J. Math. Soc. Japan
Vol. 54, No. 1, 2002

Hilbert C^*-bimodules and continuous Cuntz-Krieger algebras

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(Received Sept. 21, 1998)
(Revised Jun. 15, 2000)

Abstract. We consider certain correspondences on disjoint unions Ω of circles which naturally give Hilbert C^*-bimodules X over circle algebras A. The bimodules X generate C^*-algebras $\mathcal{O}_X$ which are isomorphic to a continuous version of Cuntz-Krieger algebras introduced by Deaconu using groupoid method. We study the simplicity and the ideal structure of the algebras under some conditions using (I)-freeness and (II)-freeness previously discussed by the authors. More precisely, we have a bijective correspondence between the set of closed two sided ideals of $\mathcal{O}_X$ and saturated hereditary open subsets of Ω. We also note that a formula of K-groups given by Deaconu is given without any minimality condition by just applying Pimsner's result.

1. Introduction.

Pimsner [**Pim**] introduced C^*-algebras $\mathcal{O}_X$ for Hilbert C^*-bimodules X, which are a generalization of both Cuntz-Krieger algebras and crossed products by the integers. He showed a universality of the algebras and gave a 6-term exact sequence of K-theory. Katayama also considered C^*-algebras $\mathcal{O}_X$ for inclusions of algebras in [**Ka**]. Kajiwara-Pinzari-Watatani [**KPW1**] studied the simplicity and the ideal structure of the C^*-algebras $\mathcal{O}_X$ for finitely generated bimodules X. Kajiwara-Pinzari-Watatani [**KPW2**] constructed bimodules which provide countably generated Cuntz-Krieger algebras to study their ideal structure and compare with a groupoid approach by Kumjian, Pask, Raeburn and Renault [**KPR**], [**KPRR**].

In [**D1**] Deaconu considered groupoids for self-coverings that generalize the Cuntz groupoid in Renault [**Re**] and studied the groupoid C^*-algebras. In [**D2**] he introduced a continuous version of Cuntz-Krieger algebras using the groupoid method for special embeddings of circle algebras A. The corresponding space of paths is a generalized solenoid and the resulting algebra is related to the paper [**B**] by Brenken. He showed a criterion of simplicity and a formula for K-theory under some minimality condition. In [**De**] Delaroche studied purely infinite simple C^*-algebras arising from general dynamical systems.

2000 *Mathematics Subject Classification.* Primary 46L08, 46L05, 46L80; Secondary 46L55.

Key Words and Phrases. Hilbert C^*-bimodules, C^*-algebras, K-theory.

The authors are partially supported by the Grant-in-aid for Scientific Research, Japan Society for the Promotion of Science.

In this paper we consider certain correspondences on disjoint unions Ω of circles which naturally give rise to Hilbert C^*-bimodules X over circle algebras A. The bimodules X generate C^*-algebras $\mathcal{O}_X$ which are isomorphic to continuous Cuntz-Krieger algebras considered by Deaconu. In fact we can regard the above special embeddings of circle algebras as left module actions $\phi : A \to \mathcal{L}_A(X_A)$. We study the simplicity and the ideal structure of the algebras $\mathcal{O}_X$ under some conditions using (I)-freeness and (II)-freeness defined in [**KPW1**]. More precisely, we have a bijective correspondence between the set of closed two sided ideals of $\mathcal{O}_X$ and saturated hereditary open subsets of Ω. We also note that a formula of K-groups in [**D2**] is given without any minimality condition by just applying Pimsner's result.

While completing our paper, Deaconu let us know his preprint [**D3**], where he also shows that his continuous Cuntz-Krieger algebra of a continuous graph could be thought as a C^*-algebra considered by Pimsner.

2. Correspondences and bimodules.

Let A be a C^*-algebra. Throughout the paper a Hilbert C^*-bimodule is a right Hilbert A-module X endowed with a non-degenerate isometric *-homomorphism ϕ of A into $\mathcal{L}_A(X_A)$, the algebra of adjointable right Hilbert A-module maps. A finite set $\{u_i\}_i$ of X is called a finite basis if $x = \sum_i u_i(u_i|x)_A$ for $x \in X$. We denote by $\theta_{x,y}$ the "rank one" operators on X defined by $\theta_{x,y}(z) = x(y|z)_A$. The closed linear span of "rank one" operators is denoted by $\mathcal{K}_A(X_A)$ and called the algebra of "compact operators". If the right A-module X has a finite basis, then $\mathcal{L}_A(X_A) = \mathcal{K}_A(X_A)$. We refer for other definitions and basic facts to [**KPW1**].

We start to consider correspondences in general. Let Ω be a compact Hausdorff space. Most Hilbert C^*-bimodules over a commutative C^*-algebra $A = C(\Omega)$ naturally arise from correspondences (i.e. closed subsets $\mathcal{C}$ of $\Omega \times \Omega$) similar to the case of commutative von Neumann algebras as in [**Co**].

Definition 1. We call a pair $(\mathcal{C}, \mu)$ a (multiplicity free) correspondence on a compact Hausdorff space Ω if $\mathcal{C}$ is a (closed) subset of $\Omega \times \Omega$ and $\mu = (\mu^y)_{y\in\Omega}$ is a family of finite regular Borel measure on Ω satisfying the following conditions:

(1) (faithfulness) the support $\operatorname{supp}\mu^y$ of the measure μ^y is the y-section $\mathcal{C}^y = \{x \in \Omega \mid (x, y) \in \mathcal{C}\}$,

(2) (continuity) for any $f \in C(\mathcal{C})$, the map $y \in \Omega \to \int_{\mathcal{C}^y} f(x, y)\, d\mu^y(x) \in \boldsymbol{C}$ is continuous.

The vector space $X_0 = C(\mathcal{C})$ is an A-A bimodule by

$$(a \cdot f \cdot b)(x, y) = a(x)f(x, y)b(y)$$

for $a, b \in A$, $f \in X_0$ and $(x, y) \in \mathscr{C}$.

We define an A-valued inner product on X_0 by

$$(f|g)_A(y) = \int_{\mathscr{C}^y} \overline{f(x, y)} g(x, y)\, d\mu^y(x)$$

for $f, g \in X_0$. By the faithfulness and continuity of μ, X_0 is a right pre-Hilbert A-module. We usually assume that for any $x \in \Omega$ (resp. $y \in \Omega$), there exists $y \in \Omega$ (resp. $x \in \Omega$) with $(x, y) \in \mathscr{C}$.

We denote by X the completion X_0. The left A-action on X_0 can be extended to an injective *-homomorphism $\phi : A \to \mathscr{L}_A(X_A)$. Thus we obtain a Hilbert C^*-bimodule X over A (or Hilbert A-A bimodule) from the correspondence $(\mathscr{C}, \mu)$.

Let $\Omega = \bigcup_{i=1}^n \Omega_i$ be a disjoint union of circles $\Omega_i \simeq \boldsymbol{T}$. Let p and q be integers which are not equal to zero. We consider closed subsets $\mathscr{C}^{p,1} = \{(z, z^p) \mid z \in \boldsymbol{T}\} \subset \boldsymbol{T} \times \boldsymbol{T}$ and $\mathscr{C}^{q,-1} = \{(z^q, z) \mid z \in \boldsymbol{T}\} \subset \boldsymbol{T} \times \boldsymbol{T}$.

Definition 2. We call $(\mathscr{C}, \mu)$ a circle correspondence if $\mathscr{C}$ is a closed subset of $\Omega \times \Omega$ and $\mu = (\mu^y)_{y \in \Omega}$ is a family of counting measures satisfying the following condition:

For any $i, j = 1, \dots, n$ the set $\mathscr{C}_{i,j} = \mathscr{C} \cap (\Omega_i \times \Omega_j)$ is of the form $\mathscr{C}^{p,1}$, $\mathscr{C}^{q,-1}$ or $\emptyset$, and μ^y is the counting measure on the y-section $\mathscr{C}^y$.

We denote $(\mathscr{C}, \mu)$ briefly by $\mathscr{C}$.

We study only circle correspondences in this paper. Consider a circle algebra $A = C(\Omega) = \bigoplus_{i=1}^n A_i$, where each $A_i = C(\Omega_i) \cong C(\boldsymbol{T})$. Let $X = C(\mathscr{C})$ and $X_{i,j} = C(\mathscr{C}_{i,j})$. We regard $X_{i,j} = \{0\}$ if $\mathscr{C}_{i,j} = \emptyset$. Then $X = \bigoplus_{i,j} X_{i,j}$ is a pre-Hilbert C^*-bimodule over A obtained from the circle correspondence $(\mathscr{C}, \mu)$. The inner product of X is given explicitly by

$$(f_1|f_2)_A(\omega) = \sum_{\{\omega' \in \Omega;\, (\omega', \omega) \in \mathscr{C}\}} \overline{f_1(\omega', \omega)} f_2(\omega', \omega)$$

for $f_1, f_2 \in X$, $\omega, \omega' \in \Omega$.

We note that $X_{i,j} = C(\mathscr{C}_{i,j})$ is a right Hilbert A_j-module with a left A_i-module action. For $f = (f_{ij})_{ij}$, $g = (g_{ij})_{ij} \in X = \bigoplus_{i,j} X_{i,j}$ and $a = (a_i)_i$, $b = (b_j)_j \in A = \bigoplus_i A_i$, we have

$$(f|g)_A = \left(\sum_i (f_{ij}|g_{ij})_{A_j} \right)_j$$

and

$$af = (a_i f_{ij})_{ij}, \quad fb = (f_{ij} b_j)_{ij}.$$

Proposition 3. *The above X has a finite right basis and becomes a full Hilbert C^*-bimodule over A.*

Proof. We construct a basis for each $X_{i,j} = C(\mathscr{C}_{i,j})$. If $\mathscr{C}_{i,j}$ is of -1 type, $\{1_{\mathscr{C}_{i,j}}\}$ is a natural basis. If $\mathscr{C}_{i,j}$ is of 1 type, we may take $\{g_j\}_{j=0}^{|p|-1}$ where $g(z, z^p) = (1/\sqrt{|p|})z^j$ as in [**Ku**]. We denote by $\{u_k^{i,j}\}_k$ the above basis of $X_{i,j}$ and by $U_k^{i,j}$ the corresponding element embedded in X, that is, $(U_k^{i,j})_{r,s} = \delta_{i,r}\delta_{j,s}u_k^{i,j}$. Then their union $\{U_k^{i,j}; i, j, k\}$ form a finite basis for X. In fact,

$$\sum_{i,j,k} U_k^{i,j}(U_k^{i,j}|f)_A = \sum_{i,j,k} U_k^{i,j}\left(\sum_r (\delta_{i,r}\delta_{j,s}u_k^{i,j} \mid f_{rs})_{A_s}\right)_s$$
$$= \sum_{i,j,k} U_k^{i,j}((\delta_{j,s}u_k^{i,j} \mid f_{is})_{A_s})_s = \left(\sum_k u_k^{i,j}(u_k^{i,j}|f_{ij})_{A_j}\right)_{ij} = (f_{ij})_{ij} = f.$$

Therefore X is already complete, see [**KW1**; Lemma 1.11] for example. It is clear that X is full. $\square$

Definition 4. Consider a circle algebra $A = C(\Omega)$. We say that X is a circle bimodule if $X = C(\mathscr{C})$ for some circle correspondence $\mathscr{C}$.

Proposition 3 shows that X is a Hilbert C^*-bimodule over A without completion.

