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Exponential decay of stochastic oscillatory integrals on classical Wiener spaces

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Abstract. An exponential decay of a stochastic oscillatory integral with phase function determined as a stochastic line integral of a 1-form is studied. A sufficient condition for such an integral to decay exponentially fast is given in terms of the exterior derivative of the 1-form, i.e., the magnetic field.

1. Introduction and statement of main result.

A stochastic oscillatory integral is an integral defined by

$$I(\lambda) := \int_X \exp[\sqrt{-1}\lambda q]\psi \, d\nu,$$

where X is a real abstract Wiener space with Wiener measure ν, and q, ψ are real valued Wiener functionals. In general, X is an infinite dimensional vector space. Studies of oscillatory integrals on infinite dimensional spaces, including stochastic ones, have a long history (for example, see [**1**]–[**7**], [**10**], [**11**], [**16**]–[**21**], [**23**], [**28**]–[**31**] and references therein), and are motivated by and closely related to Feynman path integrals. One of the goals of such studies of oscillatory integrals is to establish *a principle of stationary phase*. Recently a new approach to study a stochastic oscillatory integral was introduced by P. Malliavin and the author [**23**] by using *complex transformations* on a *complexified* Wiener space. In particular, a general machinery to study an exponential decay of $I(\lambda)$ as $\lambda \to \infty$ was achieved in [**23**], [**29**]. Even though many contributions were made for the studies of oscillatory integrals on infinite dimensional spaces as mentioned above, a general scheme to deal with the principle has not been established yet except for a case where q is a quadratic Wiener functional. For example, see [**4**], [**28**] and the references therein. Then one may ask a question if the principle of station-

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ary phase is really available on an infinite dimensional space. The exponential decay of $I(\lambda)$ as $\lambda \to \infty$ can be thought of as a positive answer to the question, as an evidence for a principle of stationary phase to hold on an infinite dimensional space. Namely, the principle of stationary phase on $\boldsymbol{R}^n$ insists that the oscillatory integral decays at the order of $\lambda^{-n/2}$ and the exponential decay guarantees to substitute ∞ for n. In this paper, we run the machinery developed in [**23**], [**29**] to show an exponential decay of $I(\lambda)$ as $\lambda \to \infty$ when the phase function of stochastic oscillatory integral is given by a stochastic line integral of a 1-form along the Brownian motion. We shall polish up the abstract machinery given in [**23**], [**29**], and then run it in a concrete setting where we can take advantage of the powerful Itô calculus. A sufficient condition for $I(\lambda)$ to decay exponentially fast in terms of the exterior derivative of the 1-form, i.e. the magnetic field, will be established.

We shall state a main result of the present paper more precisely, and then make some comments on it. Let $D \in \boldsymbol{N}$ and (W, H, μ) be the D-dimensional classical Wiener space over $[0,1]$[1];

$$W = \{w : [0,1] \to \boldsymbol{R}^D : w \text{ is continuous and } w(0) = 0\},$$

$$H = \left\{ h \in W : \begin{array}{l} h \text{ is absolutely continuous and has a} \\ \text{square integrable derivative } \dot{h} \text{ on } [0,1] \end{array} \right\},$$

and μ be the Wiener measure on W. The Cameron-Martin subspace H is a real separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ given by

$$\langle h, k \rangle_H = \int_0^1 \langle \dot{h}(t), \dot{k}(t) \rangle \, dt,$$

where $\langle x, y \rangle = \sum_{\alpha=1}^D x^\alpha y^\alpha$ for $x = (x^1, \ldots, x^D)$, $y = (y^1, \ldots, y^D) \in \boldsymbol{R}^D$. For $x_0 \in \boldsymbol{R}^D$, $N \in \boldsymbol{Z}_+ := \boldsymbol{N} \cup \{0\}$, $\Theta = (\theta^0, \ldots, \theta^N) \in C^\infty(\boldsymbol{R}^D; (\boldsymbol{R}^D)^{N+1})$, $V = (v^0, \ldots, v^N) \in C^\infty(\boldsymbol{R}^D; \boldsymbol{R}^{N+1})$, $\boldsymbol{a} = (a_1, \ldots, a_N) \in [0,1)^N$, and $\lambda > 0$, we define

$$\begin{aligned} q_{x_0}^\lambda[\Theta, V, \boldsymbol{a}] &= \int_0^1 \langle \theta^0(w_{x_0}(t)), \circ dw(t) \rangle + \int_0^1 v^0(w_{x_0}(t)) \, dt \\ &\quad + \sum_{i=1}^N \lambda^{-(1-a_i)} \left\{ \int_0^1 \langle \theta^i(w_{x_0}(t)), \circ dw(t) \rangle + \int_0^1 v^i(w_{x_0}(t)) \, dt \right\}, \end{aligned}$$

where $w_{x_0}(t) = x_0 + w(t)$, $w(t) = (w^1(t), \ldots, w^D(t))$ is the position of $w \in W$ at time t, and

[1] In this paper, the lowercase d is used to indicate both exterior and stochastic differentiations. To emphasize the dimension, we use the uppercase D instead of the standard letter d.

$$\int_0^1 \langle \theta^j(w_{x_0}(t)), \circ dw(t)\rangle = \sum_{\alpha=1}^{D} \int_0^1 \theta^j_\alpha(w_{x_0}(t)) \circ dw^\alpha(t),$$

θ^j_α being the αth component of θ^j, and $\circ dw^\alpha(t)$ being the Stratonovich integral with respect to $w^\alpha(t)$ under μ. In the present paper, we investigate an exponential decay as $\lambda \to \infty$ of

$$I(\lambda) := p_1^\lambda(x_0, x_1) := \int_W \exp[\sqrt{-1}\lambda q^\lambda_{x_0}[\Theta, V, \boldsymbol{a}](w)]\delta_{x_1}(w_{x_0}(1))\mu(dw),$$

where $x_1 \in \boldsymbol{R}^D$ and $\delta_{x_1}(w_{x_0}(1))\mu(dw)$ denotes the integration with respect to Watanabe's pull-back of Dirac's delta function δ_{x_1} concentrating at x_1 via $w_{x_0}(1)$ ([**22**]). Thus, we are dealing with a stochastic oscillatory integral with the phase function $q^\lambda_{x_0}[\Theta, V, \boldsymbol{a}](w)$ and the amplitude function $\delta_{x_1}(w_{x_0}(1))$ on (W, H, μ), and we will show its exponential decay as $\lambda \to \infty$, which is governed by the term $\int_0^1 \langle \theta^0(w_{x_0}(t)), \circ dw(t)\rangle + \int_0^1 v^0(w_{x_0}(t))\, dt$. Our main result is

Theorem 1.1. *Let $x_0, x_1 \in \boldsymbol{R}^D$ and $\Theta = (\theta^0, \dots, \theta^N) \in C^\infty(\boldsymbol{R}^D; (\boldsymbol{R}^D)^{N+1})$. Suppose that*

(A.1) *all components $(d\theta^i)_{\alpha\beta}$ of the exterior derivative $d\theta^i$ of θ^i, and v^i $(i = 0, \dots, N, \alpha, \beta = 1, \dots, D)$ are polynomials on $\boldsymbol{R}^D$,*

(A.2) $d\theta^0(x_0) \neq 0$.

Put

$$K = \max\{(\deg(d\theta^i)_{\alpha\beta}) + 1, (\deg v^i) - 1 : i = 0, \dots, N, \alpha, \beta = 1, \dots, D\}.$$

Then there exist $C_1, C_2 > 0$ such that

$$|p_1^\lambda(x_0, x_1)| \le C_1 \exp[-C_2 \lambda^{1/(K+5)}] \quad \text{for any } \lambda > 0. \tag{1.1}$$

We shall make several remarks on the theorem.

