK_X|$ and $|-mK_{X'}|$ are very ample for $m \gg 0$, it follows that $\varphi$ is actually an isomorphism. $\square$

Corollary 1.18. *Let $(X, Y)$ be a model which is Zariski equivalent to a model $(X_0, Y_0)$ with $\rho(X_0) = 1$. Then any two smooth minimal Zariski models of $(X, Y)$ are isomorphic.*

Proof. Any two smooth minimal Zariski models of $(X, Y)$ are isomorphic in codimension one. In particular, their base number is the same. Since a model with $\rho = 1$ is both minimal and anticanonical, the result follows from the proposition. $\square$

Corollary 1.19. *If $\rho(X) = 1$, any birational automorphism of $X$ inducing a Zariski equivalence between two models $(X, Y)$ and $(X, Y')$ is a biregular automorphism of $X$ (so $(X, Y) \simeq (X, Y')$).*

Remark 1.20. Recall from Theorem 1.2 (v) that any smooth model is simply connected. In general, a singular minimal Zariski model need not be simply connected. Indeed, the examples constructed in [**1**], Example 2.7 and denoted there by $(Z, Y)$ are minimal (by construction) but they are not simply connected. Otherwise, by minimality, an open Zariski neighbourhood of $Y$ in $Z$ would be simply connected (since it is got by throwing out a closed subset of real codimension $\geq 4$) and this is not the case.

The following proposition shows that in case $n = 2$ we have a clear and complete picture of all models, together with their minimal Zariski models; in particular, it appears that our "minimal Zariski models" are an attempt to generalize the classical "relative minimal models" of rational surfaces. The notation follows that of [**11**], Chapter V, Section 2.

Proposition 1.21. *Assume that $n = 2$. Given a model $(X, Y)$, there is a unique minimal Zariski model $(X_0, Y_0)$ and a birational morphism $\varphi : X \to X_0$ inducing a Zariski equivalence between $(X, Y)$ and $(X_0, Y_0)$. Moreover, $(X_0, Y_0)$ is one of the following*:

(i) $X_0 \simeq \boldsymbol{F}_e$, $e \geq 0$, $Y_0 \in |C_0 + \beta f|$, $\beta > e$;

(ii) $X_0 \simeq \boldsymbol{P}^2$, $Y_0$ *is a line or a conic*;

(iii) $X_0$ *is the cone in* $\boldsymbol{P}^{e+1}$ *over the rational normal curve of degree* $e$ *in* $\boldsymbol{P}^e$, $e \geq 2$, $Y_0$ *is a hyperplane section.*

$a(X, Y) = (Y^2)$ *can take on any positive value*, $Y$ *is always* G3 *in* $X$ (*so* $b(X, Y) = 1$) *and* $c(X, Y) \in \{1, 2\}$.

PROOF. $Y$ is nef and big, so $H^1(X, \mathcal{O}_X(-Y)) = 0$ by the vanishing theorem. The exact sequence:

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_{\boldsymbol{P}^1} \to 0$$

shows that $H^1(X, \mathcal{O}_X) = 0$ (of course, this is a special case of Theorem 1.2 (v)). Let $a =: (Y^2) > 0$. The exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \to \mathcal{O}_{\boldsymbol{P}^1}(a) \to 0$$

together with the vanishing of $H^1(X, \mathcal{O}_X)$ show that the linear system $|Y|$ is base points free and $h^0(X, \mathcal{O}_X(Y)) = a + 2$. Therefore we may consider the morphism $\varphi =: \varphi_{|Y|} : X \to \boldsymbol{P}^{a+1}$ and we denote by $X_0$ its image. We have $\deg X_0 \cdot \deg \varphi = a$. As $\dim X_0 = 2$ and $X_0$ is nondegenerate in $\boldsymbol{P}^{a+1}$, we have moreover:

$$a \geq \deg X_0 \geq \operatorname{codim} X_0 + 1 = a.$$

So $\varphi$ is birational and $X_0$ is a surface of minimal degree. By a classical result, going back to Del Pezzo, (see e.g. [**4**] for a modern proof) $X_0$ is either:

(i) a rational normal scroll, or

(ii) $\boldsymbol{P}^2$ or its Veronese embedding $v_2(\boldsymbol{P}^2) \subset \boldsymbol{P}^5$, or

(iii) a cone over the Veronese embedding of $\boldsymbol{P}^1$ in $\boldsymbol{P}^a$.

Both the fact that $\varphi$ is an isomorphism along $Y$ and the uniqueness of minimal Zariski models follow in our case from Zariski's Main Theorem, since $X_0$ is normal. See [**10**], p. 208 for the fact that $Y$ is G3 in $X$. The rest is standard (see [**11**], Chapter V, Section 2). □

Let us point out that, for $n \geq 3$, there are examples of models $(X, Y)$ whose minimal Zariski model is $(\boldsymbol{P}^n, \text{line})$, such that there is no birational *morphism* from $X$ to $\boldsymbol{P}^n$ (see [**1**], Example 4.7).

## 2. Models of $(\boldsymbol{P}^n, \text{line})$.

Let $(X, Y)$ be a model and consider some effective divisor $D \in \operatorname{Div}(X)$. As $Y$ moves, we have $(D \cdot Y) \geq 0$ and moreover, $(D \cdot Y) > 0$ if $\dim|D| \geq 1$. The following result goes back to Oxbury ([**26**]), who stated and proved it in a special case.