We recall the C^*-algebra $\mathcal{O}_X$ generated by a bimodule X introduced by Pimsner [**Pim**]. Assume that the Hilbert C^*-bimodule X over A is full and has a finite right A-basis $\{u_i\}_i$ and A is unital. Then $\mathcal{O}_X$ is the universal C^*-algebra generated by $\{S_x; x \in X\}$ satisfying the relations $S_x^*S_y = (x|y)_A$, $S_{xa} = S_x a$, $aS_x = S_{\phi(a)x}$, and $\sum_i S_{u_i}S_{u_i}^* = I$ for $a \in A$ and $x \in X$. The gauge action β of $\boldsymbol{T}$ on $\mathcal{O}_X$ is given by $\beta_z(S_x) = zS_x$ for $z \in \boldsymbol{T}$. The fixed point algebra under the gauge action is denoted by $\mathscr{F}_\infty^{(0)}$ and is identified with the inductive limit $\varinjlim \mathscr{K}(X^{\otimes n})$. In particular if X is a circle bimodule, then we call $\mathscr{F}_\infty^{(0)}$ the "AT-part" of $\mathcal{O}_X$. Actually we see that $\mathscr{F}_\infty^{(0)}$ is an AT-algebra. We denote by $\mathcal{O}_X^0$ the *-subalgebra of $\mathcal{O}_X$ generated algebraically by $\{S_x; x \in X\}$.

Let $u_1, \dots, u_n$ be a finite basis of X_A. We define a completely positive map σ on $\mathcal{O}_X$ by $\sigma(T) = \sum_{i=1}^n u_i T u_i^*$ for $T \in \mathcal{O}_X$.

We recall the following facts:

Lemma 5 ([**KPW1**]). (i) *The restriction of σ to $A' \cap \mathcal{O}_X$ is a *-homomorphism and does not depend on the choice of the basis.*

(ii) *For $T \in A' \cap \mathcal{O}_X$ and $x_1, \dots, x_m \in X$, we have*

$$\sigma^m(T)x_1 \cdots x_m = x_1 \cdots x_m T$$

and $\sigma^m(T)$ commutes with elements in $\mathscr{K}_A(X_A^{\otimes m})$.

Following Deaconu [**D2**], it is useful to regard a disjoint union Ω of circles as the set of vertices and a circle correspondence $\mathscr{C}$ considered in section 1 as the set of edges so that we can draw pictures of the diagram as in [**D2**]. The resulting graph $\mathscr{G}$ is called a continuous graph. We call an element $(\omega_0, \omega_1, \ldots, \omega_r) \in \Omega^{(r+1)}$ a continuous path if $(\omega_{i-1}, \omega_i) \in \mathscr{C}$ for $1 \leq i \leq r$, and r is called the length of the path. A continuous path $(\omega_0, \omega_1, \ldots, \omega_r)$ is called a continuous loop if $\omega_r = \omega_0$.

As in [**D2**] we also shrink each circle to a point and collapse the corresponding edges to get a discrete graph $\mathscr{G}^d$. Thus we consider the set $\Sigma = \{1, 2, \ldots, n\}$ of discrete vertices and the set $E = \{(i, j) \in \Sigma \times \Sigma; \mathscr{C}_{i,j} \neq \phi\}$ of discrete edges. Discrete paths and loops are defined as paths and loops in the discrete graph $\mathscr{G}^d$. We regard the set E of edges as the discrete correspondence $\mathscr{C}^d \subset \Sigma \times \Sigma$ of the original correspondence $\mathscr{C}$.

A discrete path $(i_0, i_1, \ldots, i_r) \in \Sigma^{(r+1)}$ is a discrete loop if $i_r = i_0$ and it is called a simple discrete loop if furthermore the vertices i_k $(0 \leq k \leq r-1)$ are all different. A discrete vertex j is called an exit of a simple discrete loop L if j is not contained in the loop L and there exists a vertex i in the loop L so that (i, j) is an edge.

We denote by $\mathscr{C}_r$ the set of continuous paths with length r, that is,

$$C_r = \{(\omega_0, \ldots, \omega_r) \in \Omega^{r+1}; (\omega_0, \omega_1) \in \mathscr{C}, \ldots, (\omega_{r-1}, \omega_r) \in \mathscr{C}\}.$$

The set $\mathscr{C}_r$ has the relative topology from Ω^{r+1}. The set $C(\mathscr{C}_r)$ of continuous functions on $\mathscr{C}_r$ has a pre-Hilbert C^*-bimodule structure similar to $X = C(\mathscr{C})$: For $a, b \in A$, $f \in C(\mathscr{C}_r)$,

$$(a \cdot f \cdot b)(\omega_0, \ldots, \omega_r) = a(\omega_0) f(\omega_0, \ldots, \omega_r) b(\omega_r)$$

and for $f, g \in C(\mathscr{C}_r)$, $\omega_r \in \Omega$,

$$(f|g)_A(\omega_r) = \sum_{\{(\omega_0, \ldots, \omega_{r-1}) \in \mathscr{C}_{r-1}; (\omega_0, \ldots, \omega_r) \in \mathscr{C}_r\}} \overline{f(\omega_0, \ldots, \omega_r)} g(\omega_0, \ldots, \omega_r).$$

Lemma 6. *Let X be a circle correspondence. Then the relative tensor product $X_A^{\otimes r}$ is isomorphic to $C(\mathscr{C}_r)$ as Hilbert C^*-bimodules over A.*

Proof. We see that there exists a linear map $\varphi : X_A^{\otimes r} \to C(\mathscr{C}_r)$ such that

$$\varphi(f_1 \otimes f_2 \otimes \cdots \otimes f_r)(\omega_0, \omega_1, \ldots, \omega_r) = f_1(\omega_0, \omega_1) f_2(\omega_1, \omega_2) \cdots f_r(\omega_{r-1}, \omega_r)$$

for $f_1, f_2, \ldots, f_r \in X$. It is easy to show that φ is bi-linear and preserves A-valued inner product. Thus φ is also injective. The only non-trivial part is to show that φ is onto. Let $\{u_i\}_i$ be a finite basis of X_A. Then for $f \in C(\mathscr{C})$ we have

$$
\begin{aligned}
f(\omega_0,\omega_1) &= \left(\sum_{i=1}^m u_i(u_i|f)_A\right)(\omega_0,\omega_1) \\
&= \sum_{i=1}^m u_i(\omega_0,\omega_1)(u_i|f)_A(\omega_1) \\
&= \sum_{i=1}^m u_i(\omega_0,\omega_1) \sum_{\{\tilde\omega_0\in\Omega;\,(\tilde\omega_0,\omega_1)\in\mathscr{C}\}} \overline{u_i(\tilde\omega_0,\omega_1)} f(\tilde\omega_0,\omega_1) \\
&= \sum_{\{\tilde\omega_0\in\Omega;\,(\tilde\omega_0,\omega_1)\in\mathscr{C}\}} \left(\sum_{i=1}^m u_i(\omega_0,\omega_1)\overline{u_i(\tilde\omega_0,\omega_1)}\right) f(\tilde\omega_0,\omega_1).
\end{aligned}
$$

If $\omega_1 \in \Omega$ is fixed, the number of $\tilde\omega_0 \in \Omega$ such that $(\tilde\omega_0,\omega_1) \in \mathscr{C}$ is finite. Separating points by functions $f \in C(\mathscr{C})$, we have

$$
\sum_{i=1}^m u_i(\omega_0,\omega_1)\overline{u_i(\tilde\omega_0,\omega_1)} = \delta_{\omega_0,\tilde\omega_0}.
$$

Now for any $f \in C(\mathscr{C}_r)$, we have

$$
\begin{aligned}
&\sum_{i_1,i_2,\dots,i_r} \varphi(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_r})(\varphi(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_r})\,|\,f)_A(\omega_0,\omega_1,\omega_2,\dots,\omega_r) \\
&= \sum_{i_1,i_2,\dots,i_r} \varphi(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_r})(\omega_0,\omega_1,\omega_2,\dots,\omega_r) \\
&\quad \times (\varphi(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_r})\,|\,f)_A(\omega_r) \\
&= \sum_{i_1,i_2,\dots,i_r} u_{i_1}(\omega_0,\omega_1)u_{i_2}(\omega_1,\omega_2)\cdots u_{i_r}(\omega_{r-1}\omega_r) \\
&\quad \times \sum_{\{(\tilde\omega_0,\tilde\omega_1,\dots,\tilde\omega_{r-1})\in\mathscr{C}_{r-1};\,(\tilde\omega_0,\tilde\omega_1,\dots,\tilde\omega_{r-1},\omega_r)\in\mathscr{C}_r\}} \overline{u_{i_1}(\tilde\omega_0,\tilde\omega_1)u_{i_2}(\tilde\omega_1,\tilde\omega_2)\cdots u_{i_r}(\tilde\omega_{r-1},\omega_r)} \\
&\quad \times f(\tilde\omega_0,\tilde\omega_1,\dots,\tilde\omega_{r-1},\omega_r) \\
&= \sum_{\{(\tilde\omega_0,\tilde\omega_1,\dots,\tilde\omega_{r-1})\in\mathscr{C}_{r-1};\,(\tilde\omega_0,\tilde\omega_1,\dots,\tilde\omega_{r-1},\omega_r)\in\mathscr{C}_r\}} \sum_{i_1,\dots,i_{r-1}} \left(\sum_{i_r} u_{i_r}(\omega_{r-1},\omega_r)\overline{u_{i_r}(\tilde\omega_{r-1},\omega_r)}\right) \\
&\quad \times u_{i_1}(\omega_0,\omega_1)\overline{u_{i_1}(\tilde\omega_0,\tilde\omega_1)}\cdots u_{i_{r-1}}(\omega_{r-2},\omega_{r-1})\overline{u_{i_{r-1}}(\tilde\omega_{r-2},\tilde\omega_{r-1})} f(\tilde\omega_0,\dots,\tilde\omega_{r-1},\omega_r) \\
&= f(\omega_1,\omega_2,\dots,\omega_r).
\end{aligned}
$$

Thus $\{\varphi(u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_r});\, i_1,\dots,i_r\}$ constitutes a basis of the Hilbert right module $C(\mathscr{C}_r)$. This implies that φ is surjective. □

Consider the commutative C^*-algebra $D_X^{r-1} = C^*(A, \sigma(A), \sigma^2(A), \dots, \sigma^{r-1}(A))$ generated by $\sigma^k(A)$, $k = 0, \dots, r-1$.