Remark 1.2. (i) Set

$$\Theta^{\lambda, \boldsymbol{a}} = \lambda\theta^0 + \sum_{i=1}^{N} \lambda^{a_i}\theta^i, \quad V^{\lambda, \boldsymbol{a}} = \lambda v^0 + \sum_{i=1}^{N} \lambda^{a_i} v^i$$

and define a Schrödinger operator $S^{\lambda, \boldsymbol{a}}$ relative to the magnetic field $d\Theta^{\lambda, \boldsymbol{a}}$ (the exterior derivative of $\Theta^{\lambda, \boldsymbol{a}}$) and $V^{\lambda, \boldsymbol{a}}$ by

$$S^{\lambda, \boldsymbol{a}} u = (1/2)\Delta u + \sqrt{-1}\langle \Theta^{\lambda, \boldsymbol{a}}, du\rangle + \{\sqrt{-1}((1/2)\, d^*\Theta^{\lambda, \boldsymbol{a}} + V^{\lambda, \boldsymbol{a}}) - (1/2)|\Theta^{\lambda, \boldsymbol{a}}|^2\}u,$$

where $u \in C^\infty(\boldsymbol{R}^D; \boldsymbol{R})$, Δ is the Laplacian on $\boldsymbol{R}^D$, and

$$d^*\Theta^{\lambda, \boldsymbol{a}} = \sum_{\alpha=1}^{D} \frac{\partial \Theta^{\lambda, \boldsymbol{a}}_\alpha}{\partial x^\alpha}.$$

As is well-known, $p_1^\lambda(x_0,x_1)$ is the value of the heat kernel corresponding to $S^{\lambda,\boldsymbol{a}}$ at time 1.

(ii) According to the previous remark, the assumptions (A.1) and (A.2) are of interest from the point of view of the gauge invariance of Schrödinger operators.

(iii) Let $N=0$ and $v^0\equiv 0$. Suppose (A.1) and (A.2). Due to the gauge invariance, without loss of generality, we may assume that each component θ_α^0 of θ^0 is a polynomial on $\boldsymbol{R}^D$ (also see the proof of Theorem 1.1). Decomposing θ_α^0 as $\theta_\alpha^0(x_0+*)=\sum_{i=0}^L\theta_\alpha^{(i)}$, where $L=\max\{\deg\theta_\alpha^0 : 1\le\alpha\le D\}$ and $\theta_\alpha^{(i)}$ is a homogeneous polynomial of order i if $i\le\deg\theta_\alpha^0$ and equal to 0 if $i>\deg\theta_\alpha^0$, we then have

$$p_1^\lambda(x_0,x_1)=e^{\sqrt{-1}\lambda\langle\theta^{(0)},x_1-x_0\rangle}\times\int_W\exp\left[\sqrt{-1}\lambda\left(\sum_{i=1}^L\int_0^1\langle\theta^{(i)}(w(t)),\circ dw(t)\rangle\right)\right]\delta_{x_1-x_0}(w(1))\mu(dw).$$

Intuitively speaking, each $\int_0^1\langle\theta^{(i)}(w(t)),\circ dw(t)\rangle$ is a homogeneous polynomial of order $i+1$ on W (cf. [**28**]). Moreover, $d\theta^0(x_0)=d\theta^{(1)}(0)$. Thus the assumption (A.2) may be thought of as a condition which indicates that the asymptotic behaviour is dominated by the quadratic term $\int_0^1\langle\theta^{(1)}(w(t)),\circ dw(t)\rangle$ of the phase function $\sum_{i=1}^L\int_0^1\langle\theta^{(i)}(w(t)),\circ dw(t)\rangle$. Such a domination by the quadratic term of the phase function is one of the key ingredients to achieve a principle of stationary phase in $\boldsymbol{R}^n$ ([**8**]).

(iv) We shall give an explanation about the rather complicated forms of the phase function $q_{x_0}^\lambda[\Theta,V,\boldsymbol{a}]$ and the corresponding 1 form $\Theta^{\lambda,\boldsymbol{a}}$. To do this, consider a long time asymptotic behavior of stochastic oscillatory integral

$$J(T):=\int_\Omega\exp\left[\sqrt{-1}\int_0^T\langle\theta^0(x+b(t)),\circ db(t)\rangle\right]\delta_x(x+b(T))\,dP,$$

where $b(t)$ is a D-dimensional Brownian motion on a probability space $(\Omega,\mathscr{F},P)$ and $x\in\boldsymbol{R}^D$. Suppose that every component θ_α^0 of θ^0 is a polynomial, and decompose them as in the previous paragraph. Then, by the space-time scaling property of the Brownian motion, we have

$$J(T)=T^{-D/2}\int_W\exp\left[\sqrt{-1}\sum_{i=0}^L T^{(i+1)/2}\int_0^1\langle\theta^{(i)}(w(t)),\circ dw(t)\rangle\right]\delta_0(w(1))\mu(dw).$$

Thus such stochastic oscillatory integrals as discussed in the theorem appear naturally in a study of long time asymptotic behavior of heat kernels associated with Schrödinger operators with magnetic fields. The theorem may be regarded as a first step toward the study in such a direction via the complex analysis on the Wiener space W introduced in [**23**], [**29**].

(v) Consider the case where $N = 0$. If $v^0 = 0$, then $p_1^\lambda(x_0, x_1)$ is the heat kernel at time $t = 1$ corresponding to the Schrödinger operator with vector potential $\lambda\theta^0$. In this case, upper estimations of the heat kernel is closely related to asymptotic behaviors of the infimum of the spectra of the Schrödinger operator, and, via a precise analytic estimation of the asymptotic behavior of its infimum of the spectra, Ueki [**30**], [**31**] obtained much sharper exponential decay than ours. When v^0 is general, assuming the uniform positivity of $d\theta^0$, Prat [**25**] obtained a precise estimation of derivatives of the distribution of $\int_0^T \langle \theta^0(w(t)), dw(t)\rangle + \int_0^T v(w(t))\,dt$ on $\boldsymbol{R}$ under $\delta_y(x + w(t))\mu(dw)$ and showed that the distribution admits an analytic density function with respect to the Lebesgue measure. Theorem 1.1 implies that the distribution of $\int_0^T \langle \theta^0(w(t)), dw(t)\rangle + \int_0^T v(w(t))\,dt$ possesses a density function of the Gevrey class of order $K + 5$.

2. General estimation on a pinned Wiener space.

In this section, we develop a general scheme to estimate asymptotic behaviors of stochastic oscillatory integrals on a pinned Wiener space, which is an extension of the estimation obtained in [**29**]. To do this, let (W_0, H_0, μ_0) be a pinned Wiener space;

$$W_0 = \{w \in W : w(1) = 0\}, \quad H_0 = H \cap W_0$$

and μ_0 is the pinned Wiener measure on W_0. H_0 is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{H_0}$ inherited from H.

We shall review a definition of analytic functions on W_0. For a separable Hilbert space E, $n \in \boldsymbol{N}$, and $p \in (1, \infty)$, we denote by $\boldsymbol{D}^{n,p}(W_0; E)$ a Sobolev space of E-valued n-times differentiable functions on W_0 in the sense of the Malliavin calculus, whose derivatives of order up to n are all p-th integrable with respect to μ_0. The m-th Malliavin gradient of $F \in \boldsymbol{D}^{n,p}(W_0; E)$ $(m \le n)$ is denoted by $\nabla_0^m F$. We use ∇_0^* to denote the adjoint operator of ∇_0. For details, see [**22**]. Set

$$\boldsymbol{D}^{\infty,\infty-}(W_0; E) = \bigcap_{n \in \boldsymbol{N}} \bigcap_{p \in (1,\infty)} \boldsymbol{D}^{n,p}(W_0; E).$$

We say $F \in \boldsymbol{D}^{\infty,\infty-}(W_0; \boldsymbol{R})$ is *analytic* ($F \in C^\omega(W_0)$ in notation) if there is $p \in (1, \infty)$ such that

$$\sum_{n=0}^\infty \frac{r^n}{n!} \|\nabla_0^n F\|_{L^p(\mu_0; H_0^{\otimes n})} < \infty, \quad \text{for every } r > 0,$$

where $H_0^{\otimes n}$ is a Hilbert space of n-linear mappings of H_0^n to $\boldsymbol{R}$ of Hilbert-Schmidt type, and $\|G\|_{L^p(\mu_0;E)}$ is the L^p-norm of $G : W_0 \to E$ with respect to μ_0. For properties of analytic functions on W_0, see [**23**], [**28**]. For $r > 0$, $j \in \boldsymbol{Z}_+$, $N \in \boldsymbol{Z}_+$, $\phi \in C^\omega(W_0)$, and $q = (q_0, \dots, q_N) \in (C^\omega(W_0))^{N+1}$, we define random variables

$$M_j[\phi, r] = \sum_{n=j}^{\infty} \frac{r^n}{n!} \|\nabla_0^n \phi\|_{H_0^{\otimes n}}^2, \quad \text{and}$$

$$L_j[q, r] = \sum_{i=0}^{N} \{M_{j+1}[q_i, r] + M_{j+1}[\nabla_0^* \nabla_0 q_i, r]\} + \left(1 + \sum_{i=0}^{N} (\nabla_0^* \nabla_0 q_i)^2\right)^{1/2}.$$