PROPOSITION 2.1. *Let $(X, Y)$ be a model with $\dim(X) = n$ and let $D$ be an effective divisor on $X$. Put $d =: (D \cdot Y)$. Then we have*

$$h^0(X, \mathcal{O}_X(D)) \leq \binom{d+n}{n}.$$

PROOF. We may assume $d > 0$. Consider the $d$-th jet bundle of $\mathcal{O}_X(D)$, denoted $\mathcal{J}_d(D)$. Consider also, as in [**26**], the natural map

$$u : H^0(X, \mathcal{O}_X(D)) \otimes \mathcal{O}_X \to \mathcal{J}_d(D)$$

which sends a section to its $d$-th jet. We claim that $u$ is generically injective, so

$$\operatorname{rank} \mathcal{J}_d(D) = \binom{d+n}{n} \geq \operatorname{rank}(H^0(X, \mathcal{O}_X(D)) \otimes \mathcal{O}_X) = h^0(X, \mathcal{O}_X(D)).$$

Let $s \in H^0(X, \mathcal{O}_X(D)) \backslash \{0\}$ and let $D' =: (s)_0$. If $x \in Y \cap \operatorname{supp}(D')$, by Theorem 1.2 (i) we may deform $Y$ to some $Y'$ by keeping $x$ fixed, so that the local intersection number $(Y' \cdot D')_x$ be defined. Assume now that $u_x(s) = 0$. This means that $(Y' \cdot D')_x > d = (Y \cdot D)$, so $s$ has to be zero. Thus $u$ is injective in the open set swept out by the deformations of $Y$ having ample normal bundle. □

Definition 2.2. For a given model $(X, Y)$ with $\dim(X) = n$, an effective divisor $D \in \operatorname{Div}(X)$ such that $d =: (D \cdot Y) > 0$ and $h^0(X, \mathcal{O}_X(D)) = \binom{d+n}{n}$ is called *extremal.*

Characterising conveniently models Zariski equivalent to $(\boldsymbol{P}^n, \text{line})$ is a natural problem. As a step towards its solution we propose the next:

Conjecture 2.3. *The following are equivalent for a model* $(X, Y)$:
(i) *some extremal divisor exists*;
(ii) $(X, Y) \sim_Z (\boldsymbol{P}^n, \text{line})$.

The implication (ii) $\Rightarrow$ (i) is quite obvious. Moreover, a necessary condition for the converse implication to hold is that $Y$ has to be a quasi-line. This is indeed the case by the following:

Lemma 2.4. *Assume that the model* $(X, Y)$ *admits an extremal divisor* $D$. *Then* $Y$ *is a quasi-line.*

Proof. In our case the map $u : H^0(X, \mathcal{O}_X(D)) \otimes \mathcal{O}_X \to \mathcal{J}_d(D)$ from the proof of Proposition 2.1 is a generically injective map from a trivial vector bundle to a vector bundle of the same rank. So its determinant is a non-zero global section of the line bundle $\det(\mathcal{J}_d(D))$. From the standard exact sequences of jet bundles:

$$0 \to S^m(\Omega_X) \otimes \mathcal{O}_X(D) \to \mathcal{J}_m(D) \to \mathcal{J}_{m-1}(D) \to 0$$

we find by successively taking determinants:

$$\det(\mathcal{J}_d(D)) = \mathcal{O}_X(r(dK + (n+1)D)),$$

where $r =: (1/d)\binom{d+n}{d-1}$. As $\det(u)$ is a non-zero global section of $\mathcal{O}_X(r(dK +$

$(n+1)D))$ it follows that $(dK+(n+1)D\cdot Y)\geq 0$, so $-(K\cdot Y)\leq n+1$, or $a(X,Y)\leq n-1$; thus $a(X,Y)=n-1$ and $Y$ is a quasi-line. □

Corollary 2.5. *The conjecture is true if* $n=2$.

Proof. Indeed, by the lemma $Y$ is a quasi-line, hence $(Y^2)=1$. It follows that $X$ admits the birational morphism $\varphi_{|Y|}$ to $\boldsymbol{P}^2$, which is an isomorphism along $Y$. □

Remark 2.6. We proved in Corollary 1.13 (ii) that in case $d=1$ the conjecture holds true after replacing the model $(X,Y)$ by $(X,Y_1)$, where $Y_1$ is a general deformation of $Y$. Note that this result can be used as a (sufficient) rationality criterion for a rationally connected manifold $X$.

Corollary 2.7. *Let* $(X,Y)$ *be a model. Assume that there is a nef and big divisor* $D\in\mathrm{Div}(X)$ *such that* $(D\cdot Y)=1$. *Then* $(X,Y')\sim_Z(\boldsymbol{P}^n,\mathrm{line})$, *where* $Y'$ *is a general deformation of* $Y$.

The proof given in [**1**], Corollary 4.6 (i) in the special case of quasi-lines works with minor changes.

It is a natural question to ask if the hypothesis of Corollary 2.7 is also *necessary* to ensure that $(X,Y)$ is Zariski equivalent to $(\boldsymbol{P}^n,\mathrm{line})$. We show that, for any $n\geq 3$, this is not the case.

Example 2.8. (cf. [**1**], Example 4.7) For each $n\geq 3$, there are models $(X,Y)$ Zariski equivalent to $(\boldsymbol{P}^n,\mathrm{line})$ such that there is no nef and big divisor $D$ on $X$ with $(D\cdot Y)=1$.

