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Geometry of decomposable directing modules over tame algebras

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Abstract. Let A be a tame algebra and M a directing A-module (there exists no sequence $M_1 \to \cdots \to \tau_A X \to * \to X \to \cdots \to M_2$ of nonzero maps between indecomposable A-modules for some indecomposable nonprojective A-module X and indecomposable direct summands M_1, M_2 of M). Then the variety $\mathrm{mod}_A(\mathbf{dim}\, M)$ of A-modules with dimension vector $\mathbf{dim}\, M$ is a complete intersection. If, in addition, M is a tilting A-module then $\mathrm{mod}_A(\mathbf{dim}\, M)$ is normal.

Introduction and the main result.

Throughout the paper K denotes a fixed algebraically closed field. By an algebra we usually mean a finite dimensional basic connected K-algebra with an identity. For an algebra A we write $\mathrm{mod}\, A$ for the category of finite dimensional left A-modules.

According to the Drozd's theorem ([**13**], see also [**11**]) finite dimensional algebras can be divided into two disjoint classes. One class is formed by the wild algebras for which the classification of indecomposable modules is equivalent to the classification of finite dimensional indecomposable modules over the free noncommutative algebra with two generators. Another class is the class of tame algebras. An algebra A is tame if, for each natural number d, there exist A-$K[X]$-bimodules $M_1, \dots, M_{n_d}$, which are free of finite rank as right $K[X]$-modules, such that all but a finite number of indecomposable A-modules of dimension d are (up to isomorphism) of the form $M_i \otimes_{K[X]} K[X]/(X-\lambda)$ for some i and $\lambda \in K$.

Let A be an algebra. Since A is basic there exist a quiver Q and an admissible ideal I in the path algebra KQ of Q such that $A \simeq KQ/I$. The category $\mathrm{mod}\, A$ is equivalent to the category $\mathrm{rep}_K(Q, I)$ of K-linear representations of Q satisfying the relations from I. Using this equivalence we get an isomorphism $K_0(A) \simeq \boldsymbol{Z}^{Q_0}$, where $K_0(A)$ denotes the Grothendieck group of A and Q_0 is the set of vertices of Q. The isomorphism is given by assigning to each A-module M its dimension vector $\mathbf{dim}\, M \in \boldsymbol{Z}^{Q_0}$, $(\mathbf{dim}\, M)_x := \dim_K e_x M$, $x \in Q_0$, where e_x denotes the trivial path at vertex x.

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Given a vector $\boldsymbol{d} \in K_0(A)$ with nonnegative entries it is an interesting task to study the module variety $\operatorname{mod}_A(\boldsymbol{d})$ of A-modules of dimension vector $\boldsymbol{d}$. It is the set of all collections $(M_\alpha)_{\alpha \in Q_1}$, where Q_1 is the set of arrows in Q, such that, for each arrow $\alpha : x \to y$, M_α is the linear map from K^{d_x} to K^{d_y} and, for each relation $\sum \lambda_i \alpha_{i,k_i} \cdots \alpha_{i,1}$ in I, we have $\sum \lambda_i M_{\alpha_{i,k_i}} \cdots M_{\alpha_{i,1}} = 0$. Using the above mentioned equivalence of $\operatorname{rep}_K(Q, I)$ and $\operatorname{mod} A$ we get that each point M of $\operatorname{mod}_A(\boldsymbol{d})$ determines the A-module structure on $\prod_{x \in Q_0} K^{d_x}$ (multiplication by an arrow α is given by M_α). We will denote this module also by M.

The reductive group $GL(\boldsymbol{d}) := \prod_{x \in Q_0} GL(d_x)$ acts on $\operatorname{mod}_A(\boldsymbol{d})$ by conjugations, that is, $(g \cdot M)_\alpha := g_y M_\alpha g_x^{-1}$ for each arrow $\alpha : x \to y$ in Q, where $M = (M_\alpha) \in \operatorname{mod}_A(\boldsymbol{d})$ and $g = (g_x) \in GL(\boldsymbol{d})$. For each $M \in \operatorname{mod}_A(\boldsymbol{d})$ we denote by $\mathcal{O}(M)$ the orbit of M under this action. We have $\mathcal{O}(M) = \mathcal{O}(N)$ if and only if $M \simeq N$.