Set

$$\boldsymbol{Q}(N, r) = \left\{ q \in (C^\omega(W_0))^{N+1} : \begin{array}{l} L_0[q, r] \in L^{\infty-}(\mu_0; \boldsymbol{R}), \quad \text{and} \\ \exp[L_1[q, r]/L_0[q, 1]^3] \in L^{0+}(\mu_0; \boldsymbol{R}) \end{array} \right\},$$

$$\boldsymbol{\Psi}(r) = \{\psi \in C^\omega(W_0) : M_0[\psi, r] \in L^{1+}(\mu_0; \boldsymbol{R})\},$$

$$A(N) = \{z = (z^0, \dots, z^N) \in \boldsymbol{R}^{N+1} : 1 \le |z|^2 \le 2\},$$

where $L^{\infty-}(\mu_0; E) = \bigcap_{p \in (1,\infty)} L^p(\mu_0; E)$ and $L^{p+}(\mu_0; E) = \bigcup_{r \in (p,\infty)} L^r(\mu_0; E)$. For $q \in \boldsymbol{Q}(N, r)$ and $z \in A(N)$, we define $q^{(z)}$ by

$$q^{(z)} = \sum_{i=0}^{N} z^i q_i.$$

We are now ready to state our general estimation, which is an extension of [**29**, Theorem 1.2].

Theorem 2.1. *There exist constants* $C_1, C_2 \ge 0$ *such that, if* $1 < r < e$, $\varepsilon > 0$, $N \in \boldsymbol{Z}_+$, $q \in \boldsymbol{Q}(N, r)$, *and* $\psi \in \boldsymbol{\Psi}(\varepsilon)$, *then there is a* $k_0 \in \boldsymbol{N}$ *so that*

$$(2.1) \quad \left| \int_{W_0} e^{\sqrt{-1}\lambda q^{(z)}} \psi \, d\mu_0 \right| \le C_1 \left(\int_{W_0} \exp\left[C_2 \frac{1}{k} \frac{L_1[q, r]}{L_0[q, 1]^3} \right] M_0\left[\psi, \frac{1}{k}\right] d\mu_0 \right)^{1/2} \times \left(\int_{W_0} \exp\left[-\lambda \frac{\min\{\|\nabla_0 q^{(z)}\|_{H_0}^2, 1\}}{54ekL_0[q, 1]} \right] d\mu_0 \right)^{1/2}$$

for any $z \in A(N)$, $k \ge k_0$ *and* $\lambda > 1$.

Proof. Let $1 < r < e$, $\varepsilon > 0$, $N \in \boldsymbol{Z}_+$, $q \in \boldsymbol{Q}(N, r)$, $\psi \in \boldsymbol{\Psi}(\varepsilon)$, and $z \in A(N)$. It is easily seen that

$$M_1[q^{(z)}, 1] \le 2 \sum_{i=0}^{N} M_1[q_i, 1].$$

Hence

$$M := 2eL_0[q, 1] \ge e(1 + M_1[q^{(z)}, 1]). \tag{2.2}$$

Due to [**23**, Lemma 8.5], we have, for $\phi \in \boldsymbol{D}^{\infty, \infty-}(W_0; \boldsymbol{R})$ and $r > 1$,

$$\|\nabla_0 M_j[\phi, 1]\|_{H_0} \le 2\left(\frac{1}{\log r}\right)^{1/2} M_j[\phi, 1]^{1/2} M_{j+1}[\phi, r]^{1/2}.$$

Moreover, it is easily seen that

$$\left\| \nabla_0 \left(\left\{ 1 + \sum_{i=0}^{N} (\nabla_0^* \nabla_0 q_i)^2 \right\}^{1/2} \right) \right\|_{H_0} \le \sum_{i=0}^{N} M_1[\nabla_0^* \nabla_0 q_i, 1]^{1/2}.$$

Since $L_1[q, r] \ge 1$, these estimates yield that

$$\|\nabla_0 L_0[q, 1]\|_{H_0} \le 5(N+1)\left(\frac{1}{\log r}\right)^{1/2} L_0[q, 1]^{1/2} L_1[q, r]^{1/2}.$$

Hence

$$\frac{\|\nabla_0 M\|_{H_0}^2}{M^4} \le \frac{25(N+1)^2}{4e^2 \log r} \frac{L_1[q, r]}{L_0[q, 1]^3}. \tag{2.3}$$

Moreover it holds that

$$\frac{|\nabla_0^* \nabla_0 q^{(z)}|}{M} \le \frac{|z| \left\{ 1 + \sum_{i=0}^{N} (\nabla_0^* \nabla_0 q_i)^2 \right\}^{1/2}}{2eL_0[q, 1]} \le 1. \tag{2.4}$$

By virtue of (2.2)–(2.4) and [**29**, Theorem 1.2], we can find $k_0 \in \boldsymbol{N}$ such that each $k \ge k_0$ possesses $\Psi_k \in L^2(\mu_0; \boldsymbol{R})$ satisfying that

$$\begin{aligned} &\left| \int_{W_0} e^{\sqrt{-1}\lambda q^{(z)}} \psi \, d\mu_0 \right| \\ &\quad \le \left(\int_{W_0} \Psi_k^2 \, d\mu_0 \right)^{1/2} \left(\int_{W_0} \exp\left[-\lambda \frac{\min\{\|\nabla_0 q^{(z)}\|_{H_0}^2, 1\}}{54ekL_0[q, 1]} \right] d\mu_0 \right)^{1/2} \end{aligned} \tag{2.5}$$

for any $\lambda > 1$. While Theorem 1.2 in [**29**] does not state explicitly about dependence of Ψ_k and k_0 on $q^{(z)}$ and ψ, tracing its proof carefully and making use of (2.3) and (2.4), we can find constants $C_3, C_4 \ge 0$, which are independent of q, z, r, ε, N and ψ, such that

$$|\Psi_k|^2 \le C_3 \exp\left[C_4 \frac{1}{k^2} \frac{25(N+1)^2}{4e^2 \log r} \frac{L_1[q,r]}{L_0[q,1]^3}\right] M_0\left[\psi, \frac{1}{k}\right].$$

At the same time, we also notice that what was required of k_0 was only that the RHS of the above inequality is in $L^1(\mu_0; \boldsymbol{R})$ if $k \ge k_0$ (see the proofs of Lemmas 2.16 and 2.19 in [**29**]). Hence k_0 can be taken to be independent of $z \in A(N)$. Thus, we obtain the desired constants $C_1, C_2 \ge 0$ and estimation (2.1). □

In repetition of the argument employed in the proof of [**23**, Scholium 8.1], we can conclude from Theorem 2.1 the following.

Corollary 2.2. *Let $1 < r < e$, $\varepsilon > 0$, $N \in \boldsymbol{Z}_+$, $q \in \boldsymbol{Q}(N,r)$, $\psi \in \boldsymbol{\Psi}(\varepsilon)$ and $A \subset A(N)$. Suppose that, for some $\delta, a, b > 0$, it holds that*

$$\sup_{z \in A} \int_{W_0} \exp[\delta \|\nabla_0 q^{(z)}\|_{H_0}^{-a}]\, d\mu_0 < \infty \quad \textit{and} \quad \int_{W_0} \exp[\delta L_0[q,1]^b]\, d\mu_0 < \infty. \tag{2.6}$$

Then there are $C_1, C_2 > 0$ such that

$$\sup_{z \in A} \left| \int_{W_0} e^{\sqrt{-1}\lambda q^{(z)}} \psi\, d\mu_0 \right| \le C_1 \exp[-C_2 |\lambda|^\gamma] \quad \textit{for every } \lambda \in \boldsymbol{R}, \tag{2.7}$$

where $\gamma = ab/(a + ab + 2b)$.