For the proof we need the following useful lemma, due to Fano (see [**16**], Chapter V, Proposition 2.9).

Lemma 2.9. (Fano) *Let* $D$ *be a nef divisor on the projective n-dimensional manifold* $X$. *Assume that for some* $d>0$ *we have a point* $x\in X$ *and an algebraic family of irreducible curves* $\{Y_t\}_t$ *joining* $x$ *and a general point* $t\in X$, *and such that* $(D\cdot Y_t)\leq d$. *Then* $(D^n)\leq d^n$.

Consider now the examples $(X,Y)$ constructed in [**1**], Example 4.7. If there were a nef and big divisor $D$ on $X$ such that $(D\cdot Y)=1$, by Thm 1.2 (i) and the above lemma it would follow $0<(D^n)\leq 1$, so $(D^n)=1$. But it was proved in [**1**], Example 4.7 that the self-intersection of a nef divisor on $X$ cannot be one.

The following proposition shows the usefulness of the existence of a smooth minimal model in a given Zariski equivalence class.

Proposition 2.10. *Assume that the model* $(X,Y)$ *admits a smooth minimal Zariski model. Then Conjecture* 2.3 *is true for* $(X,Y)$.

PROOF. It is easy to see that an extremal divisor $D$ descends to a smooth minimal Zariski model, so we may assume that $(X, Y)$ is minimal. We shall prove that $X \simeq \boldsymbol{P}^n$. We look again at the map

$$u : H^0(X, \mathcal{O}_X(D)) \otimes \mathcal{O}_X \to \mathcal{J}_d(D)$$

from the proof of Proposition 2.1, keeping the same notations. By the proof of Lemma 2.4 and our minimality hypothesis, $u$ is actually an isomorphism. Thus the jet bundle $\mathcal{J}_d(D)$ is trivial. Consider now the exact sequences:

$$(*) \qquad 0 \to S^i(\Omega_X) \otimes \mathcal{O}_X(D) \to \mathcal{J}_i(D) \to \mathcal{J}_{i-1}(D) \to 0,$$

for $0 < i \leq d$. As $\mathcal{J}_d(D)$ is trivial, it follows that $\mathcal{O}_X(D) = \mathcal{J}_0(D)$ is spanned by global sections. Dualizing the above sequence for $i = d$, it follows that $S^d(T_X)$ is also spanned by global sections, so $T_X$ is a nef vector bundle. Recall from Theorem 1.2 (v) that $\chi(\mathcal{O}_X) = 1$. From the next lemma we deduce that $X$ is a Fano manifold. But we have seen in the proof of Lemma 2.4 that $\det(\mathcal{J}_d(D)) \simeq \mathcal{O}_X(r(dK + (n+1)D))$. As $\det(\mathcal{J}_d(D))$ is trivial and $-K$ is ample, it follows that $\mathcal{O}_X(D)$ is ample. This in turn implies (as above, by dualising the sequence $(*)$ for $i = d$) that $S^d(T_X)$ is ample, so $T_X$ is ample. We conclude by Mori's theorem (see [**23**]) that $X \simeq \boldsymbol{P}^n$. □

We have extracted the following lemma from [**3**].

LEMMA 2.11. (Demailly-Peternell-Schneider) *Assume that for a given projective manifold $X$ with nef tangent bundle we have $\chi(\mathcal{O}_X) \neq 0$. Then $X$ is Fano.*

PROOF. If $T_X$ is nef, $-K$ is nef, so we have $(-1)^n(K^n) \geq 0$. If $(K^n) = 0$, using the Fulton-Lazarsfeld inequalities and Riemann-Roch theorem as in [**3**] p. 332 we get $\chi(\mathcal{O}_X) = 0$ and this was excluded. So $(-1)^n(K^n) > 0$. Since $-K$ is nef, this means that $-K$ is big. But on a manifold with $T_X$ nef any effective divisor is nef (see [**3**], p. 319); consequently, any big divisor is ample ([**3**], p. 320). So $-K$ is ample. □

REMARK 2.12. Sommese ([**27**]) proved (modulo Hartshorne's conjecture, now Mori's theorem) that, if a compact complex manifold possesses a line bundle $L$ such that $\mathcal{J}_d(L)$ is trivial for some $d > 0$, then one of the following holds:

(i) $T_X$ and $L$ are trivial;

(ii) $X \simeq \boldsymbol{P}^n$.

Alternatively, one may use his result in order to prove Proposition 2.10.

## 3. Polarizations and deformations of models.

We begin by introducing a global numerical invariant of a model $(X, Y)$:

$$d(X, Y) =: \min\{(Y \cdot H) \mid H \text{ ample divisor on } X\}.$$

Note that, clearly, we have $d(X,Y) \geq c(X,Y)$; moreover, equality holds if $\rho(X) = 1$. The following proposition illustrates the use of polarizations.

Proposition 3.1. *For a model $(X,Y)$ and an ample divisor $H \in \mathrm{Div}(X)$, let $d =: (H \cdot Y)$.*
(i) *If $dK + (a(X,Y)+2)H$ is nef, then $X$ is Fano;*
(ii) *if $d(X,Y) = 1$, then $(X,Y) \simeq (\boldsymbol{P}^n, \text{line})$.*

Proof. (i) If $dK + (a(X,Y)+2)H$ is nef, by the Kawamata-Reid-Shokurov Theorem (see [**15**]) it follows that $|m(dK + (a(X,Y)+2)H)|$ is base-points free for some $m > 0$. If $D \in |m(dK + (a(X,Y)+2)H)|$, it follows that $(D \cdot Y) = 0$ and this forces $\dim|D| = 0$, so $D$ is trivial and $X$ is Fano.