The path in $\operatorname{mod} A$ is a sequence $X_0 \to X_1 \to \cdots \to X_t$ of nonzero maps between indecomposable A-modules. An A-module M is called directing if there exists no path of the form $M_1 \to \cdots \to \tau_A X \to * \to X \to \cdots \to M_2$ for some indecomposable direct summands M_1 and M_2 of M and an indecomposable nonprojective A-module X. By τ_A we denote the Auslander-Reiten translate in $\operatorname{mod} A$ (see [**20**, (2.4)]). It is known due to Happel and Ringel [**16**, Section 1] that if M is indecomposable then the above definition coincides with the usual one, that is, M is directing if and only if there exists no path $X_0 \to X_1 \to \cdots \to X_t$ of nonzero nonisomorphisms between indecomposable modules in $\operatorname{mod} A$ such that $X_0 \simeq M \simeq X_t$, $t > 0$.

Let q_A denote the Tits form of the algebra A, that is, the quadratic form on $K_0(A)$ defined by the formula

$$q_A(\boldsymbol{d}) := \sum_{x \in Q_0} d_x^2 - \sum_{\alpha : x \to y \in Q_1} d_x d_y + \sum_{x, y \in Q_0} \dim_K \operatorname{Ext}_A^2(S_A(x), S_A(y)) d_x d_y.$$

Here, $S_A(x)$ denotes the simple A-module at vertex x given by the representation (V_y, V_α) such that $V_x = K$, $V_y = 0$ if $y \neq x$ and $V_\alpha = 0$ for any $\alpha \in Q_1$. We also put $a_A(\boldsymbol{d}) := \dim GL(\boldsymbol{d}) - q_A(\boldsymbol{d})$.

The following theorem, which generalizes [**3**, Theorem 1] to decomposable directing modules, is the main result of the paper.

Main Theorem. *Let A be a tame algebra, M a directing A-module and $\boldsymbol{d} :=$ $\mathbf{dim}\, M$. The variety $\operatorname{mod}_A(\boldsymbol{d})$ has the following properties:*

(1) *$\operatorname{mod}_A(\boldsymbol{d})$ is a complete intersection of dimension $a_A(\boldsymbol{d})$;*
(2) *$\overline{\mathcal{O}(M)}$ is an irreducible component of $\operatorname{mod}_A(\boldsymbol{d})$ and $\mathcal{O}(M)$ is the unique orbit of maximal dimension in $\operatorname{mod}_A(\boldsymbol{d})$;*
(3) *maximal orbits in $\operatorname{mod}_A(\boldsymbol{d})$ consist of nonsingular points in $\operatorname{mod}_A(\boldsymbol{d})$;*

(4) *there is only a finite number of orbits of codimension* 1 *in* $\mathrm{mod}_A(\boldsymbol{d})$, *they are contained in* $\overline{\mathcal{O}(M)}$ *and consist of nonsingular points in* $\mathrm{mod}_A(\boldsymbol{d})$;

(5) $N \in \mathrm{mod}_A(\boldsymbol{d})$ *is nonsingular in* $\mathrm{mod}_A(\boldsymbol{d})$ *if and only if* $\mathrm{Ext}_A^2(N, N) = 0$.

Moreover, if M *is tilting then* $\mathrm{mod}_A(\boldsymbol{d}) = \overline{\mathcal{O}(M)}$, *is irreducible and normal.*

Recall that an A-module T is called tilting if $\mathrm{pd}_A T \le 1$, $\mathrm{Ext}_A^1(T, T) = 0$ and there exists a short exact sequence $0 \to A \to T' \to T'' \to 0$, where $T', T'' \in \mathrm{add}\, T$. Here, by $\mathrm{add}\, T$ we denote the full subcategory of $\mathrm{mod}\, A$ whose objects are finite direct sums of direct summands of T. For the definitions of a complete intersection, a normal variety, etc., we refer to [**14**].