Lemma 7. *Let X be a circle bimodule. Then there exists an isomorphism $\psi_{r-1} : D_X^{r-1} \to C(\mathscr{C}_{r-1})$ as C^*-algebras such that*

$$(\psi_{r-1}(a_1\sigma(a_2)\cdots\sigma^{r-1}(a_r)))(\omega_0, \dots, \omega_{r-1}) = a_1(\omega_0)a_2(\omega_1)\cdots a_r(\omega_{r-1}).$$

Proof. For $a_1, a_2, \dots, a_r \in A$ and $f_1 \otimes f_2 \otimes \cdots \otimes f_r \in X \otimes_A X \otimes_A \cdots \otimes_A X$, we have

$$a_1\sigma(a_2)\cdots\sigma^{r-1}(a_r)(f_1 \otimes \cdots \otimes f_r) = a_1 f_1 \otimes a_2 f_2 \otimes \cdots \otimes a_r f_r.$$

Consider a bimodule isometry $\varphi : X_A^{\otimes r} \to C(\mathscr{C}_r)$ constructed in Lemma 6. For $T \in D_X^{r-1}$, we put $\pi(T) = \varphi T \varphi^{-1}$. Then $\pi : D_X^{r-1} \to \mathscr{L}_A(C(\mathscr{C}_r)_A)$ is an injective *-homomorphism. For $f \in C(\mathscr{C}_r)$ we have

$$(\pi(a_1\sigma(a_2)\cdots\sigma^{r-1}(a_r))f)(\omega_0, \dots, \omega_r) = a_1(\omega_0)a_2(\omega_1)\cdots a_r(\omega_{r-1})f(\omega_0, \dots, \omega_r).$$

Thus $\pi(D_X^{r-1})$ can be identified with a *-subalgebra of the C^*-algebra $C(\mathscr{C}_{r-1})$ acting on the Hilbert C^*-module $C(\mathscr{C}_r)$ by left multiplication. Since $\pi(D_X^{r-1})$ separates the points of $\mathscr{C}_{r-1}$, we have a desired isomorphism $\psi_{r-1} : D_X^{r-1} \to C(\mathscr{C}_{r-1})$ by the Stone Weierstrass Theorem. $\square$

For each $m \in \mathbf{N}$, we define an inclusion of C^*-algebras $b_m : C(\mathscr{C}_m) \to C(\mathscr{C}_{m+1})$ by

$$b_m(f)(\omega_0, \omega_1, \dots, \omega_{m+1}) = f(\omega_0, \omega_1, \dots, \omega_m)$$

for $f \in C(\mathscr{C}_m)$. Let $d_m : D_X^m \to D_X^{m+1}$ be the canonical inclusion. We also note that $\sigma(D_X^m) \subset D_X^{m+1}$.

Lemma 8. *In the above setting, the isomorphisms $\psi_m : D_X^m \to C(\mathscr{C}_m)$ satisfy that $\psi_{m+1} \circ d_m = b_m \circ \psi_m$ and*

$$(\psi_m \circ \sigma \circ \psi_m^{-1}(f))(\omega_0, \omega_1, \dots, \omega_{m+1}) = f(\omega_1, \dots, \omega_{m+1})$$

for $f \in C(\mathscr{C}_m)$.

Proof. Use the above Lemma 7 and recall that σ is actually a shift on the tensor components by [**KPW1**; Lemma 3.3]. $\square$

3. Uniqueness.

In [**CK**], Cuntz and Krieger studied the condition (I) which implies that the relation between generators determines the algebra uniquely. In this section, we consider this uniqueness property for continuous Cuntz-Krieger algebras.

We say that the type of a discrete edge (i, j) in $\mathscr{G}^d$ is (p, t) if $\mathscr{C}_{i,j} = \mathscr{C}^{(p,t)}$, and will denote it by $(p(i, j), t(i, j))$. For a discrete loop $L = (i_0, i_1, \dots, i_{r-1}, i_0)$ in $\mathscr{G}^d$, a rational number $p(L)$ of the loop L is given by

$$p(L) = (p(i_0, i_1))^{t(i_0, i_1)}(p(i_1, i_2))^{t(i_1, i_2)} \cdots (p(i_{r-1}, i_0))^{t(i_{r-1}, i_0)}.$$

DEFINITION 9. Let L be a discrete loop. We call L periodic if $p(L)$ is equal to 1 or -1 and there exists some k such that $p(i_{k-1}, i_k)$ is not equal to 1 nor -1. A discrete loop L is expansive if $|p(L)| > 1$ and is contractive if $|p(L)| < 1$. We say L is trivial if $|p(i_{k-1}, i_k)| = 1$ for any k.

It is useful to investigate the iterated images of subsets of $\boldsymbol{T}$ under the transformation given by the correspondences.

We denote $\tilde{\boldsymbol{Z}}^1 = \boldsymbol{Z} \backslash \{0\}$, $\tilde{\boldsymbol{Z}}^{-1} = \{p \in \boldsymbol{R}; p = q^{-1}, q \in \tilde{\boldsymbol{Z}}^1\}$ and $\tilde{\boldsymbol{Z}} = \tilde{\boldsymbol{Z}}^1 \cup \tilde{\boldsymbol{Z}}^{-1}$. Let $p \in \tilde{\boldsymbol{Z}}$. We put $t(p) = 1$ if $p \in \tilde{\boldsymbol{Z}}^1$ and $t(p) = -1$ if $p \in \tilde{\boldsymbol{Z}}^{-1}$. For $U \subset \boldsymbol{T}$ we put

$$\Phi_p(U) = \{z \in \boldsymbol{T}; (z', z) \in \mathscr{C}^{|p|, t(p)} \quad \text{for some} \quad z' \in U\}.$$

For $V \subset \boldsymbol{R}$, we put

$$\tilde{\Phi}_p(V + \boldsymbol{Z}) = pV + p\boldsymbol{Z} + \boldsymbol{Z}.$$

We define a quotient map π from $\boldsymbol{R}$ to $\boldsymbol{T}$ by $\pi(x) = \exp(2\pi i x)$, and we identify $\boldsymbol{T}$ with $[0, 1)$ or $(0, 1]$ if necessary. Then we have $\pi(\tilde{\Phi}_p(V)) = \Phi_p(\pi(V))$.

Let $\boldsymbol{p} = (p_1, p_2, \dots, p_r)$ where all $p_i \in \tilde{\boldsymbol{Z}}$. We put

$$\Phi_{\boldsymbol{p}} = \Phi_{p_r} \circ \Phi_{p_{r-1}} \circ \cdots \circ \Phi_{p_1},$$

$$\tilde{\Phi}_{\boldsymbol{p}} = \tilde{\Phi}_{p_r} \circ \tilde{\Phi}_{p_{r-1}} \circ \cdots \circ \tilde{\Phi}_{p_1}.$$

We may assume $p_i \neq 1$. If $p_1 p_2 \cdots p_r < 0$, then we have $\Phi_{\boldsymbol{p}} = \Phi_{\boldsymbol{p}'}$ and $\tilde{\Phi}_{\boldsymbol{p}} = \tilde{\Phi}_{\boldsymbol{p}'}$ for $\boldsymbol{p}' = (|p_1|, |p_2|, \dots, |p_r|, -1)$.

We investigate the images of $\Phi_{\boldsymbol{p}}$'s using $\tilde{\Phi}_{\boldsymbol{p}}$'s. Let $V \subset \boldsymbol{R}$ be a subset. Then we have

$$\tilde{\Phi}_p(V + \boldsymbol{Z}) = pV + p\boldsymbol{Z} + \boldsymbol{Z} = pV + \tilde{\Phi}_p(\boldsymbol{Z}).$$

This shows that $\tilde{\Phi}_p(V + \boldsymbol{Z})$ is determined essentially by $\tilde{\Phi}_p(\boldsymbol{Z})$.

LEMMA 10. *We have the following*:

(1) *Suppose all $p_1, \dots, p_m$ are contained in $\tilde{\boldsymbol{Z}}^1$ or all $p_1, \dots, p_m$ are contained in $\tilde{\boldsymbol{Z}}^{-1}$. Then we have*

$$\tilde{\Phi}_{p_1 p_2 \cdots p_m} = \tilde{\Phi}_{p_m} \circ \cdots \circ \tilde{\Phi}_{p_2} \circ \tilde{\Phi}_{p_1}.$$

(2) *Let* α *and* β *be two different prime numbers. If* $p = \alpha^{\pm 1}$ *and* $q = \beta^{\pm 1}$ *then we have*

$$\tilde{\Phi}_p \circ \tilde{\Phi}_q = \tilde{\Phi}_q \circ \tilde{\Phi}_p.$$

(3) *Let* α *be a positive integer. Then we have* $\tilde{\Phi}_\alpha \circ \tilde{\Phi}_{\alpha^{-1}}(V + \boldsymbol{Z}) = V + \boldsymbol{Z}$.

Proof. (1) We have $\tilde{\Phi}_{p_1} \circ \tilde{\Phi}_{p_2}(\boldsymbol{Z}) = p_1 p_2 \boldsymbol{Z} + p_1 \boldsymbol{Z} + \boldsymbol{Z}$, and $\tilde{\Phi}_{p_1 p_2}(\boldsymbol{Z}) = p_1 p_2 \boldsymbol{Z} + \boldsymbol{Z}$. If $p_1, p_2 \in \tilde{\boldsymbol{Z}}^1$, they are equal to $\boldsymbol{Z}$. If $p_1, p_2 \in \tilde{\boldsymbol{Z}}^{-1}$, they are equal to $p_1 p_2 \boldsymbol{Z}$.

(2) For example, let $p = \alpha$ and $q = \beta^{-1}$.

$$\tilde{\Phi}_p \circ \tilde{\Phi}_q(\boldsymbol{Z}) = (\alpha/\beta)\boldsymbol{Z} + \alpha\boldsymbol{Z} + \boldsymbol{Z} = (1/\beta)\boldsymbol{Z}.$$

On the other hand,

$$\tilde{\Phi}_q \circ \tilde{\Phi}_p(\boldsymbol{Z}) = (\alpha/\beta)\boldsymbol{Z} + (1/\beta)\boldsymbol{Z} + \boldsymbol{Z} = (1/\beta)\boldsymbol{Z}.$$

Other cases are similar.

(3) $\tilde{\Phi}_\alpha \circ \tilde{\Phi}_{\alpha^{-1}}(V + \boldsymbol{Z}) = \alpha(\alpha^{-1} V + \alpha^{-1}\boldsymbol{Z} + \boldsymbol{Z}) + \boldsymbol{Z} = V + \boldsymbol{Z} + \alpha\boldsymbol{Z} + \boldsymbol{Z} = V + \boldsymbol{Z}$. □

By the above lemma, for every $\boldsymbol{p}$, there exists $\boldsymbol{p}'$ such that $\tilde{\Phi}_{\boldsymbol{p}} = \tilde{\Phi}_{\boldsymbol{p}'}$ with a form

$$\boldsymbol{p}' = (\overbrace{\alpha_1, \dots, \alpha_1}^{m_1}, \overbrace{\alpha_1^{-1}, \dots, \alpha_1^{-1}}^{l_1}, \dots, \overbrace{\alpha_t, \dots, \alpha_t}^{m_t}, \overbrace{\alpha_t^{-1}, \dots, \alpha_t^{-1}}^{l_t}).$$

In this case we write $\boldsymbol{p}' = ((\alpha_1)^{\times m_1}, (\alpha_1^{-1})^{\times l_1}, \dots, (\alpha_t)^{\times m_t}, (\alpha_t^{-1})^{\times l_t})$ for short. Put $q = \prod_{s=1}^{t} (\alpha_s)^{l_s}$, $n_s = \max(l_s - m_s, 0)$ and $q' = \prod_{s=1}^{t} (\alpha_s)^{n_s}$.