(ii) $(K + (a(X,Y)+2)H \cdot Y) = 0$, so $K + (a(X,Y)+2)H$ is not ample. As we have $a(X,Y) + 2 \geq n+1$, from [**13**], main theorem, it follows that $X \simeq \boldsymbol{P}^n$ and $Y$ has to be a line. □

Next we consider deformations of models. Let $(X,Y)$ be a model. By a *deformation* of $(X,Y)$ we mean a commutative diagram

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{i} & \mathscr{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & = & T \end{array}$$

where $p, q$ are proper smooth morphisms, $i$ is a closed embedding, $T$ is a connected scheme such that $(\mathscr{X}_t, \mathscr{Y}_t)$ is a model for each (closed) $t \in T$ and $(X,Y) \simeq (\mathscr{X}_{t_0}, \mathscr{Y}_{t_0})$ for some $t_o \in T$. Remark that the special case $\mathscr{X} = X \times T$, $p$ being the projection, was already used in the previous sections. We say that $(X,Y)$ and $(X',Y')$ are deformation equivalent if both appear as fibers of the same deformation.

Note that $\underline{a}(X,Y)$ is not invariant in a deformation (see however [**16**], Chapter II, Lemma 3.9.2, for a semicontinuity property), but $a(X,Y)$ is. The proof of Proposition 3.10 (ii) from [**1**] shows that both $c(X,Y)$ and $d(X,Y)$ behave upper semicontinuously in a deformation. As $\rho(X)$ is just the second Betti number of $X$, it is deformation invariant.

By a *polarized model* we mean a triple $(X,Y,H)$, where $(X,Y)$ is a model and $H$ is an ample divisor on $X$. Let $d =: (H \cdot Y)$. The following theorem shows that polarized models of fixed dimension and "degree" $d$ are, in principle, classifiable. See [**26**] and [**20**] for the actual classification of the first nontrivial cases: $n = 3$, $\underline{a}(X,Y) = (1,1)$ and $d(X,Y) \leq 4$.

Theorem 3.2. *Fix $n \geq 2$ and $d > 0$. There are only finitely many isomorphism classes of polarized models $(X,Y,H)$ with $\dim(X) = n$ and $(H \cdot Y) = d$, modulo deformations.*

PROOF. The proof is based on Matsusaka's theorem (see [**22**]), together with its refinement from [**17**]. According to the main result in [**22**], it is enough to show that, for fixed $n$ and $d$ as above, only finitely many Hilbert polynomials of polarized pairs $(X, H)$ can occur. Moreover, Lemma 5.2 in [**17**] shows that, in order to ensure this finiteness condition, it is enough to bound the first two top coefficients of the Hilbert polynomial. We refer the reader to Theorem 3 in [**17**] for the precise statement. So, it will be enough to prove that $(H^n)$ and $(H^{n-1} \cdot K)$ are bounded by some functions depending only on $n$ and $d$. Note that Lemma 2.9 already gives $0 < (H^n) \leq d^n$. It remains to prove that $(H^{n-1} \cdot K)$ is bounded. Set $H_1 =: K + (n+2)H$. By Mori's Cone Theorem (see [**24**]) $H_1$ is ample on $X$. Put $d_1 =: (H_1 \cdot Y) = (K \cdot Y) + (n+2)d$. As we have $(K \cdot Y) \leq -n-1$, we get $1 \leq d_1 \leq (n+2)d - n - 1$, so $d_1$ is bounded; moreover, $(H_1^n) \leq d_1^n$, again by Fano's lemma. Recall that the arithmetic genus of an ample divisor $H_1$ is defined by the formula:

$$2p_a(H_1) - 2 = ((K + (n-1)H_1) \cdot H_1^{n-1}).$$

It is proved in [**13**], Lemma 7, that $p_a(H_1) \geq 0$. It follows that $-2 \leq (K \cdot H_1^{n-1}) + (n-1)(H_1^n)$, which shows that $(K \cdot H_1^{n-1})$ is bounded from below, say by a function $\alpha(n, d)$. Moreover, we have:

$$d_1^n \geq (H_1^n) = ((K + (n+2)H) \cdot H_1^{n-1}) \geq \alpha(n, d) + (n+2)(H \cdot H_1^{n-1}).$$

It follows that $(H \cdot H_1^{n-1})$ is bounded from above.

On the other hand, a well-known generalization of the Hodge index theorem (see e.g. [**16**] p. 301) says that for any nef divisor $D$ and any ample $H$ we have

$$(H^{j+1} \cdot D^{n-j-1}) \cdot (H^{j-1} \cdot D^{n-j+1}) \leq (H^j \cdot D^{n-j})^2.$$

We apply this for $D = H_1$ to get by induction on $j$ that $(H_1 \cdot H^{n-1})$ is bounded from above.

This means that

$$1 \leq (H_1 \cdot H^{n-1}) = ((K + (n+2)H) \cdot H^{n-1})$$

is bounded from above, hence $(K \cdot H^{n-1})$ is bounded and we are done. □

## 4. Models containing quasi-lines.

In this section we consider only models $(X, Y)$ with $\underline{a}(X, Y) = (1, \ldots, 1)$, i.e. we assume $Y$ to be a quasi-line. We first investigate the cases where $X$ has the structure of a projective bundle over a smooth curve.