It should be added that it has been proved in [**8**, Proposition 6] (see also [**9**, Corollary 3.5]) that if A is any algebra and M a directing preprojective A-module then the variety $\mathrm{mod}_A(\mathbf{dim}\, M)$ is a normal and irreducible complete intersection of dimension $a_A(\mathbf{dim}\, M)$.

The paper is divided in two parts. First part is devoted to recalling and proving some preliminary facts on directing modules. There are also contained some geometric tools we use. In the second part we present the proof of the above theorem.

1. Preliminaries.

Let $A = KQ/I$ be an algebra. For an A-module M we denote by $\mathrm{supp}\, M$ the support of M, that is, the full subquiver of Q given by all vertices x such that $e_x M \neq 0$. By $\mathrm{Supp}\, M$ we denote the support algebra of M, which is the quotient of the algebra A by the ideal generated by all e_x for x not in $\mathrm{supp}\, M$. The support $\mathrm{supp}\, M$ is said to be convex in Q if, for any path $a_0 \to a_1 \to \cdots \to a_t$ in Q with a_0 and a_t in $\mathrm{supp}\, M$, a_i belongs to $\mathrm{supp}\, M$ for each i. If $\mathrm{supp}\, M$ is convex in Q then $\mathrm{Ext}_A^i(M_1, M_2) \simeq \mathrm{Ext}_{\mathrm{Supp}\, M}^i(M_1, M_2)$ for $i \ge 0$ and any A-modules M_1 and M_2 with support contained in $\mathrm{supp}\, M$. Moreover $a_A(\boldsymbol{d}) = a_{\mathrm{Supp}\, M}(\boldsymbol{d})$ for any $\boldsymbol{d} \in K_0(A)$ such that $\boldsymbol{d}_x = 0$ for x not in $\mathrm{supp}\, M$.

In view of the above remarks the following fact generalizing the result of Bongartz from [**6**] is of particular importance.

Lemma 1.1. *Let* $A = KQ/I$ *be an algebra and* M *a directing* A*-module. Then the support* $\mathrm{supp}\, M$ *of* M *is convex in* Q.

Proof. Assume this is not the case and let $a_0 \to a_1 \to \cdots \to a_{t-1} \to a_t$, $t \ge 2$, be a path in Q such that a_0 and a_t belong to $\mathrm{supp}\, M$ and $a_1, \ldots, a_{t-1}$ do not. The minimal projective presentation of $S_A(a_{t-1})$ is of the form $P_A(a_t) \oplus P \to P_A(a_{t-1}) \to S_A(a_{t-1}) \to 0$ for some projective A-module P, where $P_A(x) := Ae_x$ for $x \in Q_0$. Let $I_A(x) := D(e_x A)$, where $D(-) := \mathrm{Hom}_K(-, K)$ is the usual duality. From the construction of $\tau_A S_A(a_{t-1})$ (see [**20**, (2.4)]) it follows that we have a short exact sequence $0 \to \tau_A S_A(a_{t-1}) \to I_A(a_t) \oplus I \to$

$I_A(a_{t-1})$ for some injective A-module I. Applying $\mathrm{Hom}_A(M,-)$ we get an exact sequence $0 \to \mathrm{Hom}_A(M, \tau_A S_A(a_{t-1})) \to \mathrm{Hom}_A(M, I_A(a_t)) \oplus \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(M, I_A(a_{t-1}))$. Since $a_t \in \operatorname{supp} M$ we have $\mathrm{Hom}_A(M, I_A(a_t)) \simeq D(e_t M) \neq 0$. Similarly $\mathrm{Hom}_A(M, I_A(a_{t-1})) \simeq D(e_{t-1}M) = 0$, because $a_{t-1} \notin \operatorname{supp} M$. Thus $\mathrm{Hom}_A(M, \tau_A S_A(a_{t-1})) \neq 0$. In the same way we show $\mathrm{Hom}_A(\tau_A^- S_A(a_1), M) \neq 0$. The existence of the arrow $a_i \to a_{i+1}$ implies $\mathrm{Ext}_A^1(S_A(a_i), S_A(a_{i+1})) \neq 0$ for each i. We obtain this way a path $M_1 \to \tau_A S_A(a_{t-1}) \to * \to S_A(a_{t-1}) \to \cdots \to S_A(a_1) \to * \to \tau_A^- S_A(a_1) \to M_2$ in $\operatorname{mod} A$ for some indecomposable direct summands M_1 and M_2 of M, what contradicts our assumption. $\square$