Proposition 11. *Consider a subset* $V \subset \boldsymbol{R}$ *and* $\boldsymbol{p} = (p_1, \dots, p_r)$ *with* $p_i \in \tilde{\boldsymbol{Z}}$ *for* $i = 1, \dots, r$. *Let* q *and* q' *be the integers determined by* $\boldsymbol{p}$ *as above. Take a positive integer* $u > 1$. *Then*

$$\tilde{\Phi}_{\boldsymbol{p}}(V + \boldsymbol{Z}) = (p_1 \cdots p_r)V + \frac{1}{q}\boldsymbol{Z}$$

$$\tilde{\Phi}_{\boldsymbol{p}}^u(V + \boldsymbol{Z}) = (p_1 \cdots p_r)^u V + \frac{1}{(q')^{u-1} q}\boldsymbol{Z}.$$

Proof. It is sufficient to prove $\tilde{\Phi}_{\boldsymbol{p}}(\boldsymbol{Z}) = (1/q)\boldsymbol{Z}$ and $\tilde{\Phi}_{\boldsymbol{p}}^u(\boldsymbol{Z}) = (1/((q')^{u-1} q))\boldsymbol{Z}$. Put $\boldsymbol{p}^s = ((\alpha_s)^{\times m_s}, (\alpha_s^{-1})^{\times l_s})$. Let T be an integer which does not contain α_s as a prime factor. Then for $1 \le s \le t$ we have $\tilde{\Phi}_{\boldsymbol{p}^s}((1/T)\boldsymbol{Z}) = (1/((\alpha_s)^{l_s} T))\boldsymbol{Z}$, and by $\tilde{\Phi}_{\boldsymbol{p}} = \tilde{\Phi}_{\boldsymbol{p}^s} \circ \cdots \circ \tilde{\Phi}_{\boldsymbol{p}^1}$, we have the first equation. We write $q = (\alpha_t)^{l_s} S$, where S does not contain α_s as a prime factor. Then

$\tilde{\Phi}_{\boldsymbol{p}^s}((1/((\alpha_s)^{l_s}S))\boldsymbol{Z}) = (1/((\alpha_s)^{n_s}(\alpha_s)^{l_s}S))\boldsymbol{Z}$. Then we have the second equation. $\square$

We remark that if $|p_1 \cdots p_r| < 1$ then we have $q' \geq 2$.

Corollary 12. *Consider the same situation as above. Let $u > 1$ be an integer. Then we have the following*:

(1) *If $p_1 \cdots p_r = 1$, then we have*

$$\tilde{\Phi}_{\boldsymbol{p}}^u(V + \boldsymbol{Z}) = V + \frac{1}{q}\boldsymbol{Z}.$$

(2) *If $p_1 \cdots p_r = -1$, then we have*

$$\tilde{\Phi}_{\boldsymbol{p}}^u(V + \boldsymbol{Z}) = (-1)^u V + \frac{1}{q}\boldsymbol{Z}.$$

Let (i, j) be a discrete edge. For an open subset U of Ω_i, we put

$$\Phi_{i,j}(U) = \{\omega' \in \Omega_j \mid \text{there exists } \omega \in U \text{ such that } (\omega, \omega') \in \mathscr{C}_{i,j}\}.$$

For a closed subset V of Ω_j, we put

$$\Psi_{i,j}(V) = \{\omega \in \Omega_i \mid \text{there exists } \omega' \in V \text{ such that } (\omega, \omega') \in \mathscr{C}_{i,j}\}.$$

Let $\Sigma = (i_0, i_1, \ldots, i_{s-1}, i_0)$ be a discrete loop. For an open subset $U \in \Omega_{i_0}$ and a closed subset $V \in \Omega_{i_0}$, we put

$$\Phi_\Sigma(U) = \Phi_{i_{s-1}, i_0} \circ \Phi_{i_{s-2}, i_{s-1}} \circ \cdots \circ \Phi_{i_0, i_1}(U)$$
$$\Psi_\Sigma(V) = \Psi_{i_0, i_1} \circ \Psi_{i_1, i_2} \circ \cdots \circ \Psi_{i_{s-1}, i_0}(V).$$

For a discrete loop $L = (i_0, i_1, \ldots, i_{s-1}, i_0)$, we put

$$\boldsymbol{p} = (p(i_0, i_1)^{t(i_0, i_1)}, p(i_1, i_2)^{t(i_1, i_2)}, \ldots, p(i_{s-1}, i_0)^{t(i_{s-1}, i_s)}).$$

We denote by $q(L)$ the integer q determined by $\boldsymbol{p}$ appeared in the preceding discussion. We shall identify the group $\boldsymbol{Z}_q = \boldsymbol{Z}/q\boldsymbol{Z}$ with the finite cyclic subgroup of $\boldsymbol{T}$.

Lemma 13. *Let L be a periodic path and x be an element of $\boldsymbol{T}$. If $p(L) = 1$, then $\Phi_L(x) = x \cdot \boldsymbol{Z}_{q(L)}$. If $p(L) = -1$, then $\Phi_L(x) = x^{-1} \cdot \boldsymbol{Z}_{q(L)}$.*

Proof. This follows from Corollary 12. $\square$

Lemma 14. *Let $L = (i_0, i_1, \ldots, i_{s-1}, i_0)$ be a contractive discrete loop. For any $\omega \in \Omega_0$ and any $p \in \boldsymbol{N}$, there exists a continuous path $(\omega_0, \omega_1, \ldots, \omega_{ps})$ with length ps such that $\omega_0 = \omega$, all ω_{is}'s are different for $1 \leq i \leq p$ and the corre-*

sponding discrete path rotates over L, that is, if $k \equiv j \pmod{s}$, then $\omega_k \in \Omega_{i_j}$ for $k = 0, 1, \ldots, ps$.

Proof. Since L is contractive, $q' \geq 2$ in Proposition 11. Therefore we can choose a desired continuous path. $\square$

Recall that a discrete graph $\mathscr{G}^d$ satisfies the condition (I) in [**CK**] if and only if $\mathscr{G}^d$ satisfies the condition (L) in [**KPR**], that is, there exists no simple loop without exit, since we assume that for each i (resp. j) there exists a j (resp. i) such that $\mathscr{C}_{i,j} \neq \emptyset$. We shall modify the condition (I) in our particular circle situation as follows:

Definition 15. A circle correspondence $\mathscr{C}$ is called to satisfy the condition circle-(I) if each simple discrete loop without exit is not trivial.

Definition 16. Consider the circle correspondence $\mathscr{C}$ and its continuous graph $\mathscr{G}$. Let m and r be positive integers with $r \geq m$. A continuous path $(\omega_0, \omega_1, \ldots, \omega_r)$ with length r is m-aperiodic if the shifted m paths $(\omega_0, \omega_1, \ldots, \omega_{r-m+1})$, $(\omega_1, \omega_2, \ldots, \omega_{r-m+2}), \ldots, (\omega_{m-1}, \omega_{m+1}, \ldots, \omega_r)$ with length $r - m + 1$ are all different.

Definition 17. An element ω in Ω is called an m-aperiodic point if there exists an m-aperiodic path starting at ω.

In the following Lemma 18, we investigate typical examples first to understand the general situation. In these examples, for every positive integer m, the set of m-aperiodic points are dense in Ω.

Lemma 18. *Fix an integer p with $p \geq 2$. Assume that a circle correspondence $\mathscr{C}$ has the form $\mathscr{C} = \mathscr{C}^{(p,1)}$ or $\mathscr{C}^{(p,-1)}$. Then for every $m \in \boldsymbol{N}$, the set of m-aperiodic points is dense in Ω. The same conclusion is also true for a circle correspondence $\mathscr{C} = \bigoplus_{i,j=1}^{2} \mathscr{C}_{i,j}$ with $\mathscr{C}_{1,2} = \mathscr{C}^{(p,-1)}$, $\mathscr{C}_{2,1} = \mathscr{C}^{(p,1)}$ and $\mathscr{C}_{1,1} = \mathscr{C}_{2,2} = \emptyset$.*

Proof. (1) The case that $\mathscr{C} = \mathscr{C}^{(p,1)}$: We define a dense subset $K = \bigcap_{j=1}^{\infty} \{\omega \in \boldsymbol{T} \mid \omega^{pj} \neq 1\}$ of $\boldsymbol{T}$. Then for $\omega \in K$, ω is an m-aperiodic point for every m. In fact $(\omega, \omega^p, \ldots, \omega^{mp})$ is an m-aperiodic path starting at ω.

(2) The case that $\mathscr{C} = \mathscr{C}^{(p,-1)}$: We may identify $\Omega = \boldsymbol{T}$ with $(0, 1]$ by the exponential map. Every point $\omega = \exp(2\pi i\theta) \in \Omega = \boldsymbol{T}$, $(0 < \theta \leq 1)$, is an m-aperiodic point. In fact, $(\exp(2\pi i\theta), \exp(2\pi i\theta/p), \exp(2\pi i\theta/p^2), \ldots, \exp(2\pi i\theta/p^m))$ is an m-aperiodic path starting at ω.

(3) The case that $\mathscr{C} = \bigoplus_{i,j=1}^{2} \mathscr{C}_{i,j}$ with $\mathscr{C}_{1,2} = \mathscr{C}^{(p,-1)}$, $\mathscr{C}_{2,1} = \mathscr{C}^{(p,1)}$ and $\mathscr{C}_{1,1} = \mathscr{C}_{2,2} = \emptyset$: Every point of $\Omega = \Omega_1 \cup \Omega_2$ is an m-aperiodic point. In fact, first let $\omega = \exp(2\pi i\theta) \in \Omega_1$. Consider two different loops $\alpha = (\exp(2\pi i\theta)$, $\exp(2\pi i\theta/p), \exp(2\pi i\theta))$ and $\beta = (\exp(2\pi i\theta), \exp(2\pi i\theta/p + 2\pi i/p), \exp(2\pi i\theta))$. We

may construct an m-aperiodic path starting at $\omega = \exp(2\pi\theta)$ by concatenating the two loops α and β aperiodically. Secondly let $\omega \in \Omega_2$. We have also two different loops starting at ω similarly. By the same argument the proof is complete. $\square$

Lemma 19. *Consider a circle correspondence $\mathscr{C}$. Let L be a simple discrete loop in $\mathscr{G}^d$ which is not trivial. Then for every discrete vertex i in L and for any positive integer m, the set of m-aperiodic points in Ω_i is dense in Ω_i.*

Proof. Since the discrete loop L is not trivial, L is expansive, contractive or periodic. Each correspondence in Lemma 18 provides a typical example of expansive, contractive and periodic loops respectively. But we need to combine and arrange the argument in Lemma 18.

Assume that the length of L is s. Since $L = (i_0, i_1, \dots, i_s)$ is a simple loop, $i_0, \dots, i_{s-1}$ are all different and $i_0 = i_s$. It is enough to show that the set of m-aperiodic points in Ω_{i_0} is dense in Ω_{i_0}. If $m \leq s$, then any point $\omega_0 \in \Omega_{i_0}$ is an m-aperiodic point. In fact we may take any continuous path $P = (\omega_0, \dots, \omega_m)$ such that the corresponding discrete path is L, that is, $\omega_k \in \Omega_{i_k}$ for $k = 0, \dots, m$. Since L is simple, the sources of the shifted m paths with length one are all different and P is m-aperiodic. Thus we may assume that $s < m$. We put $M = \bigcup_{j=1}^{\infty} \{\omega \in \boldsymbol{T} \mid \omega^j = 1\}$ and we also identify M with a corresponding subset in some Ω_i

(1) The case that L is expansive: Any point $\omega_0 \in \Omega_{i_0} \backslash M$ is an m-aperiodic point. In fact we may take any continuous path $P = (\omega_0, \dots, \omega_m)$ such that the corresponding discrete path rotates over L, that is, for $k = 0, \dots, m$, we have $\omega_k \in \Omega_{i_j}$ if $k \equiv j \pmod{s}$ with $0 \leq j \leq s-1$. Consider the shifted m paths $(\omega_0, \omega_1), \dots, (\omega_{m-1}, \omega_m)$ with length one. If $k \not\equiv r \pmod{s}$, then (ω_k, ω_{k+1}) and (ω_r, ω_{r+1}) are different because the source of these paths are in different Ω_i's. Suppose that $k \equiv r \pmod{s}$ and $k \neq r$. Say $r = k + ts$ for some non-zero integer t. Then we have $\omega_r = \omega_k^{p(L)^t} z$ for some $z \in M$ if we identify Ω_k with $\boldsymbol{T}$. Suppose that $\omega_r = \omega_k$. Then $\omega_k^{p(L)^t - 1} \in M$. Since L is expansive, we have $|p(L)| > 1$ and $p(L)^t - 1 \neq 0$. Thus we have that $\omega_k \in M$. This implies that $\omega_o \in M$. This is a contradiction. Hence $\omega_r \neq \omega_k$. Therefore the shifted m paths are all different and P is m-aperiodic.