As an application of the corollary, we obtain

Corollary 2.3. *Let $N \in \boldsymbol{Z}_+$, $\varepsilon > 0$, $q = (q_0, \dots, q_N) \in \boldsymbol{D}^{\infty,\infty-}(W_0; \boldsymbol{R}^{N+1})$, $\psi \in \boldsymbol{\Psi}(\varepsilon)$, and $A \subset A(N)$. Suppose that there is $K \in \boldsymbol{N}$ such that $\nabla_0^K q_i = 0$, $0 \le i \le N$, and that (2.6) holds for some $\delta, a, b > 0$. Then, (2.7) holds.*

Proof. On account of the commutation rule

$$\nabla_0 \nabla_0^* = \nabla_0^* \nabla_0 + I \quad \text{on } \boldsymbol{D}^{\infty,\infty-}(W_0; H \otimes E),$$

E being a real separable Hilbert space, we have

$$\nabla_0^K (\nabla_0^* \nabla_0 q_i) = 0, \quad i = 0, \dots, N.$$

It is then straightforward to see that

$$M_0[q_i, r], M_0[\nabla_0^* \nabla_0 q_i, r] \in L^{\infty-}(\mu_0; \boldsymbol{R}), \quad L_1[q,r] \le r^{K-1} L_0[q,1] \quad r > 1.$$

Since $L_0[q,1] \ge 1$, this implies that

$$q \in \bigcap_{r>1} \boldsymbol{Q}(N,r).$$

The assertion then follows from Corollary 2.2. □

3. Calculus on classical Wiener spaces.

In this section, we shall develop some estimations on the classical Wiener space (W, H, μ) which will be used to show Theorem 1.1.

For a Hilbert space E, we denote by $\boldsymbol{D}^{n,p}(W; E)$ a Sobolev space of E-valued n-times differentiable p-th integrable functions on W in the sense of the Malliavin calculus with p-th integrable derivatives of all orders up to n. The m-th Malliavin derivative of $F \in \boldsymbol{D}^{n,p}(W; E)$ is denoted by $\nabla^m F$. Let P_0 be the orthogonal projection of H onto H_0. Through an inclusion $W_0 \subset W$, we can identify $P_0 \circ \nabla$ on W with ∇_0 on W_0. Namely, every $F \in \boldsymbol{D}^{\infty, \infty-}(W; E)$ admits a quasi-continuous version (cf. [**22**]), which can be evaluated on W_0 while W_0 is a μ-null set. If we write again F for the restriction of the quasi-continuous version to W_0, then $\langle \nabla_0 F, h \rangle = \langle \nabla F, h \rangle = \langle P_0 \circ \nabla F, h \rangle$ for any $h \in H_0$ with μ_0-probability 1. In what follows, we shall also write ∇_0 for $P_0 \circ \nabla$.

We shall introduce some more notations. For $x_0, x_1 \in \boldsymbol{R}^D$, set

$$w_{x_0, x_1}(t) = x_0 + w(t) + t(x_1 - x_0), \quad t \in [0, 1].$$

For $\eta \in C^\infty(\boldsymbol{R}^D; \boldsymbol{R}^D)$ and $1 \le \alpha \le D$, define $H_\alpha^\eta \in C^\infty(\boldsymbol{R}^D; \boldsymbol{R}^D)$ by

$$H_\alpha^\eta = (H_{\alpha\beta}^\eta)_{1 \le \beta \le D}, \quad H_{\alpha\beta}^\eta = \frac{\partial \eta_\alpha}{\partial x^\beta} - \frac{\partial \eta_\beta}{\partial x^\alpha}.$$

Notice that $H_{\alpha\beta}^\eta$'s are coefficients of $-2\, d\eta$, $d\eta$ being the exterior derivative of η;

$$-2\, d\eta = \sum_{\alpha, \beta = 1}^{D} H_{\alpha\beta}^\eta \, dx^\alpha \wedge dx^\beta.$$

Lemma 3.1. *Let $\phi \in C^\infty([0,1] \times \boldsymbol{R}^D; \boldsymbol{R})$. Suppose that, for every $n \in \boldsymbol{Z}_+$, there exist $C_n \ge 0$ and $K_n \in \boldsymbol{Z}_+$ such that*

$$\max\{|\partial_x^I \phi(t,x)| : I = (i_1, \ldots, i_n) \in \{1, \ldots, D\}^n\} \le C_n (1 + |x|)^{K_n} \tag{3.1}$$

for any $(t, x) \in [0,1] \times \boldsymbol{R}^D$, where $\partial_x^I = (\partial/\partial x^{i_1}) \cdots (\partial/\partial x^{i_n})$ and $\max\{\cdots\} = \phi(t,x)$ for $n = 0$. Then, for every $h_1, \ldots, h_n \in H_0$ and $1 \le \alpha \le D$, it holds that

$$\begin{aligned} &\left\langle \nabla_0^n \left(\int_0^1 \phi(t, w(t))\, dt \right), h_1 \otimes \cdots \otimes h_n \right\rangle_{H^{\otimes n}} \\ &\qquad = - \sum_{|I| = n} \int_0^1 \left(\int_0^t \partial_x^I \phi(s, w(s))\, ds \right) \frac{d}{dt} \left(\prod_{j=1}^n h_j^{i_j}(t) \right) dt \end{aligned} \tag{3.2}$$

and

$$(3.3)\qquad \left\langle \nabla_0^n\left(\int_0^1 \phi(t,w(t))\circ dw^\alpha(t)\right), h_1\otimes\cdots\otimes h_n\right\rangle_{H^{\otimes n}}$$
$$= -\sum_{|I|=n}\int_0^1\left(\int_0^t \partial_x^I\phi(s,w(s))\circ dw^\alpha(s)\right)\frac{d}{dt}\left(\prod_{j=1}^n h_j^{i_j}(t)\right)dt$$
$$+\sum_{m=1}^n\sum_{|I|=n-1}\int_0^1 \partial_x^I\phi(t,w(t))\left(\prod_{j=1}^{n-1} h_{k(m,j)}^{i_j}(t)\right)\dot h_m^\alpha(t)\,dt,$$

where $|I|=n$ *for* $I\in\{1,\dots,D\}^n$, *and* $k(m,j)=j$ *if* $j\le m-1$ *and* $=j+1$ *if* $j\ge m$.

In particular, if each component η_α *of* $\eta\in C^\infty(\boldsymbol{R}^D;\boldsymbol{R}^D)$ *and* $v\in C^\infty(\boldsymbol{R}^D;\boldsymbol{R})$ *enjoys* (3.1) *with* $\phi(t,x)=\eta_\alpha(x)$ *and* $=v(x)$ *for any* $n\in\boldsymbol{Z}_+$, *then, for every* $x_0,x_1\in\boldsymbol{R}^D$ *and* $h\in H_0$,

$$(3.4)\qquad \left\langle \nabla_0\left(\int_0^1\langle\eta(w_{x_0,x_1}(t)),\circ dw_{x_0,x_1}(t)\rangle+\int_0^1 v(w_{x_0,x_1}(t))\,dt\right),h\right\rangle_H$$
$$=\sum_{\alpha=1}^D\int_0^1\left(\int_0^t\left\{\langle H_\alpha^\eta(w_{x_0,x_1}(s)),\circ dw_{x_0,x_1}(s)\rangle-\frac{\partial v}{\partial x^\alpha}(w_{x_0,x_1}(s))\,ds\right\}\right)\dot h^\alpha(t)\,dt.$$

Finally, if η_α*'s and* v *are all polynomials and* $K=\max\{\deg\eta_\alpha,(\deg v)-1: 1\le\alpha\le D\}$, *then*

$$(3.5)\qquad \nabla_0^{K+2}\left(\int_0^1\langle\eta(w_{x_0,x_1}(t)),\circ dw_{x_0,x_1}(t)\rangle+\int_0^1 v(w_{x_0,x_1}(t))\,dt\right)=0.$$

Proof. Since $\langle\nabla_0 G,h\rangle_H=\langle\nabla G,h\rangle_H$ and $h(1)=0$ for $G\in\boldsymbol{D}^{\infty,\infty-}(W;\boldsymbol{R})$ and $h\in H_0$, (3.2) and (3.3) are obtained as an application of the derivation property of ∇_0 and an integration by parts formula on $[0,1]$. (3.4) follows from these two identities in conjunction with Itô's formula. (3.5) is an immediate consequence of (3.2) and (3.3). □

If $f:\boldsymbol{R}\to\boldsymbol{R}$ has a property that $\sup_{x\in\boldsymbol{R}}|f(x)|/(1+|x|)^L<\infty$ for some $L\in\boldsymbol{Z}_+$, then, for some $p>0$, $\exp[p|f(x)|^{2/L}]$ is integrable with respect to the normal Gaussian measure $(2\pi)^{-1/2}\exp[-x^2/2]$ on $\boldsymbol{R}$. We shall continue such an estimation to the Wiener space W.