From this point we assume that M is sincere. Then according to the result of Bakke in [**1**] the algebra A is tilted, that is, A is the endomorphism algebra of a tilting module over a hereditary algebra. In particular, A is triangular (there is no cycle in the ordinary quiver of A) of global dimension at most 2 and we have $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$ for each indecomposable A-module X (see [**15**]). It follows from the result of Bongartz [**6**, Section 1] that if $r_{x,y}$ denotes the number of relations from x to y in the minimal set of relations generating the ideal I then $r_{x,y}$ equals $\dim_K \mathrm{Ext}_A^2(S_A(x), S_A(y))$. Thus the Tits form q_A of A can be also defined by the formula

$$q_A(\boldsymbol{d}) = \sum_{x \in Q_0} d_x^2 - \sum_{\alpha : x \to y} d_x d_y + \sum_{x, y \in Q_0} r_{x,y} d_x d_y.$$

Moreover the variety $\mathrm{mod}_A(\boldsymbol{d})$ can be described in $\prod_{\alpha : x \to y} K^{d_x} \times K^{d_y}$ by $\sum_{x, y \in Q_0} r_{x,y} d_x d_y$ equations. Hence by Krull Principal Ideal Theorem it follows that each irreducible component of the variety $\mathrm{mod}_A(\boldsymbol{d})$ has dimension at least $\sum_{\alpha : x \to y} d_x d_y - \sum_{x, y \in Q_0} r_{x,y} d_x d_y = a_A(\boldsymbol{d})$.

Since A is tilted, we get using the result of Kerner [**17**, Theorem 6.2] that the algebra A is tame if and only if its Tits form q_A is weakly nonnegative, that is, $q_A(\mathbf{dim}\, M) \geq 0$ for each A-module M. We also have the bilinear nonsymmetric Euler form $\langle -, - \rangle_A$ on $K_0(A)$ such that $\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_A = \dim_K \mathrm{Hom}_A(M, N) - \dim_K \mathrm{Ext}_A^1(M, N) + \dim_K \mathrm{Ext}_A^2(M, N)$ for any A-modules M, N (see [**20**, (2.4)]). It follows from [**6**, Proposition 2.2] that $q_A(\boldsymbol{d}) = \langle \boldsymbol{d}, \boldsymbol{d} \rangle_A$ in our case.

An easy consequence of the definition of a directing module and the sincerity of M is that if $\mathrm{Hom}_A(X, M) \neq 0$ for some indecomposable A-module X then $\mathrm{pd}_A X \leq 1$. Here we use that $\mathrm{Hom}_A(M, I) \neq 0$ for each indecomposable injective A-module I and $\mathrm{pd}_A X \leq 1$ if and only if $\mathrm{Hom}_A(I, \tau_A X) = 0$ for any injective A-module I (see [**20**, (2.4)]). Similarly $\mathrm{Hom}_A(M, X) \neq 0$ implies $\mathrm{id}_A X \leq 1$. In particular, $\mathrm{pd}_A M \leq 1$ and $\mathrm{id}_A M \leq 1$. Moreover, we have $\mathrm{Hom}_A(M, \tau_A M) = 0$, thus $\mathrm{Ext}_A^1(M, M) = 0$, by the Auslander-Reiten formula (see again [**20**, (2.4)]). Using the result from [**1**] we know there exists a directing tilting module T such that $M \in \operatorname{add} T$.