PROPOSITION 4.1. *Let $(X, Y)$ be a model such that $Y$ is a quasi-line. Assume that $X \simeq \boldsymbol{P}(E) \xrightarrow{\pi} C$, where $C$ is a smooth projective curve, $E$ is a rank $n$ vector bundle on $C$ and $\pi$ is the natural projection. Then $C \simeq \boldsymbol{P}^1$,*

$$E \simeq \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a+1)$$

*for some $a \in \boldsymbol{Z}$ and $Y$ is a section for $\pi$. Moreover, $X$ identifies to the blowing-up of $\boldsymbol{P}^n$ with center a codimension two linear subspace $L$ and $Y$ identifies to the pull-back of a line in $\boldsymbol{P}^n$ not meeting $L$, so $(X, Y) \sim_Z (\boldsymbol{P}^n, \text{line})$.*

Proof. $Y$ cannot be contained in a fibre of $\pi$, hence $C$ must be rational. So $E$ is a sum of line bundles, say $E \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\boldsymbol{P}^1}(a_i)$ and we may further assume that $0 < a_1 \le a_2 \le \cdots \le a_n$. Let $F$ denote a fibre of $\pi$ and let $D$ be the divisor corresponding to the line bundle $\mathcal{O}_{\boldsymbol{P}(E)}(1)$. Adjunction formula gives:

$$K + nD = (a_1 + \cdots + a_n - 2)F.$$

Put $d =: (D \cdot Y)$ and $f =: (F \cdot Y)$; we get:

$$nd - n - 1 = (a_1 + \cdots + a_n - 2)f.$$

We find easily that $h^0(X, \mathcal{O}_X(D - a_nF)) = \operatorname{card}\{i \mid a_i = a_n\}$. It follows that $(D - a_nF) \cdot Y \ge 0$, hence $d \ge a_nf$. We shall prove that in fact $d = a_nf$. Otherwise we would have $a_nf \le d - 1$, so

$$nd - n - 1 = (a_1 + \cdots + a_n - 2)f \le (na_n - 2)f \le n(d-1) - 2f.$$

It follows $2f \le 1$, so $f = 0$; but this is absurd since both $F$ and $Y$ move. So we get that $a_nf = d$, or $(D - a_nF) \cdot Y = 0$. This in turn implies that $h^0(X, \mathcal{O}_X(D - a_nF)) = 1$, hence $a_i < a_n$ for $i = 1, 2, \ldots, n-1$. We find:

$$(a_1 + \cdots + a_n - 2)f \le ((n-1)(a_n - 1) + a_n - 2)f,$$

or

$$nd - n - 1 \le nd - (n+1)f.$$

This gives $f = 1$ and $a_1 = a_2 = \cdots = a_{n-1} = a_n - 1$. Recalling that $a_nf = d$, we find $a_1 = \cdots = a_{n-1} = d - 1$, $a_n = d$.

It is now standard to see that, regarding $X \simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1))$ as the blowing-up of $\boldsymbol{P}^n$ with center a codimension two linear subspace $L$, $Y$ identifies to the pull-back of a line in $\boldsymbol{P}^n$ not meeting $L$. □

The following result enables us to construct new models starting from a given one.

Proposition 4.2. *Let $(X, Y)$ be a model, where $Y$ is a quasi-line. If $E$ is a vector bundle of rank $r$ on $X$, denote by $X' =: \boldsymbol{P}(E)$ and by $\pi : X' \to X$ the natural projection. The following conditions are equivalent*:

(i) *there is a quasi-line $Y'$ on $X'$ such that $\pi_{|Y'} : Y' \to Y$ is an isomorphism*;

(ii) *$E_{|Y} \simeq \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(a+1)$ for some $a \in \boldsymbol{Z}$.*

Proof. Let $Z =: \pi^{-1}(Y)$.

(i) $\Rightarrow$ (ii) Since $\pi_{|Y'} : Y' \to Y$ is an isomorphism it follows that $N_{Z|X'}\big|_{Y'} \simeq N_{Y|X}$. The exact sequence:

$$(**) \qquad 0 \to N_{Y'|Z} \to N_{Y'|X'} \to N_{Z|X'}\big|_{Y'} \simeq N_{Y|X} \to 0$$

shows that $Y'$ is a quasi-line in $Z = \boldsymbol{P}(E_{|Y})$. (ii) follows now from Proposition 4.1.

(ii) $\Rightarrow$ (i) We have seen in Proposition 4.1 that $Z = \boldsymbol{P}(E_{|Y})$ identifies to the blowing-up of $\boldsymbol{P}^r$ with center a codimension two linear subspace $L$. Take $Y'$ to be the pull-back of a line in $\boldsymbol{P}^r$ not meeting $L$; remark that $Y'$ is a quasi-line on $Z$ and a section of $\pi_{|Z} : Z \to Y$. The exact sequence $(**)$ shows that $Y'$ is in fact a quasi-line on $X'$. $\square$

Remark 4.3. In the above construction, if $Y$ is an almost-line, $Y'$ is also an almost-line.

Let $E$ be a rank $r$ vector bundle on $\boldsymbol{P}^m$ and assume that there is a line $l \subset \boldsymbol{P}^m$ such that $E_{|l} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2)$. By the above proposition, we get a model $(X = \boldsymbol{P}(E), Y)$, with $Y$ an almost-line that projects isomorphically onto $l$.