Lemma 3.2. *Let* $\phi\in C^\infty([0,1]\times\boldsymbol{R}^D;\boldsymbol{R})$ *be as in Lemma* 3.1. *Then, for each* $n\in\boldsymbol{Z}_+$, *there are* $p_n,p_n'>0$ *such that*

$$(3.6)\qquad \int_W\exp\left[p_n\left\|\nabla_0^n\int_0^1\phi(t,w(t))\,dt\right\|_{H^{\otimes n}}^{2/K_n}\right]\mu(dw)<\infty,$$

$$(3.7)\qquad \int_W \exp\left[p_n'\left\|\nabla_0^n \int_0^1 \phi(t,w(t))\circ dw^\alpha(t)\right\|_{H^{\otimes n}}^{2/K_n'}\right]\mu(dw) < \infty, \quad 1 \le \alpha \le D,$$

where $K_n' = \max\{K_{n-1}, 1+K_n, K_{n+1}\}$ *and* $K_{-1} = 0$.

Proof. We first show (3.6). Since

$$\langle \nabla_0^n \phi(t,w(t)), h_1 \otimes \cdots \otimes h_n\rangle_H = \sum_{|I|=n} \partial_x^I \phi(t,w(t)) \prod_{j=1}^n h_j^{i_j}(t), \quad h_1,\dots,h_n \in H_0,$$

by virtue of (3.1), we obtain that, for $t \in [0,1]$,

$$\|\nabla_0^n \phi(t,w(t))\|_{H^{\otimes n}}^2 \le D^n t^n \sum_{|I|=n} |\partial_x^I \phi(t,w(t))|^2 \le C_n^2 D^{2n}\left(1 + \sup_{s\in[0,1]} |w(s)|\right)^{2K_n}.$$

Hence

$$\left\|\nabla_0^n\left(\int_0^1 \phi(t,w(t))\,dt\right)\right\|_H \le C_n D^n\left(1+\sup_{t\in[0,1]}|w(t)|\right)^{K_n}.$$

Remembering that a D-dimensional Brownian motion $b(t)$ on a probability space $(\Omega,\mathscr{F},P)$ enjoys a property that

$$(3.8)\qquad \exp\left[\sup_{t\in[0,T]} |b(t)|^2\right] \in L^{0+}(P;\boldsymbol{R}) \quad \text{for any } T>0,$$

we can conclude (3.6) from the above estimation.

We now show (3.7). We fix $\alpha \in \{1,\dots,D\}$. On account of (3.1)–(3.3), we have

$$\begin{aligned}
&\left\|\nabla_0^n \int_0^1 \phi(t,w(t))\circ dw^\alpha(t)\right\|_{H^{\otimes n}}^2 \\
&\quad \le 2n^2 D^n \sum_{|I|=n} \int_0^1 \left|\int_0^t \partial_x^I \phi(s,w(s))\circ dw^\alpha(s)\right|^2 t^{(n-1)}\,dt \\
&\qquad + 2n^2 D^{n-1} \sum_{|I|=n-1} \int_0^1 |\partial_x^I \phi(t,w(t))|^2 t^{(n-1)}\,dt \\
&\quad \le 4n^2 D^n \sum_{|I|=n} \int_0^1 \left|\int_0^t \partial_x^I \phi(s,w(s))\,dw^\alpha(s)\right|^2 t^{(n-1)}\,dt \\
&\qquad + n^2 D^n \sum_{|I|=n} \int_0^1 \left|\int_0^t \frac{\partial}{\partial x^\alpha}\partial_x^I \phi(s,w(s))\,ds\right|^2 t^{(n-1)}\,dt
\end{aligned}$$

$$+ 2n^2 D^{n-1} \sum_{|I|=n-1} \int_0^1 |\partial_x^I \phi(t, w(t))|^2 t^{(n-1)}\, dt$$

$$\le 4n^2 D^n \sum_{|I|=n} \sup_{t \in [0,1]} \left| \int_0^t \partial_x^I \phi(s, w(s))\, dw^\alpha(s) \right|^2$$

$$+ n^2 D^{2n} C_{n+1}^2 \left(1 + \sup_{t \in [0,1]} |w(t)| \right)^{2K_{n+1}}$$

$$+ 2n^2 D^{2n-2} C_{n-1}^2 \left(1 + \sup_{t \in [0,1]} |w(t)| \right)^{2K_{n-1}}.$$

Hence, by (3.8), in order to see (3.7), it suffices to show that

$$\exp\left[\sup_{t \in [0,1]} \left| \int_0^t \partial_x^I \phi(s, w(s))\, dw^\alpha(s) \right|^{2/K_n'} \right] \in L^{0+}(\mu; \boldsymbol{R}) \tag{3.9}$$

for any $I \in \{1, \dots, D\}^n$. To do this, fix $I \in \{1, \dots, D\}^n$ arbitrarily. By a standard time change argument, we can find a 1-dimensional Brownian motion $\{B(t)\}_{t \ge 0}$ such that

$$\int_0^t \partial_x^I \phi(s, w(s))\, dw^\alpha(s) = B\left(\int_0^t |\partial_x^I \phi(s, w(s))|^2\, ds \right).$$

By (3.1) and (3.8), we can find p_n'' such that

$$\tilde{C} := \int_W \exp\left[p_n'' \left(\int_0^1 |\partial_x^I \phi(t, w(t))|^2\, dt \right)^{1/K_n} \right] \mu(dw) < \infty,$$

which means that

$$\mu\left(\int_0^1 |\partial_x^I \phi(t, w(t))|^2\, dt \ge k^a \right) \le \tilde{C} \exp[-p_n'' k^{a/K_n}] \quad \text{for any } k \in \boldsymbol{N}, a > 0.$$

This implies that

$$\mu\left(\sup_{t \in [0,1]} \left| \int_0^t \partial_x^I \phi(s, w(s))\, dw^\alpha(s) \right| > k \right)$$

$$\le \tilde{C} \exp[-p_n'' k^{a/K_n}] + \mu\left(\sup_{0 \le t \le k^a} |B(t)| > k \right)$$

$$\le \tilde{C} \exp[-p_n'' k^{a/K_n}] + 2D \exp[-(1/2D) k^{2-a}]$$

(for the last inequality, see [**26**, Theorem 4.2.1]). Setting $a = 2K_n/(1+K_n)$, we arrive at

$$\mu\left(\sup_{t\in[0,1]}\left|\int_0^t \partial_x^I \phi(s, w(s))\, dw^\alpha(s)\right| > k\right)$$
$$\le (\tilde{C} + 2D)\exp[-\min\{p_n'', (1/2D)\}k^{2/(K_n+1)}], \quad k \in \boldsymbol{N},$$

which yields (3.9). □

We shall close this section with an estimation of the reciprocal of the pinned Malliavin covariance of a stochastic line integral.