Assume now that M is tilting. By [**5**, Theorem 2.1] it is equivalent to say that the number of isomorphism classes of indecomposable direct summands of M equals the number of isomorphism classes of simple A-modules. From the definition of a tilting module it easily follows that $\mathrm{Hom}_A(M,N)=0$ and $\mathrm{Ext}_A^1(M,N)=0$ implies $N=0$. Obviously every tilting module is sincere, hence $\mathrm{id}_A M \leq 1$. Thus M is also cotilting, that is, $D(M)$ is a tilting A^{op}-module, where A^{op} denotes the opposite algebra of A. In particular, either $\mathrm{Hom}_A(N,M)\neq 0$ or $\mathrm{Ext}_A^1(N,M)\neq 0$ for each nonzero A-module N. The module M induces the torsion pair $(\mathscr{T}(M),\mathscr{F}(M))$, where $\mathscr{T}(M)=\{N\in \mathrm{mod}\,A \mid \mathrm{Ext}_A^1(M,N)=0\}$ and $\mathscr{F}(M):=\{N\in\mathrm{mod}\,A\mid \mathrm{Hom}_A(M,N)=0\}$. Note that $\mathrm{add}\,M\subset\mathscr{T}(M)$. By [**1**] the torsion pair $(\mathscr{T}(M),\mathscr{F}(M))$ is splitting, that is, each indecomposable A-module belongs either to $\mathscr{T}(M)$ or $\mathscr{F}(M)$. Moreover, since M is cotilting it follows that $\mathscr{F}(M)$ is the class of all A-modules N such that $X\notin \mathrm{add}\,M$ for each indecomposable direct summand X of N and $\mathrm{Ext}_A^1(N,M)=0$. Similarly, an indecomposable A-module $X\notin\mathrm{add}\,M$ belongs to $\mathscr{T}(M)$ if and only if $\mathrm{Hom}_A(X,M)=0$. Thus it follows that if $\mathrm{Hom}_A(M,X)\neq 0$ and $\mathrm{Hom}_A(X,M)\neq 0$ for some indecomposable A-module X then $X\in\mathrm{add}\,M$. We also have the following useful lemma.

Lemma 1.2. *Let A be a tame algebra and M a directing tilting A-module. If $\boldsymbol{h}\in K_0(A)$ is a connected positive vector such that $q_A(\boldsymbol{h})=0$ then $|\langle \boldsymbol{d},\boldsymbol{h}\rangle_A|\geq 2$ and $|\langle \boldsymbol{h},\boldsymbol{d}\rangle_A|\geq 2$.*

Proof. It follows from [**17**] that there exist pairwise nonisomorphic indecomposable A-modules X_λ, $\lambda\in K$, such that $\mathbf{dim}\,X_\lambda=\boldsymbol{h}$ for each λ. Since the torsion pair $(\mathscr{T}(M),\mathscr{F}(M))$ is splitting there exists an infinite set $\mathscr{I}\subset K$ such that either $X_\lambda\in\mathscr{T}(M)$ for each $\lambda\in\mathscr{I}$ or $X_\lambda\in\mathscr{F}(M)$ for each $\lambda\in\mathscr{I}$. Assume the first possibility holds. Then $\mathrm{Ext}_A^1(M,X_\lambda)=0$ for each $\lambda\in\mathscr{I}$. In addition, $\mathrm{Ext}_A^2(M,X_\lambda)=0$ for any λ, because $\mathrm{pd}_A M\leq 1$. On the other hand by the Brenner-Butler theorem [**10**] (see also [**15**, Section 2]) $\mathrm{Hom}_A(M,X_\lambda)$, $\lambda\in\mathscr{I}$, are pairwise nonisomorphic indecomposable $\mathrm{End}_A(M)^{\mathrm{op}}$-modules. Since there is only a finite number of isomorphism classes of indecomposable one dimensional $\mathrm{End}_A(M)^{\mathrm{op}}$-modules we get that $\dim_K \mathrm{Hom}_A(M,X_\lambda)\geq 2$ for all but a finite number of $\lambda\in\mathscr{I}$. Fix $\lambda_0\in\mathscr{I}$ with this property. Since $\mathrm{Ext}_A^1(M,X_{\lambda_0})=0=\mathrm{Ext}_A^2(M,X_{\lambda_0})$ we have $\langle\boldsymbol{d},\boldsymbol{h}\rangle_A=\dim_K\mathrm{Hom}_A(M,X_{\lambda_0})\geq 2$. We proceed similarly if the second possibility holds. The other inequality follows by duality. $\square$

We will also need some links between properties of modules and properties of the corresponding points of module varieties. The following proposition will play a crucial role in our investigations. Recall that an orbit $\mathscr{O}(M)\subset\mathrm{mod}_A(\boldsymbol{d})$ is called maximal in $\mathrm{mod}_A(\boldsymbol{d})$ if and only if $\mathscr{O}(M)\subset\overline{\mathscr{O}(N)}$ implies $\mathscr{O}(N)=\mathscr{O}(M)$.