(2) The case that L is contractive: Let $\omega_0 \in \Omega_{i_0}$. Take an integer p with $ps > m + s$. By Lemma 14, there exists a continuous path $(\omega_0, \omega_1, \dots, \omega_s, \dots, \omega_{is}, \dots, \omega_{2s}, \dots, \omega_{ps})$ such that the corresponding discrete path rotates over L and all ω_{is}'s are different for $1 \leq i \leq p$. Consider a shorted continuous path $P = (\omega_0, \omega_1, \dots, \omega_{m+s})$. We take the shifted m paths $(\omega_0, \dots, \omega_s)$, $(\omega_1, \dots, \omega_{s+1})$, $(\omega_2, \dots, \omega_{s+2}), \dots, (\omega_m, \dots, \omega_{s+m})$. Then the shifted m paths are all different and P is m-aperiodic.

(3) The case that L is periodic: Let $\omega_0 \in \Omega_{i_0}$. Suppose that $p(L) = 1$. Then there exist two different continuous loops α and β such that corresponding discrete paths rotate over L. We construct an m-aperiodic path by concatenating α and β aperiodically. Suppose that $p(L) = -1$. Then there exit two different continuous paths α and β from ω_0 to $\overline{\omega_0}$ such that corresponding discrete path rotates over L. There also exists a continuous path γ from $\overline{\omega_0}$ to ω_0 such that corresponding discrete path rotates over L. We construct an m-aperiodic path by concatenating $\alpha \circ \gamma$ and $\beta \circ \gamma$ aperiodically. $\square$

PROPOSITION 20. *Let $\mathscr{C}$ be a circle correspondence. Suppose that $\mathscr{C}$ satisfies the condition circle-*(I). *Then for every $m \in \boldsymbol{N}$, the set of m-aperiodic points is dense in Ω.*

PROOF. Consider the discrete graph $\mathscr{G}^d$. Let Σ_0 be the set of discrete vertices $j \in \Sigma$ such that there exist at least two different discrete loops based at j. Let Σ_1 be the set of discrete vertices $j \in \Sigma$ such that there exists only one simple loop based at j. Recall an argument of the equivalence between the condition (I)$'$ and the condition (L) in [**KPW2**]. Since we assume that for each discrete vertex u (resp. v) there exists a vertex v (resp. u) such that (u, v) is a discrete edge, for every $i \in \Sigma$ there exists a discrete path from i to some $j \in \Sigma_0 \cup \Sigma_1$. If there exists a discrete path from i to some $j \in \Sigma_0$, then any point $\omega \in \Omega_i$ is an m-aperiodic points. In fact we have an m-aperiodic path starting at ω by concatenating corresponding continuous paths for two discrete loops aperiodically. If otherwise, there exists a discrete path from i to some $j \in \Sigma_1$. Since $\mathscr{C}$ satisfies the condition circle-(I), there exists only one simple discrete loop based at j which is not trivial. Then by Lemma 19, there exists a dense subset K of Ω_j such that there exists an m-aperiodic path starting at $\omega' \in K$. By concatenating finite continuous paths from Ω_i to Ω_j, we get a desired dense subset K' of m-aperiodic points in Ω_i. $\square$

LEMMA 21. *Let $\mathscr{C}$ be a circle correspondence. Suppose that $\mathscr{C}$ satisfies the condition circle-*(I). *For any $n \in \boldsymbol{N}$, any $a = (a_{ij})_{ij} \in A \otimes M_n(\boldsymbol{C})$, any $\varepsilon > 0$ and any $m \in \boldsymbol{N}$, there exist an $r \in \boldsymbol{N}$ and an operator $Q \in D_X^r$ satisfying the following*:

(1) $0 \leq Q \leq I$.

(2) $Q\sigma^j(Q) = 0$ *for* $1 \leq j \leq m - 1$.

(3) $\|(a_{ij})_{ij} \operatorname{diag}(Q^2)\| \geq \|(a_{ij})_{ij}\| - \varepsilon$, *in* $D_X^r \otimes M_n(\boldsymbol{C})$, *where* $\operatorname{diag}(T)$ *is the diagonal matrix whose diagonal elements are all equal to* T.

PROOF. Since $C(\Omega) \otimes M_n(\boldsymbol{C})$ is isomorphic to $C(\Omega, M_n(\boldsymbol{C}))$ as C^*-algebras, we may identify a with a continuous $M_n(\boldsymbol{C})$-valued function on Ω. Therefore for any $\varepsilon > 0$ and any $m \in \boldsymbol{N}$ there exists an m-aperiodic point $\omega_0 \in \Omega$ such that

$$\|(a_{ij})_{ij}(\omega_0)\| \geq \|(a_{ij})_{ij}\| - \varepsilon,$$

since the set of m-aperiodic points is dense in Ω by Proposition 20.

Let $P = (\omega_0, \omega_1, \dots, \omega_r)$ be an m-aperiodic path with length r $(\geq m)$ starting from ω_0. Since the shifted m paths $R_0 = (\omega_0, \omega_1, \dots, \omega_{r-m+1})$, $R_1 = (\omega_1, \omega_2, \dots, \omega_{r-m+2}), \dots, R_{m-1} = (\omega_{m-1}, \omega_{m+1}, \dots, \omega_r)$ with length $r - m + 1$ are all different point in $\mathscr{C}_{r-m+1}$, there exist disjoint open neighborhoods U_i of R_i, $(i = 0, \dots, m-1)$ in $\mathscr{C}_{r-m+1}$. For each $i = 0, \dots, m-1$, define an open neighborhood V_i of $P = (\omega_0, \omega_1, \dots, \omega_r)$ in $\mathscr{C}_r$ by

$$V_i = \{(\mu_0, \mu_1, \dots, \mu_r) \in \mathscr{C}_r; (\mu_i, \mu_{i+1}, \dots, \mu_{r-m+i+1}) \in U_i\}.$$

Consider an open neighborhood $V = \bigcap_{i=0}^{m-1} V_i$ of P. Choose a continuous function $f \in C(\mathscr{C}_r)$ such that $0 \leq f \leq 1$, $f(P) = 1$ and $f(W) = 0$ for $W \in V^c$. Recall the isomorphism $\psi_r : D_X^r \to C(\mathscr{C}_r)$ in Lemma 7. Let $Q = \psi_m^{-1}(f)$. Then the operator Q satisfies (1) and (2) by Lemma 8 and the construction of f. We identify an element $(a_{ij})_{ij} \operatorname{diag}(Q^2) \in D_X^r \otimes M_n(\boldsymbol{C})$ with an element $(a_{ij})_{ij} \operatorname{diag}(f^2) \in C(\mathscr{C}_r, M_n(\boldsymbol{C}))$. Then we have

$$\|((a_{ij})_{ij} \operatorname{diag}(f^2))(P)\| = \|(a_{ij})_{ij}(\omega_0) \operatorname{diag}(f(P)^2)\| \geq \|(a_{ij})_{ij}(\omega_0)\| \geq \|(a_{ij})_{ij}\| - \varepsilon.$$

This implies (3). $\square$

Lemma 22. *Let $\mathscr{C}$ be a circle correspondence. Suppose that $\mathscr{C}$ satisfies the condition circle-*(I). *For any $p \in \boldsymbol{N}$, any $B_0 \in \mathscr{K}_A(X_A^{\otimes p})$, any $\varepsilon > 0$ and any $m \in \boldsymbol{N}$, there exist an $r \in \boldsymbol{N}$ and an operator $Q \in D_X^r$ satisfying* (1), (2) *of Lemma* 21 *and the following condition*:

$$\|\sigma^p(Q^2)B_0\| \geq \|B_0\| - \varepsilon.$$

Proof. Choose $r \in \boldsymbol{N}$ and an operator Q in D_X^r as in Lemma 21. Recall a fact in [**KPW1**] that, for A-A bimodule Z and Y, $T \in \mathscr{K}_A(Y_A)$ and $x_i, y_i \in Z$,

$$\left\|\sum_{i=1}^n \theta_{x_i, y_i} \otimes T\right\| = \|((x_i|x_j)_A)_{i,j}^{1/2}((y_i|y_j)_A)_{i,j}^{1/2} \operatorname{diag}((T^*T)^{1/2})\|.$$

Apply the fact putting $Z = X^{\otimes p}$, $B = \sum_{i=1}^n \theta_{x_i, y_i}$, $Y = X^{\otimes r}$ and $T = Q^2$. $\square$

Definition 23. A Hilbert A-A bimodule X is called to be (I)-free if there exists a dense subset $\mathscr{D} \subset \mathscr{O}_X^0$ such that for each $B \in \mathscr{D}$ with $B = \sum_{j=-m}^m B_j$, $(B_j \in \mathscr{K}(X^{\otimes p+j}, X^{\otimes p}))$, for every $\varepsilon > 0$, there exists a contraction $P \in \mathscr{O}_X$ in a spectral subspace under the gauge action of $\boldsymbol{T}$ on $\mathscr{O}_X$ satisfying the following:

1. For $j \neq 0$, $\|PB_jP^*\| \leq \varepsilon$.
2. $\|PB_0P^*\| \geq \|B_0\| - \varepsilon$.

PROPOSITION 24. *Let $\mathscr{C}$ be a circle correspondence and X the circle bimodule obtained from $\mathscr{C}$. If $\mathscr{C}$ satisfies the condition circle-*(I)*, then the bimodule X is* (I)*-free.*

PROOF. Let $\mathscr{D} = \mathcal{O}_X^0$. Take any $B = \sum_{j=-m}^{m} B_j \in \mathscr{D}$, with $B_j \in \mathscr{K}(X^{\otimes p+j}, X^{\otimes p})$. For B_0 and any ε we choose a positive contraction Q in Lemma 22. Put $P = \sigma^p(Q)$. Since $Q \in \mathcal{O}_X^{\boldsymbol{T}}$, P is also contained in $\mathcal{O}_X^{\boldsymbol{T}}$ which is a spectral subspace of the gauge action. It is clear that P is also a positive contraction.