Lemma 4.4. *With the above assumptions and notations, the resulting model* $(X = \boldsymbol{P}(E), Y)$ *is a minimal Zariski model if and only if* $H^0(\boldsymbol{P}^m, E(-2)) = 0$.

Proof. We first remark that $E|_{l'} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2)$ for a *general* line $l'$ in $\boldsymbol{P}^m$ (this fact was kindly pointed out to us by I. Coandă). Indeed, $E(-2)|_l$ has degree $1 - r$ and $h^0(l, E(-2)|_l) = 1$; by semicontinuity, the same holds true in a neighbourhood of $l$. Consequently, for $l'$ in that neighbourhood, the splitting-type of $E|_{l'}$ is the same as that of $E|_l$. Now, let $\pi : X = \boldsymbol{P}(E) \to \boldsymbol{P}^m$ be the projection and let $Z = \pi^{-1}(l) = \boldsymbol{P}(E_{|l})$. Regard also $Z$ as the blowing-up $\sigma : Z \to \boldsymbol{P}^r$ with center a codimension two linear subspace $L$, so that $Y$ identifies to the pull-back of a line in $\boldsymbol{P}^r$ not meeting $L$. Let $D$ denote the divisor class on $Z$ corresponding to $\mathcal{O}_Z(1)$, let $F$ denote the class of a fibre of $\pi_{|Z}$ and let $\Sigma$ denote the exceptional divisor of $\sigma$. With these notations we find easily that $\Sigma$ is linearly equivalent to $D - 2F$. If we remark moreover that the natural restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism, it follows that $\Sigma$ is induced by the divisor $T - \pi^*(2H)$, where $T$ corresponds to $\mathcal{O}_X(1)$ and $H$ is a hyperplane in $\boldsymbol{P}^m$. Now, if $A$ is an (effective) prime divisor on $X$ such that $(A \cdot Y) = 0$, it follows that the restriction of $A$ to $Z$ has to be of the form $a\Sigma$, for some $a > 0$. The fact that

$$E_{|l} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2)$$

for a *general* line $l$ implies via a Bertini-type argument that $a = 1$. Putting everything together, it follows that such an effective divisor $A$ exists if and only if $|T - \pi^*(2H)| \neq \emptyset$, which is equivalent to $H^0(\boldsymbol{P}^m, E(-2)) \neq 0$. $\square$

LEMMA 4.5. *Keeping the notations and the assumptions preceeding the statement of Lemma* 4.4, *suppose moreover that* $m = r$. *Then* $(X = \boldsymbol{P}(E), Y)$ *is a Fano model if and only if either*

$$E \simeq \mathcal{O}_{\boldsymbol{P}^m}(1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^m}(1) \oplus \mathcal{O}_{\boldsymbol{P}^m}(2)$$

*or*

$$E \simeq T_{\boldsymbol{P}^m}.$$

PROOF. We have that $\det(E) \simeq \mathcal{O}_{\boldsymbol{P}^m}(m+1)$ and that

$$\omega_X \simeq \mathcal{O}_X(-m) \otimes \pi^*(\omega_{\boldsymbol{P}^m} \otimes \det(E)) \simeq \mathcal{O}_X(-m).$$

So $\omega_X^{-1}$ is ample if and only if $E$ is ample; in this case, as $E_{|l}$ has degree $m+1$ and rank $m$, $E$ has to be uniform (i.e. $E_{|l} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2)$ for *any* line $l \subset \boldsymbol{P}^m$). Now the result follows from [**5**]. $\square$

The next question has been asked in [**1**], Question 5.7: is any smooth minimal Zariski model $(X, Y)$, with $Y$ a quasi-line, necessarily Fano? Proposition 4.6 below shows that, in general, the answer is *no*, even for threefolds.

Indeed, consider a rank-two vector bundle on $\boldsymbol{P}^2$ constructed by Serre's method (cf. [**25**]) as an extension of the form:

$$0 \to \mathcal{O}_{\boldsymbol{P}^2}(1) \to E \to \mathcal{I}_{\{P_1, P_2\}}(2) \to 0,$$

where $P_1$, $P_2$ are two points in $\boldsymbol{P}^2$. Note that for a general line $l \subset \boldsymbol{P}^2$, $E_{|l} \simeq \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2)$.

PROPOSITION 4.6. *The model* $(X = \boldsymbol{P}(E), Y)$ *constructed as above is a minimal Zariski model*, *but it is not Fano*.

PROOF. The minimality follows from Lemma 4.4, since $H^0(\boldsymbol{P}^2, E(-2)) = 0$. $E$ is neither $\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2)$ nor $T_{\boldsymbol{P}^2}$, so $\boldsymbol{P}(E)$ is not Fano by Lemma 4.5. $\square$

Our next purpose is to answer in the negative Questions 5.2 and 5.3 from [**1**]. To this end we need the following simple lemma in order to compute the cohomological dimension of $X \backslash Y$, for a certain model $(X, Y)$.

LEMMA 4.7. *Let* $X$ *be a projective scheme and* $Y_1, Y_2 \subset X$ *two closed subsets. Put* $Y = Y_1 \cap Y_2$, $U_1 = X \backslash Y_1$, $U_2 = X \backslash Y_2$, $X_0 = X \backslash Y$. *Assume that there is some* $a > 0$ *such that*:

(i) $\mathrm{cd}(U_i) \le a$, *for* $i = 1, 2$;
(ii) $\mathrm{cd}(U_1 \cap U_2) \le a - 1$.
*Then* $\mathrm{cd}(X_0) \le a$.