Lemma 3.3. *Let* $x_0, x_1 \in \boldsymbol{R}^D$, $N \in \boldsymbol{Z}_+$, $\eta^0, \dots, \eta^N \in C^\infty(\boldsymbol{R}^D; \boldsymbol{R}^D)$, *and* $\phi = (\phi^0, \dots, \phi^N) \in C^\infty(\boldsymbol{R}^D; \boldsymbol{R}^{N+1})$. *Suppose that all* α*th components* η_α^i *of* η^i, *and* ϕ^i, $\alpha = 1, \dots, D$, $i = 0, \dots, N$ *satisfy the condition* (3.1) *with* $\phi = \eta_\alpha^i$ *or* $= \phi^i$ *for any* $n \in \boldsymbol{Z}_+$, *and that*

$$d\eta^0(x_0) \neq 0. \tag{3.10}$$

For $\xi = (\xi^1, \dots, \xi^N) \in \boldsymbol{R}^N$, *define* $\eta^{(\xi)} = \eta^0 + \sum_{i=1}^N \xi^i \eta^i$, $\phi^{(\xi)} = \phi^0 + \sum_{i=1}^N \xi^i \phi^i$, *and*

$$F_{x_0,x_1}^\xi(w) = \int_0^1 \langle \eta^{(\xi)}(w_{x_0,x_1}(t)), \circ dw_{x_0,x_1}(t)\rangle + \int_0^1 \phi^{(\xi)}(w_{x_0,x_1}(t))\, dt.$$

Then there are $\varepsilon, \delta > 0$ *such that*

$$\sup_{|\xi|\le\varepsilon} \int_W \exp[\delta \|\nabla_0 F_{x_0,x_1}^\xi(w)\|_H^{-2/3}]\mu(dw) < \infty. \tag{3.11}$$

Proof. Define

$$f_{x_0,x_1;\alpha}^\xi(w;t) = \int_0^t \left\{\langle H_\alpha^{\eta^{(\xi)}}(w_{x_0,x_1}(s)), \circ dw_{x_0,x_1}(s)\rangle - \frac{\partial \phi^{(\xi)}}{\partial x^\alpha}(w_{x_0,x_1}(s))\, ds\right\}.$$

Due to (3.4), we have

$$\|\nabla_0 F_{x_0,x_1}^\xi(w)\|_H^2 = \sum_{\alpha=1}^D v_{[0,1]}(f_{x_0,x_1;\alpha}^\xi(w;\bullet)),$$

where

$$v_{[0,T]}(f) = \frac{1}{T}\int_0^T \left(f(t) - \frac{1}{T}\int_0^T f(s)\, ds\right)^2 dt, \quad f \in L^2([0,T]; \boldsymbol{R}), T > 0.$$

By the assumption (3.10), there exists $\alpha_0 \in \{1, \dots, D\}$ such that $H_{\alpha_0}^{\eta_0}(x_0) \neq 0$. Then, to show (3.11), it suffices to find constants $\varepsilon, C_1, C_2 > 0$ such that

$$\sup_{|\xi| \le \varepsilon} \mu\left[v_{[0,1]}(f^{\xi}_{x_0,x_1;\alpha_0}(w;\bullet)) \le \frac{1}{k} \right] \le C_1 \exp[-C_2 k^{1/3}], \quad k \in \boldsymbol{N}. \tag{3.12}$$

In what follows, for the sake of simplicity, we shall write H^{ξ} and $\hat{\phi}^{\xi}$ for $H_{\alpha_0}^{\eta^{(\xi)}}$ and $\partial \phi^{(\xi)}/\partial x^{\alpha_0}$, respectively. Then $H^{\xi}(x) = (H_1^{\xi}(x), \dots, H_D^{\xi}(x)) \in \boldsymbol{R}^D$ and $\hat{\phi}^{\xi}(x) \in \boldsymbol{R}$. Since $H^0(x_0) = H_{\alpha_0}^{\eta^0}(x_0) \neq 0$ and the mapping $(x, \xi) \mapsto H^{\xi}(x)$ is continuous, there exist $\varepsilon_0, \delta_0, \delta_1 > 0$ so that

$$\inf_{|\xi| \le \varepsilon_0, |x - x_0| \le \delta_0} |H^{\xi}(x)| \ge \delta_1. \tag{3.13}$$

In the remainder of the proof, we shall use C_j, $j = 1, 2, \dots$, to denote constants which are independent of ξ with $|\xi| \le \varepsilon_0$ and $k \in \boldsymbol{N}$.

Define

$$\tau_k = \min\{\inf\{t \ge 0 : |w_{x_0,x_1}(t) - x_0| > \delta_0\}, k^{-2/3}\}, \quad k \in \boldsymbol{N}.$$

Then it holds ([**26**, Theorem 4.2.1]) that

$$\mu(\tau_k < k^{-2/3}) \le 2D \exp\left[-\frac{\delta_0^2}{8D} k^{2/3} \right] \quad \text{for any } k \ge \left(\frac{2|x_1 - x_0|}{\delta_0} \right)^{3/2}. \tag{3.14}$$

Observe that

$$\int_0^T \langle H^{\xi}(w_{x_0,x_1}(t)), \circ dw_{x_0,x_1}(t) \rangle = \int_0^T \langle H^{\xi}(w_{x_0,x_1}(t)), dw(t) \rangle + \int_0^T \tilde{H}^{\xi}_{x_0,x_1}(w_{x_0,x_1}(t))\, dt, \quad T \in [0,1],$$

where $\tilde{H}^{\xi}_{x_0,x_1} = \langle H^{\xi}, x_1 - x_0 \rangle + (1/2)\, d^* H^{\xi}$ (recall that $d^* H^{\xi} = \sum_{\alpha=1}^{D} (\partial H_{\alpha}^{\xi}/\partial x^{\alpha})$). Since

$$v_{[0,T]}(f) \ge \frac{S}{T} v_{[0,S]}(f), \quad 0 < S \le T, \tag{3.15}$$

the above identity implies that

$$v_{[0,1]}^{1/2}(f^{\xi}_{x_0,x_1;\alpha_0}(w;\bullet)) \ge \tau_k^{1/2} \left\{ v_{[0,\tau_k]}^{1/2}\left(\int_0^{\bullet} \langle H^{\xi}(w_{x_0,x_1}(t)), dw(t) \rangle \right) - v_{[0,\tau_k]}^{1/2}\left(\int_0^{\bullet} \{\tilde{H}^{\xi}_{x_0,x_1} - \hat{\phi}^{\xi}\}(w_{x_0,x_1}(t))\, dt \right) \right\}.$$

Notice that

$$(3.16)\qquad C_3 := \sup_{|\xi|\le\varepsilon_0,\,|x-x_0|\le\delta_0}\{|\hat{\phi}^{\xi}(x)| + |H^{\xi}(x)| + |d^*H^{\xi}(x)|\} < \infty$$

to see that

$$v^{1/2}_{[0,\tau_k]}\left(\int_0^{\bullet}\{\tilde{H}^{\xi}_{x_0,x_1} - \hat{\phi}^{\xi}\}(w_{x_0,x_1}(t))\,dt\right) \le C_4k^{-2/3} \quad \text{on } \{\tau_k = k^{-2/3}\},$$

where $C_4 = C_3(|x_1 - x_0| + 2)$. Thus, we have

$$(3.17)\quad \{w : v^{1/2}_{[0,1]}(f^{\xi}_{x_0,x_1;\alpha_0}(w;\bullet)) \le k^{-1}, \tau_k(w) = k^{-2/3}\}$$
$$\subset \left\{w : v^{1/2}_{[0,\tau_k]}\left(\int_0^{\bullet}\langle H^{\xi}(w_{x_0,x_1}(t)), dw(t)\rangle\right) \le (1 + C_4)k^{-2/3}, \tau_k(w) = k^{-2/3}\right\}$$

for any $k \in \boldsymbol{N}$.