By Lemma 21(2) and the commutation relation of σ as in [**KPW1**; Lemma 3.2], we have, for $1 \le j \le m$,

$$PB_jP^* = \sigma^p(Q)B_j\sigma^p(Q) = \sigma^p(\sigma^j(Q)Q)B_j = 0.$$

The case that $j < 0$ is similar. Lemma 22 shows that

$$\|PB_0P^*\| = \|\sigma^p(Q^2)B_0\| \ge \|B_0\| - \varepsilon. \qquad \square$$

The following theorem implies that if a circle bimodule $\mathscr{C}$ satisfies the condition circle-(I), then the C^*-algebra $\mathcal{O}_X$ generated by the circle bimodule X is uniquely determined by the commutation relations of the generators $\{S_x; x \in X\}$ such that $S_x^* S_y = (x|y)_A$, $S_{xa} = S_x a$, $aS_x = S_{\phi(a)}x$, and $\sum_i S_{u_i} S_{u_i}^* = I$ for $a \in A$ and $x \in X$.

PROPOSITION 25. *Let X be a Hilbert C^*-bimodule over A which has a finite right basis. Suppose that X is* (I)*-free. Let φ be a unital *-homomorphism of $\mathcal{O}_X$ to a C^*-algebra R. If φ is faithful on A, then φ is faithful on $\mathcal{O}_X$.*

PROOF. Let $\mathscr{D}$ be as in Definition 23, and $B = \sum_{j=-m}^{m} B_j \in \mathscr{D}$ with $B_j \in \mathscr{K}_A(X_A^{\otimes r+j}, X_A^{\otimes r})$. For any ε and, for $B_0 \in \mathscr{K}_A(X_A^{\otimes r})$, choose a contraction P as in Definition 23. Since φ is faithful on A, φ is faithful on $\mathcal{O}_X^{\boldsymbol{T}} = \mathscr{F}_\infty^{(0)}$ as in [**KPW1**; Lemma 2.2]. Therefore we have $\|\varphi(PB_0P^*)\| \ge \|\varphi(B_0)\| - \varepsilon$. Then

$$\|\varphi(B_0)\| \le \|\varphi(PB_0P^*)\| + \varepsilon = \left\| \varphi(PBP^*) - \sum_{j=-m, j\neq 0}^{m} \varphi(PB_jP^*) \right\| + \varepsilon$$

$$\le \|\varphi(P)\varphi(B)\varphi(P^*)\| + 2m\varepsilon + \varepsilon \le \|\varphi(B)\| + (2m+1)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\|\varphi(B_0)\| \le \|\varphi(B)\|.$$

Because $\mathscr{D}$ is dense in $\mathcal{O}_X$, we have for all $B \in \mathcal{O}_X$,

$$\|\varphi(B_0)\| \le \|\varphi(B)\|.$$

Hence there exists a conditional expectation E^{φ} from $\varphi(\mathcal{O}_X)$ onto $\varphi(\mathcal{F}_{\infty}^{(0)})$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\varphi"] \arrow[d, "E^X"'] & \varphi(\mathcal{O}_X) \arrow[d, "E^{\varphi}"] \\ \mathcal{F}_{\infty}^{(0)} \arrow[r, "\varphi"] & \varphi(\mathcal{F}_{\infty}^{(0)}) \end{tikzcd}$$

Since φ is faithful on $\mathcal{F}_{\infty}^{(0)}$, φ is faithful on $\mathcal{O}_X$. $\square$

By Proposition 24 and Proposition 25, we have the following theorem.

THEOREM 26. *Let $\mathscr{C}$ be a circle correspondence and X the circle bimodule obtained from $\mathscr{C}$. Assume that $\mathscr{C}$ satisfies the condition circle-*(I). *Let φ be a unital *-homomorphism of $\mathcal{O}_X$ to a C^*-algebra R. If φ is faithful on A, then φ is faithful on $\mathcal{O}_X$.*

REMARK 3.1. Without the assumption of the condition circle-(I), if φ is faithful on A and we know that there exists a conditional expectation E^{φ} from $\varphi(\mathcal{O}_X)$ to $\varphi(\mathcal{F}_{\infty}^{(0)})$ such that the above diagram commutes, we may conclude that φ is also faithful.

4. Groupoids and bimodules.

We shall compare the construction of Deaconu [**D2**] using groupoid theory and our construction using bimodule theory. We recall his construction in [**D2**]. For $\omega = (\tau_0, \tau_1) \in \mathscr{C}$, we put $r(\omega) = \tau_1$ and $s(\omega) = \tau_0$. Let Ξ be the set of right infinite paths of $\mathscr{C}$ with product topology, which is a compact Hausdorff space. For $\xi = (\omega_1\omega_2\omega_3\cdots)$, put $\sigma(\xi) = (\omega_2\omega_3\omega_4\cdots)$. Then σ is a continuous surjective map on Ξ and it is also a local homeomorphism. We put $s(\xi) = s(\omega_1)$. Let

$$\Gamma = \{(\xi, k, \eta) \in \Xi \times \mathbf{Z} \times \Xi \mid \sigma^{n+k}(\xi) = \sigma^n(\eta) \ \exists N \ \forall n \geq N\}.$$

The range map r and the source map s of Γ are given by

$$r(\xi, k, \eta) = \eta, \quad s(\xi, k, \eta) = \xi.$$

Two elements $\gamma = (\xi, k, \eta)$ and $\gamma' = (\eta', l, \zeta)$ in Γ are composable if and only if $s(\gamma) = r(\gamma')$ and the product is given by

$$(\xi, k, \eta)(\eta, l, \zeta) = (\xi, k + l, \zeta)$$

and the inverse of (ξ, k, η) is given by $(\eta, -k, \xi)$. Then Γ becomes a locally compact topological groupoid whose Haar system is given by the counting measures. We denote by $C_c(\Gamma)$ the set of all continuous functions on Γ with compact supports, and we denote by $C^*(\Gamma)$ the full groupoid C^*-algebra of Γ and denote by $C_r^*(\Gamma)$ the reduced groupoid C^*-algebra of Γ.

PROPOSITION 27. *Let $\mathscr{C}$ be a circle correspondence, X the circle bimodule obtained from $\mathscr{C}$ and Γ the groupoid defined by Deaconu. Then the C^*-algebra $\mathcal{O}_X$ is isomorphic to $C^*(\Gamma)$ and $C_r^*(\Gamma)$.*

PROOF. We shall construct a *-homomorphism ρ of $A = C(\Omega)$ to $C_c(\Gamma)$ and a linear map V of $X = C(\mathscr{C})$ to $C_c(\Gamma)$ as follows:

For ω in $\mathscr{C}$ and ξ in Ξ, we denote by $\omega \cdot \xi$ the concatenation of them if $r(\omega) = s(\xi)$. For $f \in X$ we define $V(f)(\omega \cdot \xi, 1, \xi) = f(\omega)$ and zero for other elements in Γ. For $k \in A$ we define $\rho(k)(\eta, 0, \eta) = a(s(\eta))$ and zero for other elements in Γ.

Then V and ρ satisfy the following relations:

$$V(f)^* * V(g) = \rho((f|g)_A)$$

$$V(f) * \rho(k) = V(f \cdot k)$$

$$\rho(k) * V(f) = V(\phi(k)f)$$

for every f, g in X and k in A.

Let $\{u_i\}_{i=1}^m$ be a finite basis of X_A constructed in Proposition 3. Then we have

$$\sum_{i=1}^m u_i(\omega)\overline{u_i(\tilde{\omega})} = \delta_{\omega,\tilde{\omega}}.$$

By the calculation in $C_{c(\Gamma)}$, for $F \in C_c(\Gamma)$, we have

$$(F * F^*)(\gamma) = \sum_{\gamma_1\gamma_2 = \gamma} F(\gamma_1)F^*(\gamma_2) = \sum_{\gamma_1\gamma_2^{-1} = \gamma} F(\gamma_1)\overline{F(\gamma_2)}.$$

If $F = V(f)$ for $f \in X$, $(F * F^*)(\gamma) = 0$ if $\gamma \neq (\omega_1 \cdot \eta, 0, \omega_2 \cdot \eta)$ for ω_1, ω_2 in $\mathscr{C}$, and

$$(V(f) * V(f)^*)((\omega_1 \cdot \eta, 0, \omega_2 \cdot \eta)) = f(\omega_1)\overline{f(\omega_2)}.$$

Therefore we have

$$\sum_{i=1}^m V(u_i) * V(u_i)^*(\omega_1 \cdot \eta, 0, \omega_2 \cdot \eta) = \delta_{\omega_1,\omega_2}.$$

This shows that $\sum_{i=1}^m V(u_i) * V(u_i)^* = I$.

By the universality of $\mathcal{O}_X$ [**Pim**], there exists a *-homomorphisms φ (resp. φ_r)

of $\mathcal{O}_X$ to $C^*(\Gamma)$ (resp. $C_r^*(\Gamma)$) such that $\varphi(S_f) = V(f)$ and $\varphi(k) = \rho(k)$ for $f \in X$ and $k \in A$. Let $f_1, f_2, \dots, f_r$ and $g_1, g_2, \dots, g_s$ be elements in X.

For non negative integers r, s, the set

$$\{V(f_1)V(f_2)\cdots V(f_r)V(g_s)^*\cdots V(g_2)^*V(g_2)^* \mid f_1, f_2, \dots, f_r, g_1, g_2, \dots, g_s \in X\}$$

is in the set of the functions whose support are contained in

$$\Gamma^{r,s} = \{(\xi, s-r, \eta) \in \Xi \times \mathbf{Z} \times \Xi \mid \sigma^r(\xi) = \sigma^s(\eta)\}$$

and which depends only on the first r edges of ξ and on the first s edges of η. When r, s varies, these elements generate dense *-subalgebra of $C_c(\Gamma)$, which is dense in $C^*(\Gamma)$.

By [**Re**; Proposition 5.1], there exists an action β' of $\boldsymbol{T}$ on $C^*(\Gamma)$ such that $\varphi(\beta_t(T)) = \beta_t'(\varphi(T))$, for $T \in \mathcal{O}_X$, where β is the gauge action of $\boldsymbol{T}$ on $\mathcal{O}_X$. The restriction of φ to $\mathcal{F}_\infty^{(0)}$ is an isomorphism of $\mathcal{F}_\infty^{(0)}$ onto the fixed point subalgebra under β'. There exists a conditional expectation $E^{\beta'}$ from $C^*(\Gamma)$ to the fixed point subalgebra under β' such that $\varphi(E^X(T)) = E^{\beta'}(\varphi(T))$ for $T \in \mathcal{O}_X$. By the usual argument as in the Remark 3.1, φ is an onto isomorphism. Similarly φ_r is also an onto isomorphism of $\mathcal{O}_X$ to $C_r^*(\Gamma)$. $\square$

5. Ideal theory.

We recall that there is a bijective lattice correspondence between the set of closed two sided ideals in A and the set of open subsets of Ω as follows: Let $J \subset A$ be a closed two sided ideal of A. Then $U_J = \Omega \backslash \bigcap_{f \in J} \ker f$ is an open subset of A. Conversely, for an open subset U in Ω, $J_U = \{f \in A \mid f(\omega) = 0$ for $\omega \in \Omega \backslash U\}$ is a closed two sided ideal of A.

DEFINITION 28 ([**KPW2**]). Let X be a Hilbert bimodule over a C^*-algebra A. A closed two sided ideal J of A is called X-invariant if $(x \mid \phi(a)y)_A \in J$ for any x, $y \in X$ and $a \in J$. And J is called X-saturated if $(x \mid \phi(a)y)_A \in J$ for all $x, y \in X$ implies $a \in J$.