Proof. Let $Z_j = X_0 \cap Y_j$ for $j = 1, 2$. For any $i \ge a + 1$ and any coherent sheaf $F$ on $X_0$ we have by (ii) the exact sequence of local cohomology (see [**8**]):

$$0 \to H^i_{Z_1 \cup Z_2}(X_0, F) \to H^i(X_0, F) \to H^i(U_1 \cap U_2, F_{|U_1 \cap U_2}) = 0.$$

Since $Z_1 \cap Z_2 = \varnothing$, we have by the Mayer-Vietoris exact sequence (see [**8**]):

$$H^i_{Z_1 \cup Z_2}(X_0, F) \simeq H^i_{Z_1}(X_0, F) \oplus H^i_{Z_2}(X_0, F).$$

Moreover, by excision (see [**8**]) we get:

$$H^i_{Z_1}(X_0, F) \simeq H^i_{Z_1}(U_2, F_{|U_2})$$

and

$$H^i_{Z_2}(X_0, F) \simeq H^i_{Z_2}(U_1, F_{|U_1}).$$

By (ii) and the exact sequence of local cohomology we get the exact sequences:

$$0 \to H^i_{Z_1}(U_2, F_{|U_2}) \to H^i(U_2, F_{|U_2}) \to H^i(U_1 \cap U_2, F_{|U_1 \cap U_2}) = 0$$

and

$$0 \to H^i_{Z_2}(U_1, F_{|U_1}) \to H^i(U_1, F_{|U_1}) \to H^i(U_1 \cap U_2, F_{|U_1 \cap U_2}) = 0.$$

Combining these exact sequences and using also (i) we get the result. □

Proposition 4.8. *Consider the model* $(X = \boldsymbol{P}(T_{\boldsymbol{P}^2}), Y)$ *constructed above. Then* $Y$ *is* G3 *in* $X$.

Proof. Regard $X \subset \boldsymbol{P}^2 \times \check{\boldsymbol{P}}^2$ as the incidence variety. We may write $Y = Y_1 \cap Y_2$, with $Y_1 = p_1^{-1}(l)$, $Y_2 = p_2^{-1}(\check{l})$, $l \subset \boldsymbol{P}^2$ and $\check{l} \subset \check{\boldsymbol{P}}^2$ being suitably chosen lines and $p_1 : X \to \boldsymbol{P}^2$, $p_2 : X \to \check{\boldsymbol{P}}^2$ being the restricted projections. Let $U_i = X \setminus Y_i$ for $i = 1, 2$; we have $U_i \simeq \boldsymbol{A}^2 \times \boldsymbol{P}^1$ for $i = 1, 2$. Moreover, $U_1 \cap U_2$ is affine, being the complement in $X$ of a hyperplane section. We may apply Lemma 4.7 with $a = 1$ to get that $\mathrm{cd}(X \setminus Y) \le 1$ (actually equality holds). The result follows now from Proposition 1.10 (i). □

Corollary 4.9. *The formal completions* $\boldsymbol{P}(\widehat{T_{\boldsymbol{P}^2}})_{|Y}$ *and* $\hat{\boldsymbol{P}}^3_{|\mathrm{line}}$ *are not isomorphic.*

Proof. Since both $Y$ in $X = \boldsymbol{P}(T_{\boldsymbol{P}^2})$ and a line in $\boldsymbol{P}^3$ are G3, if the formal completions were isomorphic, it would follow from Proposition 1.8 that $(X, Y) \sim_Z (\boldsymbol{P}^3, \mathrm{line})$. But both are anticanonical minimal Zariski models. So, by Proposition 1.17, they would be isomorphic, which is clearly absurd. □

Next we introduce another local numerical invariant of a model $(X, Y)$ in case $Y$ is a quasi-line. Recall from [**1**], Lemma 3.4 that there are only finitely many deformations of $Y$ passing through two general points of $X$.

Let

$$e(X, Y) =: \mathrm{card}\{Y' \mid Y' \text{ is a deformation of } Y \text{ passing through two general points of } X\}.$$

We close up the paper by exhibiting a rather surprising example of two anti-canonical models $(X, Y)$ and $(X', Y')$ such that $\hat{X}_{|Y}$ and $\hat{X}'_{|Y'}$ are not isomorphic although all their numerical invariants are the same. Indeed, let $(X = \boldsymbol{P}(T_{\boldsymbol{P}^2}), Y)$ be the above constructed model; let $X'$ be the blowing-up of $\boldsymbol{P}^3$ with center a point $P$ (so $X' = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2))$) and let $Y'$ be the inverse image of a line in $\boldsymbol{P}^3$ not passing through $P$.