By a standard time change argument, we can find a 1-dimensional Brownian motion $\{B^{\xi}(t)\}_{t\ge0}$, which may depend on ξ, such that

$$\int_0^T \langle H^{\xi}(w_{x_0,x_1}(t)), dw(t)\rangle = B^{\xi}\left(\int_0^T |H^{\xi}(w_{x_0,x_1}(t))|^2\,dt\right), \quad T \ge 0.$$

Observe that, if $T > 0$, $\phi \in C^1([0,T];[0,\infty))$, $\phi(0) = 0$, and $\phi' > 0$, then

$$(3.18)\qquad v_{[0,T]}(f(\phi)) \ge \frac{\phi(T)}{T\sup_{0\le t\le T}\phi'(t)}v_{[0,\phi(T)]}(f).$$

Due to (3.13),

$$|H^{\xi}(w_{x_0,x_1}(t))| \ge \delta_1, \quad t \in [0,\tau_k], \quad \text{and} \quad \int_0^{\tau_k}|H^{\xi}(w_{x_0,x_1}(t))|^2\,dt \ge \delta_1^2\tau_k.$$

Hence, by virtue of (3.15), (3.16) and (3.18), we have

$$v_{[0,\tau_k]}\left(\int_0^{\bullet}\langle H^{\xi}(w_{x_0,x_1}(t)), dw(t)\rangle\right)$$
$$\ge \frac{\int_0^{\tau_k}|H^{\xi}(w_{x_0,x_1}(t))|^2\,dt}{\tau_k\sup_{0\le t\le\tau_k}|H^{\xi}(w_{x_0,x_1}(t))|^2}v_{\left[0,\int_0^{\tau_k}|H^{\xi}(w_{x_0,x_1}(t))|^2\,dt\right]}(B^{\xi}) \ge \frac{\delta_1^2}{C_3^2}v_{[0,\delta_1^2k^{-2/3}]}(B^{\xi}).$$

Plugging this into (3.17), we obtain

$$(3.19)\qquad \{w : v^{1/2}_{[0,1]}(f^{\xi}_{x_0,x_1;\alpha_0}(w;\bullet)) \le k^{-1}, \tau_k(w) = k^{-2/3}\}$$
$$\subset \left\{v^{1/2}_{[0,\delta_1^2k^{-2/3}]}(B^{\xi}) \le \frac{C_3(1 + C_4)}{\delta_1}k^{-2/3}\right\}, \quad k \in \boldsymbol{N}.$$

Remember (cf. [**14**, Lemma V.10.6]) that

$$\mu(v_{[0,T]}^{1/2}(B^{\xi}) \leq \varepsilon) \leq \sqrt{2}\exp\left[-\frac{T}{2^7\varepsilon^2}\right], \quad T, \varepsilon > 0.$$

Combining this with (3.19), we obtain a constant $C_5 > 0$ such that

$$\mu[v_{[0,1]}^{1/2}(f_{x_0,x_1;\alpha_0}^{\xi}(w;\bullet)) \leq k^{-1}, \tau_k = k^{-2/3}] \leq \sqrt{2}\exp[-C_5k^{2/3}], \quad k \in \boldsymbol{N},$$

which, in conjunction with (3.14), implies that there are $C_6, C_7 > 0$ such that

$$\mu[v_{[0,1]}^{1/2}(f_{x_0,x_1;\alpha_0}^{\xi}(w;\bullet)) \leq k^{-1}] \leq C_6\exp[-C_7k^{2/3}], \quad k \in \boldsymbol{N}.$$

Thus (3.12) has been verified and the proof completes. □

4. Proof of Theorem 1.1.

In this section, we give a proof of Theorem 1.1. A key ingredient to combine observations made in the preceding two sections is Watanabe's pull-back of tempered distributions via non-degenerate Wiener functionals. Namely, as a measure on W_0, the following identification holds;

$$\delta_0(w(1))\mu(dw) = \frac{1}{\sqrt{2\pi}^D}\mu_0(dw). \tag{4.1}$$

By virtue of the quasi-sure analysis ([**22**]), we can restrict $\Phi \in \boldsymbol{D}^{\infty,\infty-}(W;\boldsymbol{R})$ to W_0 and may think of it as a functional on W_0, even though W_0 is a μ-null set. Namely, Φ admits a quasi-continuous version, which can be evaluated on W_0. Then, because of (4.1), several L^p-estimations on W_0 follow from those on W.

LEMMA 4.1. (i) *There exists* $C \geq 0$ *such that*

$$\int_{W_0} e^{\Phi}\,d\mu_0 \leq C(1 + \|\Phi\|_{D+2,4(D+2)})^{D+2}\left(\int_W e^{4\Phi}\,d\mu\right)^{1/4} \tag{4.2}$$

for any $\Phi \in \boldsymbol{D}^{\infty,\infty-}(W;\boldsymbol{R})$ *with* $\int_W e^{4\Phi}\,d\mu < \infty$, *where*

$$\|\Phi\|_{n,p} = \sum_{m=0}^{n} \|\nabla^m\Phi\|_{L^p(\mu;H^{\otimes m})}.$$

(ii) *Let* $a > 0$. *There is* $C' \geq 0$ *such that*

$$\int_{W_0} e^{\Psi^{-a}}\,d\mu_0 \leq C'(1 + \|\Psi\|_{D+2,4(D+2)})^{D+2}\left(\int_W e^{8\Psi^{-a}}\,d\mu\right)^{1/4} \tag{4.3}$$

for any non-negative $\Psi \in \boldsymbol{D}^{\infty,\infty-}(W;\boldsymbol{R})$ *with* $\int_W e^{8\Psi^{-a}}\,d\mu < \infty$.

Proof. Recall ([**27**]) that $\delta_0(w(1)) \in \boldsymbol{D}^{-(D+2),2}(W;\boldsymbol{R})$, the dual space of $\boldsymbol{D}^{D+2,2}(W;\boldsymbol{R})$. Hence, by (4.1), to show the assertions, it suffices to estimate the $\boldsymbol{D}^{D+2,2}$-norms of e^{Φ} and $e^{\Psi^{-a}}$. To do this, notice that there are universal constants $c^n_{i_1\cdots i_j}$ and $\tilde{c}^n_{k_1k_2i_1\cdots i_j}$ such that

$$\nabla^n e^{\Phi} = \sum_{j=1}^{n} \sum_{i_1+\cdots+i_j=n} c^n_{i_1\cdots i_j}(\nabla^{i_1}\Phi \otimes \cdots \otimes \nabla^{i_j}\Phi)e^{\Phi},$$

$$\nabla^n e^{\Psi^{-a}} = \sum_{j=1}^{n} \sum_{i_1+\cdots+i_j=n} \sum_{k_1,k_2=1}^{n} \tilde{c}^n_{k_1k_2i_1\cdots i_j}(\nabla^{i_1}\Psi \otimes \cdots \otimes \nabla^{i_j}\Psi)\Psi^{-k_1a-k_2}e^{\Psi^{-a}}$$

for any $\Phi, \Psi \in \boldsymbol{D}^{\infty,\infty-}(W;\boldsymbol{R})$ with $\Psi > 0$ μ-a.s.. Since

$$\Psi^{-k_1a-k_2} \le \left(k_1 + \left[\frac{k_2}{a}\right] + 1\right)! e^{\Psi^{-a}},$$

where $[b] = \max\{m \in \boldsymbol{Z} : m \le b\}$, applying Hölder's inequalities repeatedly, we obtain constants $C_1, C_2 \ge 0$ so that

$$\|e^{\Phi}\|_{D+2,2} \le C_1(1 + \|\Phi\|_{D+2,4(D+2)})^{D+2}\|e^{\Phi}\|_{L^4(\mu;\boldsymbol{R})}$$

$$\|e^{\Psi^{-a}}\|_{D+2,2} \le C_2(1 + \|\Psi\|_{D+2,4(D+2)})^{D+2}\|e^{2\Psi^{-a}}\|_{L^4(\mu;\boldsymbol{R})},$$

which completes the proof. □

We proceed to the proof of the theorem.

Proof of Theorem 1.1. Assume (A.1) and (A.2). For each θ^i, define $\tilde{\theta}^i \in C^{\infty}(\boldsymbol{R}^D;\boldsymbol{R}^D)$ by

$$\tilde{\theta}^i(x) = \int_0^1 t(d\theta^i)(tx)[x]\,dt, \quad x \in \boldsymbol{R}^D, \quad i = 0,\ldots,N,$$

where we are thinking of the exterior derivative $d\theta^i(y)$ at $y \in \boldsymbol{R}^D$ as an element of $\boldsymbol{R}^D \otimes \boldsymbol{R}^D$, and $d\theta^i(y)[x]$ denotes its action on x. There are $\psi^i \in C^{\infty}(\boldsymbol{R}^D;\boldsymbol{R})$, $i = 0,\ldots,N$, such that $d\psi^i = \theta^i - \tilde{\theta}^i$. See [**15**, p. 135]. We then have

$$p_1^{\lambda}(x_0,x_1) = \exp\left[\sqrt{-1}\left\{\lambda(\psi^0(x_1) - \psi^0(x_0)) + \sum_{i=1}^{N}\lambda^{a_i}(\psi^i(x_1) - \psi^i(x_0))\right\}\right]$$
$$\times \int_W \exp[\sqrt{-1}\lambda q^{\lambda}_{x_0}[\tilde{\Theta}, V, \boldsymbol{a}](w)]\delta_{x_1}(w_{x_0}(1))\mu(dw),$$

where $\tilde{\Theta} = (\tilde{\theta}^0,\ldots,\tilde{\theta}^N)$. Thus, to have the estimation as described in the theorem, using $\tilde{\theta}^j$'s instead of θ^j's if necessary, we may and will assume that

(A.3) all components of θ^i and v^i, $i = 0, \dots, N$, are polynomials on $\boldsymbol{R}^D$, and $\deg \theta^i_\alpha \le K$, $0 \le i \le N$, $1 \le \alpha \le D$.