DEFINITION 29. Let $\mathscr{C}$ be a circle correspondence. A subset U of Ω is called hereditary if $(\omega, \omega') \in \mathscr{C}$ and $\omega \in U$ implies $\omega' \in U$. A subset U of Ω is called saturated if the condition that all ω' with $(\omega, \omega') \in \mathscr{C}$ are contained in U implies that ω is also contained in U.

LEMMA 30. *Let $\mathscr{C}$ be a circle correspondence and X its circle bimodule. Let J be a closed ideal of A and U the open subset of Ω corresponding to J. Then J is X-invariant and X-saturated if and only if U is hereditary and saturated.*

PROOF. Put $U_i = U \cap \Omega_i$. Assume that J is X-invariant and X-saturated. Take any edge $(\omega_1, \omega_2) \in \mathscr{C}$ with $\omega_1 \in U_i$. Choose $\varphi \in C(\Omega_i)$ satisfying that

$\varphi(\omega_1) \neq 0$ and the support $\operatorname{supp} \varphi$ of φ is contained in U_i. Then $\varphi \in J$. Choose ξ in $C(\mathscr{C}_{i,j})$ such that $\xi(\omega_1, \omega_2) = 1$ and

$$\{\omega \in \Omega; (\omega, \omega_2) \in \operatorname{supp} \xi \cap \mathscr{C}\} = \{\omega_1\}.$$

Since J is X-invariant, we have $(\xi \mid \varphi\xi)_A \in J$. Then we have

$$\begin{aligned}(\xi \mid \varphi\xi)_A(\omega_2) &= \sum_{(\omega, \omega_2)} \overline{\xi(\omega, \omega_2)}\varphi(\omega)\xi(\omega, \omega_2) \\ &= \overline{\xi(\omega_1, \omega_2)}\varphi(\omega_1)\xi(\omega_1, \omega_2) \\ &\neq 0.\end{aligned}$$

This implies that $\omega_2 \in U_j$. Therefore U is hereditary.

Let $\omega_0 \in \Omega_{j_0}$ and $\omega_1, \ldots, \omega_p$ be the all the continuous vertices such that (ω_0, ω_i) is a continuous edge. Assume that each $\omega_i \in U$. We may put $\omega_i \in U_{j_i}$ for $1 \leq i \leq p$. For each i, there exists an open neighborhood W_i of ω_0 such that

$$\{\mu \in \Omega_{j_i};\ \text{there exists } \omega \in W_i \text{ with } (\omega, \mu) \in \mathscr{C}\} \subset U_{j_i}$$

and

$$\{\mu \in \Omega;\ \text{there exists } \omega \in W_i \text{ with } (\omega, \mu) \in \mathscr{C}\} \subset \bigcup_{k=1}^{p} U_{j_k}.$$

Consider an open neighborhood $W = \bigcap_{i=1}^{p} W_i$ of ω_0. Choose $\varphi \in C(\Omega_{j_0})$ such that $\operatorname{supp} \varphi \subset W$, $\varphi \geq 0$ and $\varphi(\omega_0) = 1$. For any ξ and η in X,

$$(\xi \mid \varphi\eta)_A(\mu) = \sum_{\{\omega \in \Omega;\, (\omega, \mu) \in \mathscr{C}\}} \overline{\xi(\omega, \mu)}\varphi(\omega)\eta(\omega, \mu).$$

If $\mu \in \Omega$ satisfies that $(\xi \mid \varphi\eta)_A(\mu) \neq 0$, then there exists $\omega \in \Omega$ such that $(\omega, \mu) \in \mathscr{C}$ and $\varphi(\omega) \neq 0$. Since $\omega \in \operatorname{supp} \varphi \subset W \subset W_i$, we have $\mu \in \bigcup_{k=1}^{p} U_{j_k}$. If $\mu \in \Omega_{j_i}$, then $\mu \in U_{j_i}$. This implies that $(\xi \mid \varphi\eta)_A \in J$. Since J is saturated, we have $\varphi \in J$. Because $\varphi(\omega_0) \neq 0$, $\omega_0 \in U$. Thus U is saturated.

Conversely assume that U is saturated and hereditary. For any $\varphi \in J$, $\xi, \eta \in X$ and $\mu \in \Omega$, we have

$$(\xi \mid \varphi\eta)_A(\mu) = \sum_{\{\omega \in \Omega;\, (\omega, \mu) \in \mathscr{C}\}} \overline{\xi(\omega, \mu)}\varphi(\omega)\eta(\omega, \mu).$$

Suppose that $(\xi \mid \varphi\eta)_A(\mu) \neq 0$. Then there exists $\omega \in \Omega$ such that $(\omega, \mu) \in \mathscr{C}$ and $\varphi(\omega) \neq 0$. Since $\varphi \in J$, ω is in U. Then μ is also in U, because U is hereditary. This shows that for any $\mu \in \Omega \backslash U$ we have $(\xi \mid \varphi\eta)_A(\mu) = 0$. Thus $(\xi \mid \varphi\eta)_A$ is in J. We have shown that J is X-invariant.

Assume that $\varphi \in A$ satisfies that $(\xi \mid \varphi\eta)_A \in J$ for all $\xi, \eta \in X$. For any $\omega \in \Omega \backslash U$ there exists $\mu \in \Omega \backslash U$ with $(\omega, \mu) \in \mathscr{C}$, because U is saturated. Let $\omega_1 = \omega, \omega_2, \dots, \omega_p$ be all the continuous vertices such that (ω_i, μ) is a continuous edge. Then there exists $\xi \in C(\mathscr{C})$ such that $\xi(\omega, \mu) = 1$ and $\xi(\omega_i, \mu) = 0$ for $i = 2, \dots, p$. Since $(\xi \mid \varphi\xi)_A \in J$, we have $(\xi \mid \varphi\xi)_A(\mu) = 0$. Then

$$\varphi(\omega) = \overline{\xi(\omega, \mu)}\varphi(\omega)\xi(\omega, \mu) = \sum_{i=1}^{p} \overline{\xi(\omega_i, \mu)} = \varphi(\omega_i)\xi(\omega_i, \mu)(\xi \mid \varphi\xi)_A(\mu) = 0.$$

This shows that φ is in J. Thus J is X-saturated. $\square$

By Theorem 26 and the above lemma, we have the following criterion of simplicity of $\mathcal{O}_X$.

Corollary 31. *Let $\mathscr{C}$ be a circle bimodule and X its circle bimodule. Assume that $\mathscr{C}$ satisfies the condition circle-*(I)*. Then $\mathcal{O}_X$ is simple if and only if any saturated and hereditary open subset of Ω is Ω or the empty set $\varnothing$.*

Proof. Assume that any saturated and hereditary open subset of Ω is Ω or $\varnothing$. Then any X-saturated and X-invariant ideals of A is A or 0 by the above Lemma. Let $\mathscr{J}$ be an ideal of $\mathcal{O}_X$ with $\mathscr{J} \neq \mathcal{O}_X$. Consider a quotient map $\varphi : \mathcal{O}_X \to \mathcal{O}_X/\mathscr{J}$. Then $A \cap \ker\varphi$ is an X-invariant and X-saturated ideal of A. Since $\varphi(I) = I$, I is not in $A \cap \ker\varphi$. Thus $A \cap \ker\varphi = \{0\}$. Therefore the restriction of φ to A is faithful map. By Theorem 26, φ is faithful. Thus $\mathscr{J} = \{0\}$. This shows that $\mathcal{O}_X$ is simple. On the contrary, we may assume that there exists an X-invariant and X-saturated ideal J of A which is not equal to A nor 0 by the above Lemma. Then by the proof of (ii) (1) of Theorem 4.3 in [**KPW1**], or Proposition 17 in [**KPW2**], the ideal in $\mathcal{O}_X$ generated by J is not equal to $\mathcal{O}_X$ or $\{0\}$. Thus $\mathcal{O}_X$ is not simple. $\square$

For $i, j \in \Sigma$, we write $i \geq j$ if there exists a finite discrete path from i to j.

If $i \in \Sigma$ satisfies $i \leq i$, then i is called a class discrete vertex. We denote $[i] = \{j \in \Sigma \mid i \leq j \leq i\}$ as in [**C**]. If $i \in \Sigma$ is not a class discrete vertex, then i is called a transit discrete vertex.

We need the following lemmas to prove Theorem 40.

Lemma 32. *Let (i, j) be a discrete edge.*

(i) *Let U be a saturated hereditary open set in Ω. Put $U_i = U \cap \Omega_i$ for any i. Then $\Phi_{i,j}(U_i) \subset U_j$.*

(ii) *Let V be the complement of a saturated hereditary open subset in Ω. Put $V_i = V \cap \Omega_i$. Then $\Psi_{i,j}(V_j) \subset V_i$.*

Proof. Obvious. $\square$

Lemma 33. *Let $L = (i_0, i_1, \dots, i_{n-1}, i_0)$ be a periodic or trivial discrete loop. Then for a saturated hereditary open subset U, we have $\Phi_L(U_i) = U_i$ and $\Psi_L(\Omega \backslash U_i) = \Omega \backslash U_i$.*

Proof. If L is trivial, the lemma is clear. Suppose that $p(L) = 1$. Since U is hereditary, $\Phi_L(U_i) \subset U_i$. On the other hand since $\Phi_L(U_i) = U_i \cdot \boldsymbol{Z}_{q(L)}$ by Lemma 13, $U_i \subset \Phi_L(U_i)$. Thus we have $\Phi_L(U_i) = U_i$. Suppose that $p(L) = -1$. Then by Lemma 13, we have $\Phi_L(U_i) = U_i^{-1} \cdot \boldsymbol{Z}_q \subset U_i$ and $\Phi_L^2(U_i) = U_i \cdot \boldsymbol{Z}_{q(L)} \subset U_i$. Thus U_i is $\boldsymbol{Z}_{q(L)}$ invariant. Then $U_i^{-1} \subset U_i$. This shows that $U_i^{-1} = U^i$ and $\Phi_L(U_i) = U_i$. Therefore U_i is $\boldsymbol{Z}_{q(L)} \rtimes \boldsymbol{Z}_2$ invariant, where $\boldsymbol{Z}_{q(L)} \rtimes \boldsymbol{Z}_2$ is a dihedral group. □

Lemma 34. *Let L be an expansive discrete loop and U a saturated hereditary open subset of Ω. Then $U_i = \Omega_i$ or $U_i = \phi$ for any i in the equivalence class of a discrete vertex in L.*

Proof. Suppose that U is non empty open saturated hereditary subset of Ω_i. Since U is open, U contains an interval V. We identify V with an interval in $\boldsymbol{R}$ contained in $[0,1)$. Since L is expansive, the length of $\tilde{\Phi}_L^n(V)$ becomes greater than 1 for a sufficiently large n by Proposition 11. Therefore $\Phi_L^n(U) = \Omega_i$. □

As in [**KPW1**], for an X-invariant and X-saturated ideal J in A, we put $X_J = \{x \in X \mid (x|x)_A \in J\}$. Then X/X_J is a Hilbert A/J-A/J bimodule. Let π be a quotient map from X to X/X_J. Let $\{u_i\}_i$ be a right A basis of X. Then the family $\{\pi(u_i)\}_i$ constitutes a right A/J basis of X/X_J. The completely positive map σ_J for X/X_J is given by $\sigma_J(x) = \sum_i \pi(u_i) x \pi(u_i)^*$ as before Lemma 5. Let $D_{X/J}^r$ be the C^*-subalgebra of $\mathcal{O}_{X/J}$ generated by $(A/J), \sigma_J(A/J), \sigma_J^2(A/J), \dots, \sigma_J^r(A/J)$.