PROPOSITION 4.10. *$\hat{X}_{|Y}$ and $\hat{X}'_{|Y'}$ are not isomorphic (in particular $(X, Y)$ and $(X', Y')$ are not Zariski equivalent); moreover, $(X, Y)$ and $(X', Y')$ are not deformation equivalent. However, all their numerical invariants considered in this paper coincide*:

$$\underline{a}(X, Y) = \underline{a}(X', Y') = (1, 1),$$

$$b(X, Y) = b(X', Y') = 1,$$

$$c(X, Y) = c(X', Y') = 1,$$

$$d(X, Y) = d(X', Y') = 2,$$

$$e(X, Y) = e(X', Y') = 1$$

*and*

$$\rho(X) = \rho(X') = 2.$$

PROOF. The fact that $\hat{X}_{|Y}$ and $\hat{X}'_{|Y'}$ are not isomorphic was already proved in Corollary 4.9. We compute easily $(K_X^3) = -48$ and $(K_{X'}^3) = -56$, so $X$ and $X'$ are not deformation equivalent. The computation of the numerical invariants is not difficult and is left to the reader. $\square$

REMARK 4.11. We can distinguish between the two models from Proposition 4.10 via another *global* invariant. Indeed, as we have already seen, we have $\mathrm{cd}(X \setminus Y) = 1$, but one computes easily that $\mathrm{cd}(X' \setminus Y') = 2$. Note that, if $l$ is a line in $\boldsymbol{P}^3$, we have $\mathrm{cd}(\boldsymbol{P}^3 \setminus l) = 1$.

ACKNOWLEDGEMENTS. 1. *We are pleased to thank Lucian Bădescu for useful conversations on the subject of this paper.*

2. *The first named author acknowledges a very pleasant one-semester stay at the Department of Mathematics of the University Montpellier II while working on this paper. He is particularly grateful to Robert Silhol for inviting him and for making his stay very enjoyable.*

3. *We thank the referee for his comments that helped improving the presentation.*

## References

[1] L. Bădescu, M. C. Beltrametti and P. Ionescu, Almost-lines and quasi-lines on projective manifolds, in Complex Analysis and Algebraic Geometry-in memory of Michael Schneider de Gruyter (2000), 1–27.

[2] F. Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. École Norm. Sup., **25** (1992), 539–545.

[3] J.-P. Demailly, T. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles, J. Algebraic Geom., **3** (1994), 295–345.

[4] D. Eisenbud and J. Harris, On varieties of minimal degree, in Algebraic Geometry, Bowdoin, 1985, Proc. Sympos. Pure Math., vol. 46, Part 1, Amer. Math. Soc., 1987, 3–13.

[5] G. Elencwajg, A. Hirschowitz and M. Schneider, Les fibrés uniformes de rang au plus n sur $\boldsymbol{P}^n$ sont ceux qu'on croit, in Proc. Nice Conference on Vector Bundles and Differential Equations, 1979, Birkhäuser, 1980.

[6] D. Gieseker, On two theorems of Griffiths about embeddings with ample normal bundles, Amer. J. Math., **99** (1977), 1137–1150.

[7] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer. J. Math., **79** (1957), 121–138.

[8] A. Grothendieck, Local Cohomology (notes by R. Hartshorne), Lecture Notes in Math., **41**, Springer, 1967.

[9] R. Hartshorne, Cohomological dimension of algebraic varieties, Ann. of Math., **88** (1968), 403–450.

[10] R. Hartshorne, Ample Subvarieties of Algebraic Varieties, Lecture Notes in Math., **156**, Springer, 1970.

[11] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math., **52**, Springer, 1977.

[12] H. Hironaka and H. Matsumura, Formal functions and formal embeddings, J. Math. Soc. Japan, **20** (1968), 52–82.

[13] P. Ionescu, Generalized adjunction and applications, Math. Proc. Cambridge Philos. Soc., **99** (1986), 457–472.

[14] V. A. Iskovskih and Y. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Math. USSR-Sbornik, **15** (1971), 141–166.

[15] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, in Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math., Vol. 10, Kinokunya-North-Holland, 1987, 283–360.

[16] J. Kollár, Rational Curves on Algebraic Varieties, Ergeb. Math. Grenzgeb., **32**, Springer, 1996.

[17] J. Kollár and T. Matsusaka, Riemann-Roch type inequalities, Amer. J. Math., **105** (1983), 229–252.

[18] J. Kollár, Y. Miyaoka and S. Mori, Rationally connected varieties, J. Algebraic Geom., **1** (1992), 429–448.

[19] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Differential Geom., **36** (1992), 765–769.

[20] A. Langer, On twistor curves of small degree, Quart. J. Math. Oxford, **49** (1998), 43–58.

[21] H. Matsumura and P. Monsky, On the orders of automorphism groups of hypersurfaces, J. Math. Kyoto Univ., **3** (1964), 347–361.
[22] T. Matsusaka, Polarised varieties with a given Hilbert polynomial, Amer. J. Math., **94** (1972), 1027–1077.
[23] S. Mori, Projective manifolds with ample tangent bundles, Ann. of Math., **110** (1979), 593–606.
[24] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math., **116** (1982), 133–176.
[25] C. Okonek, M. Schneider and H. Spindler, Vector Bundles on Complex Projective Spaces, Progr. Math., 3, Birkhäuser 1980.
[26] W. M. Oxbury, Twistor spaces and Fano threefolds, Quart. J. Math. Oxford, **45** (1994), 343–366.
[27] A. J. Sommese, Compact complex manifolds possessing a line bundle with trivial jet bundle, Abh. Math. Sem. Univ. Hamburg, **47** (1978), 79–91.

Paltin IONESCU
Department of Mathematics
University of Bucharest
14, Academiei Street
70109 Bucharest
Romania
E-mail: pionescu@pompeiu.imar.ro

Cristian VOICA
Department of Mathematics
University of Bucharest
14, Academiei Street
70109 Bucharest
Romania
E-mail: voica@gta.unibuc.ro