In the sequel, we set

$$K' = \max\{\deg \theta^i_\alpha, (\deg v^i) - 1 : i = 0, \dots, N, \alpha = 1, \dots, D\}.$$

Then $K' \le K$.

Applying Cameron-Martin's formula to the shift given by $h(t) = t(x_1 - x_0)$, on account of (4.1), we have

$$p_1^\lambda(x_0, x_1) = \frac{e^{-|x_1 - x_0|^2/2}}{\sqrt{2\pi}^D} \int_{W_0} \exp[\sqrt{-1}\lambda q^\lambda_{x_0, x_1}] \, d\mu_0, \tag{4.4}$$

where

$$q^\lambda_{x_0, x_1}(w) = q_{x_0, x_1; 0}(w) + \sum_{i=1}^N \lambda^{-(1-a_i)} q_{x_0, x_1; i}(w),$$

$$q_{x_0, x_1; i}(w) = \int_0^1 \langle \theta^i(w_{x_0, x_1}(t)), \circ dw_{x_0, x_1}(t) \rangle + \int_0^1 v^i(w_{x_0, x_1}(t)) \, dt,$$

$i = 0, 1, \dots, N$. We shall show the theorem by applying Corollary 2.3 to

$$q_{x_0, x_1} := (q_{x_0, x_1; 0}, \dots, q_{x_0, x_1; N}) \in \boldsymbol{D}^{\infty, \infty-}(W; \boldsymbol{R}^{N+1}).$$

By Lemma 3.1, we see that

$$\nabla_0^{K'+2} q_{x_0, x_1; i} = 0, \quad i = 0, \dots, N. \tag{4.5}$$

Applying Lemma 3.3 with $\eta^i = \theta^i$, $\phi^i = v^i$, $i = 0, \dots, N$, we can find $\lambda_0, \delta > 0$ such that

$$\sum_{i=1}^N \lambda_0^{-2(1-a_i)} < 1 \quad \text{and} \quad \sup_{\lambda \ge \lambda_0} \int_W \exp[\delta \|\nabla_0 q^\lambda_{x_0, x_1}\|_H^{-2/3}] \, d\mu < \infty.$$

Combined with Lemma 4.1, the second estimation yields that

$$\sup_{\lambda \ge \lambda_0} \int_{W_0} \exp\left[\frac{\delta}{8} \|\nabla_0 q^\lambda_{x_0, x_1}\|_{H_0}^{-2/3}\right] d\mu_0 < \infty.$$

Thus, if we set

$$A = \{(1, \lambda^{-(1-a_1)}, \dots, \lambda^{-(1-a_N)}) \in \boldsymbol{R}^{N+1} : \lambda \ge \lambda_0\} \subset A(N),$$

then

$$\sup_{z \in A} \int_{W_0} \exp\left[\frac{\delta}{8} \|\nabla_0 q^{(z)}_{x_0, x_1}\|_{H_0}^{-2/3}\right] d\mu_0 < \infty, \quad \text{where } q^{(z)}_{x_0, x_1} = \sum_{i=0}^N z^i q_{x_0, x_1; i}. \tag{4.6}$$

Denote by ∇^* the adjoint operator of ∇ on (W, H, μ). Since $\nabla_0(\delta_0(w(1)) = 0$, it is then easily seen that

$$\int_W \langle \nabla_0 F, \nabla_0 G \rangle_H \delta_0(w(1))\mu(dw) = \int_W (\nabla^* \nabla_0 F) G \delta_0(w(1))\mu(dw)$$

for any $F, G \in \boldsymbol{D}^{\infty,\infty-}(W; \boldsymbol{R})$. Hence, by (4.1), the adjoint ∇_0^* of ∇_0 on W_0 enjoys that

$$\nabla_0^* \nabla_0 F = \nabla^* \nabla_0 F \quad \mu_0\text{-a.s.}$$

Remember [**24**] that, if $F \in \boldsymbol{D}^{\infty,\infty-}(W; H)$ and $\overset{\bullet}{\overbrace{F(w)}}(t)$ is adapted, then

$$\nabla^* F = \int_0^1 \langle \overset{\bullet}{\overbrace{F(w)}}(t), dw(t) \rangle,$$

and that $\nabla^*(GF) = G\nabla^* F - \langle \nabla G, F \rangle_H$, $G \in \boldsymbol{D}^{\infty,\infty-}(W; \boldsymbol{R})$. Then, by virtue of Lemma 3.1 and a direct computation, we obtain

$$\begin{aligned}
&\nabla^* \nabla_0 q_{x_0,x_1;i}(w) \\
&\quad = -\sum_{\alpha=1}^D \int_0^1 \left\langle \left\{ w^\alpha(t) H_\alpha^{\theta^i}(w_{x_0,x_1}(t)) + t(t-1) \frac{\partial H_\alpha^{\theta^i}}{\partial x^\alpha}(w_{x_0,x_1}(t)) \right\}, \circ dw_{x_0,x_1}(t) \right\rangle \\
&\qquad + \sum_{\alpha=1}^D \int_0^1 \left(w^\alpha(t) \frac{\partial v^i}{\partial x^\alpha}(w_{x_0,x_1}(t)) + t(t-1) \frac{\partial^2 v^i}{\partial (x^\alpha)^2}(w_{x_0,x_1}(t)) \right. \\
&\qquad\qquad \left. - t H_{\alpha\alpha}^{\theta^i}(w_{x_0,x_1}(t)) \right) dt
\end{aligned}$$

for μ-a.e. $w \in W$, where $\partial H_\alpha^{\theta_i} / \partial x^\alpha = (\partial H_{\alpha\beta}^{\theta^i} / \partial x^\alpha)_{1 \le \beta \le D}$. Hence, due to Lemma 3.2, there is a $p > 0$ such that

$$\int_W \exp[p L_0[q_{x_0,x_1}, 1]^{1/(K'+1)}] \, d\mu < \infty.$$

Since $L_0[q_{x_0,x_1}, 1] \ge 1$, this implies that $L_0[q_{x_0,x_1}, 1]^{1/(K'+1)} \in \boldsymbol{D}^{\infty,\infty-}(W; \boldsymbol{R})$. Then, by Lemma 4.1, this also yields that

$$\int_{W_0} \exp\left[\frac{p}{4} L_0[q_{x_0,x_1}, 1]^{1/(K'+1)}\right] d\mu_0 < \infty. \tag{4.7}$$

It follows from (4.5)–(4.7) that all assumptions in Corollary 2.3 are fulfilled with $q = q_{x_0,x_1}$, $a = 2/3$ and $b = 1/(K'+1)$. Then, applying Corollary 2.3 and recalling that $K' \le K$, we obtain constants $L_1, L_2 > 0$ such that

$$\sup_{z\in A}\left|\int_{W_0}\exp[\sqrt{-1}\lambda q^{(z)}_{x_0,x_1}]\,d\mu_0\right| \le L_1\exp[-L_2|\lambda|^{1/(K+5)}] \quad \text{for every } \lambda\in \boldsymbol{R}.$$

By (4.4) and the very definition of A, we see that

$$|p_1^{\lambda}(x_0,x_1)| \le \frac{1}{\sqrt{2\pi}^D} \quad \text{for any } \lambda>0, \quad \text{and}$$

$$|p_1^{\lambda}(x_0,x_1)| \le L_1\exp[-L_2\lambda^{1/(K+5)}] \quad \text{for every } \lambda\ge\lambda_0.$$

We therefore arrive at (1.1). □

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