Let U be an open saturated hereditary subset of Ω and J an X-invariant and X-saturated ideal of A corresponding to U. Let V be the complement of U in Ω. Then V is a compact subset. We denote by $\mathscr{C} \backslash U$ the correspondence on V obtained from $\mathscr{C}$ removing all edges whose ranges are contained in U. Since the range map r is continuous, $\mathscr{C} \backslash U$ is closed and a compact Hausdorff space with respect to the topology relative to $\Omega \times \Omega$.

It is direct to show that X/X_J is the bimodule obtained from the correspondence $\mathscr{C} \backslash U$. As in Lemma 7, the C^*-algebra $D_{X/J}^r$ is isomorphic to the commutative C^*-algebra of continuous functions on the set of paths in $\mathscr{C} \backslash U$ with length r.

Definition 35. Let $\mathscr{C}$ be a circle correspondence. We say that $\mathscr{C}$ satisfies the condition circle-(II) if for every saturated hereditary open subset U in Ω, there exists no open subset in $\Omega \backslash U$ such that there exists an integer m such that there exist no m aperiodic paths starting from any point in this subset.

Definition 36 ([**KPW1**]). A Hilbert bimodule X over A is called (II)-free if for any X-saturated and X-invariant ideal J, Hilbert bimodule X/X_J over A/J is (I)-free.

Proposition 37. *Let $\mathscr{C}$ be a circle correspondence and X its circle bimodule. Assume that each class in the discrete graph $\mathscr{G}^d$ has an expansive loop or has neither contractive nor trivial loop. Then the correspondence $\mathscr{C}$ satisfies condition circle* (II).

Proof. Let U be a saturated hereditary open subset of Ω, and V be the complement of U in Ω. Let $\omega \in V$ and ω be contained in some Ω_i.

Suppose that i is a class discrete vertex. At first we assume that there exists an expansive loop L in $[i]$. Since V_i is not empty, for each $j \in [i]$, we have $V_j = \Omega_j$ by Lemma 34. Then by the same method as in Lemma 19, we can show that for every positive integer m, the set of m-aperiodic points is dense in V_i. We next assume that all simple discrete loops in $[j]$ are periodic. Then there exists more than two loops based at $\omega \in V \cap \Omega_j$ contained in $\mathscr{C}'$. Hence for every m, we can construct an m-aperiodic path for every $\omega \in V_i$ in $\mathscr{C}'$.

Suppose that i is a transit vertex. Let $\omega \in V_i$. Then there exists a path Σ_1 from ω to some ω' such that ω' is contained in V_j and j is a class discrete vertex. If there exists a periodic discrete loop through j, we can construct an m-aperiodic path Σ_2 starting at ω'. Then $\Sigma_1 \circ \Sigma_2$ is an m-aperiodic path. If $[j]$ contains expansive discrete loop, then $V_j = \Omega_j$ and there exists a dense subset K_j of Ω_j such that we may construct an m-aperiodic path starting at each point in K_j. Then V_i is also Ω_i and there exists a dense subset K_i such that there exists a finite path from $\omega \in K_i$ to a point in K_j. We can also construct an m-aperiodic path starting at points in K_i. □

Lemma 38. *There exists a circle correspondence $\mathscr{C}$ such that its circle bimodule X is not* (II)*-free but the corresponding discrete correspondence $\mathscr{C}^d$ is* (II)*-free in the sense of Cuntz* [**C**].

Proof. We present an example. Let $n = 2$ and $\mathscr{C}_{1,2}$ and $\mathscr{C}_{2,1}$ be trivial, $\mathscr{C}_{2,2} = \mathscr{C}^{2,-1}$ and $\mathscr{C}_{1,1} = \varnothing$. We take a hereditary saturated open subset U such that $U_1 = U_2 = \boldsymbol{T} \setminus \{1, -1\}$. Then the graph corresponding $\mathscr{C} \setminus U$ is a finite graph containing a closed circuit without exit. But $\mathscr{C}^d$ clearly satisfies the condition (II) in the sense of [**C**] and [**KPW2**]. □

Proposition 39. *Let $\mathscr{C}$ be a circle correspondence and X its circle bimodule. If $\mathscr{C}$ satisfies the condition circle-*(II)*, then for every X-invariant, X-saturated ideal J in A, X/X_J is* (I)*-free.*

Proof. If $\mathscr{C}$ satisfies the condition circle-(II), for every hereditary saturated open subset U, for every positive integer r, r-aperiodic points are dense in $\mathscr{C} \setminus U$.

Then for every $a \in A/J$, $\varepsilon > 0$, we can construct an operator Q satisfying (1)–(3) in Lemma 21. The rest is the same as Lemma 22 and Proposition 24. $\square$

THEOREM 40. *Let $\mathscr{C}$ be a circle correspondence and X its circle bimodule. If $\mathscr{C}$ satisfies the condition circle-*(II)*, then there exists a bijective correspondence between the lattice of closed two sided ideal of $\mathcal{O}_X$ and the lattice of saturated hereditary open subsets of Ω.*

PROOF. This follows from the previous proposition, Proposition 25 and the proof of Proposition 21 in [**KPW2**]. $\square$

6. *K*-theory.

Let $\mathscr{C}$ be a circle correspondence and X its circle bimodule. Recall that the left bimodule action is given by a *-homomorphism $\phi : A \to \mathscr{K}(X)$. We simply apply the following 6-term exact sequence obtained by Pimsner [**Pim**] for computing the K-theory of $\mathcal{O}_X$ directly. We denote by $[\mathbf{X}]$ the Kasparov element $[\phi, X, 0] \in KK(A, A)$. As in [**Pim**], $[\mathbf{X}]$ gives the transformation from $K_*(A)$ to $K_*(A)$, where $* = 0, 1$.

PROPOSITION 41 (Pimsner [**Pim**]). *In the same situation, we have the following* 6*-term exact sequence.*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{i-[X]} & K_0(A) & \longrightarrow & K_0(\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_X) & \longleftarrow & K_1(A) & \xleftarrow{i-[X]} & K_1(A) \end{array}$$

We just apply the Pimsner formula as above and get the following 6-term exact sequence:

$$\begin{array}{ccccc} \boldsymbol{Z}^n & \xrightarrow{I-D_0} & \boldsymbol{Z}^n & \longrightarrow & K_0(\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_X) & \longleftarrow & \boldsymbol{Z}^n & \xleftarrow{I-D_1} & \boldsymbol{Z}^n \end{array}$$

where D_* is the linear transformation on $\boldsymbol{Z}^n \simeq K_*(A)$ $(* = 0, 1)$ induced by $[\mathbf{X}]$. Deaconu [**D2**] obtained this 6-term exact sequence under a minimality assumption. But we do not need his minimality assumption by applying Pimsner's result directly.

EXAMPLE 6.1 ([**D1**]). Let $X = C(\mathscr{C}^{p,1})$ $(p \geq 2)$. Then there exists only one discrete loop and it is expansive. Since $\mathscr{C}$ satisfies the condition circle-(I), $\mathcal{O}_X$ is simple.

$\mathscr{F}_{\infty}^{(0)}$ is isomorphic to a Bunce-Deddense algebra. The C^*-algebra $\mathcal{O}_X$ is nuclear, purely infinite and simple. Moreover, there exists a unique KMS state under the gauge action.

D_0 is given by $n \mapsto pn$, and D_1 is given by $n \mapsto n$. The K-groups are calculated as follows:

$$K_0(\mathcal{O}_X) \simeq \boldsymbol{Z}_{p-1} \oplus \boldsymbol{Z} \quad K_1(\mathcal{O}_X) \simeq \boldsymbol{Z}.$$

Example 6.2 ([**D2**]). Let $X = C(\mathscr{C}^{p,-1})$ $(p \geq 2)$. There exists only one discrete loop and it is contractive. $\mathscr{C}$ satisfies the condition circle-(I), but do not satisfy the condition circle-(II)-free. There are many ideals in $\mathcal{O}_X$ which do not correspond to hereditary open subsets of Ω.

We recall that a solenoid S is a compact Hausdorff space defined by

$$S = \{z = (z_k)_{k \in \boldsymbol{N}} \mid z_k \in \boldsymbol{T},\ z_{k+1}^p = z_k,\ k \in \boldsymbol{N}\}.$$

Let $S_n = \{z = (z_0, z_1, \dots, z_n) \in \boldsymbol{T}^n \mid z_{k+1}^p = z_k, k = 0, 1, \dots, n\}$. Consider a continuous onto map $\pi_n : S_{n+1} \to S_n$ such that $\pi_n(z_0, z_1, \dots, z_{n+1}) = (z_0, z_1, \dots, z_n)$. Then S is the projective limit of the system $(S_n, \pi_n)_n$. Since S_n is isomorphic to the set $(\mathscr{C}^{p,-1})_n$ of continuous paths with length n, the relative tensor product $X_A^{\otimes n}$ is isomorphic to $C(S_n)$ $(\simeq A)$ by Lemma 6. If we identify $\mathscr{L}(X_A^{\otimes n})$ with $C(S_n)$ $(\simeq A)$, then the canonical inclusion $i_n : \mathscr{L}(X^{\otimes n}) \to (\mathscr{L}^{\otimes(n+1)})$ can be identified with $\pi_n^* : C(S_n) \to C(S_{n+1})$. Therefore the "AT-part" $\mathscr{F}_{\infty}^{(0)}$ is a commutative C^*-algebra which is isomorphic to the algebra of continuous functions on a solenoid. C^*-algebra $\mathcal{O}_X$ has not unique tracial states which are KMS states under the gauge action.

D_0 is given by $n \mapsto n$ and D_1 is given by $n \mapsto pn$. The K-group is calculated as follows:

$$K_0(\mathcal{O}_X) \simeq \boldsymbol{Z} \quad K_1(\mathcal{O}_X) \simeq \boldsymbol{Z}_{p-1} \oplus \boldsymbol{Z}.$$

We should note that the bimodules of Example 6.1 and Example 6.2 are conjugate to each other, but the structures of the corresponding bimodule algebras are completely different.

Example 6.3. Consider a circle correspondence $\mathscr{C}$ on $\boldsymbol{T} \cup \boldsymbol{T} \cup \boldsymbol{T}$ as follows: Put $\mathscr{C}_{1,2} = \mathscr{C}^{2,-1}$, $\mathscr{C}_{2,1} = \mathscr{C}^{2,1}$, $\mathscr{C}_{2,3}$ is trivial and $\mathscr{C}_{3,3} = \mathscr{C}^{2,1}$. Let other $\mathscr{C}_{i,j}$'s be $\varnothing$. Then the correspondence $\mathscr{C}$ satisfies the condition circle-(II) but it is not minimal.

We write down all closed two sided ideals of $\mathcal{O}_X$ using open hereditary saturated subsets of $\Omega = \boldsymbol{T} \cup \boldsymbol{T} \cup \boldsymbol{T}$. Let U be an open hereditary saturated subset of Ω. Since $\mathscr{C}_{3,3}$ is type 1, U_3 is Ω_3 or $\varnothing$. If U_3 is equal to $\varnothing$, U_1 and U_2 are also $\varnothing$. If U_3 is Ω_3, then U_1 is a $\boldsymbol{Z}_2$-invariant closed subset of $\boldsymbol{T}$, and $U_2 = \{z^2 \mid z \in U_1\}$.

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