Proposition 1.3. *Let A be a triangular algebra of global dimension at most* 2 *and $\boldsymbol{d} \in K_0(A)$ a positive vector. If* $\mathrm{Ext}_A^2(M,M) = 0$ *for each maximal orbit $\mathcal{O}(M)$ in* $\mathrm{mod}_A(\boldsymbol{d})$, *then the variety* $\mathrm{mod}_A(\boldsymbol{d})$ *is a complete intersection of dimension $a_A(\boldsymbol{d})$. Moreover, for each $N \in \mathrm{mod}_A(\boldsymbol{d})$, we have* $\dim \mathcal{O}(N) = a_A(\boldsymbol{d}) - \dim_K \mathrm{Ext}_A^1(N,N) + \dim_K \mathrm{Ext}_A^2(N,N)$, *and N is nonsingular in* $\mathrm{mod}_A(\boldsymbol{d})$ *if and only if* $\mathrm{Ext}_A^2(N,N) = 0$.

Proof. Using the arguments presented in the proof of [**7**, Proposition 2] we get that if $\mathrm{Ext}_A^2(M,M) = 0$ for any maximal orbit $\mathcal{O}(M)$ in $\mathrm{mod}_A(\boldsymbol{d})$ then the corresponding module scheme is reduced and $\mathrm{mod}_A(\boldsymbol{d})$ is a complete intersection of dimension $a_A(\boldsymbol{d})$. The formula $\dim \mathcal{O}(N) = a_A(\boldsymbol{d}) - \dim_K \mathrm{Ext}_A^1(N,N) + \dim_K \mathrm{Ext}_A^2(N,N)$ now follows from the calculations presented in the same proof. Finally, these calculations also imply that we have $\dim_K T_N(\mathrm{mod}_A(\boldsymbol{d})) = a_A(\boldsymbol{d}) + \dim_K \mathrm{Ext}_A^2(N,N)$, hence N is nonsingular if and only if $\mathrm{Ext}_A^2(N,N) = 0$. □

We see from the above proposition that an important role is played by maximal orbits in a module variety. The following characterization of maximal orbits is a direct consequence of [**21**, Corollary 6].

Proposition 1.4. *Let A be a tame tilted algebra and $\boldsymbol{d} \in K_0(A)$ a positive vector. An orbit $\mathcal{O}(N)$ is maximal in* $\mathrm{mod}_A(\boldsymbol{d})$ *if and only if we have* $\mathrm{Ext}_A^1(N_1,N_2) = 0$ *for any decomposition $N \simeq N_1 \oplus N_2$.*

2. Proof of the main result.

Fix a tame algebra A and a directing A-module M. Put $\boldsymbol{d} := \mathbf{dim}\, M$. Using Lemma 1.1 we may assume that M is sincere. Then $\mathrm{pd}_A M \leq 1$ and $\mathrm{id}_A M \leq 1$. Moreover A is tilted, hence the Tits form q_A is weakly nonnegative and $\mathrm{Ext}_A^2(X,X) = 0$ for each indecomposable A-module X.

We start with the following lemma.

Lemma 2.1. *Let $\mathcal{O}(N)$ be a maximal orbit in* $\mathrm{mod}_A(\boldsymbol{d})$.

(1) *If X is an indecomposable direct summand of N with the property* $\mathrm{Hom}_A(X,M) = 0$ *then*

$$\langle \mathbf{dim}\, X, \boldsymbol{d} \rangle_A = 0, q_A(\mathbf{dim}\, X) = 0 \quad \textit{and} \quad \mathrm{Ext}_A^2(X,N) = 0.$$

(2) *If X is an indecomposable direct summand of N with the property* $\mathrm{Hom}_A(M,X) = 0$ *then*

$$\langle \boldsymbol{d}, \mathbf{dim}\, X \rangle_A = 0, q_A(\mathbf{dim}\, X) = 0 \quad \textit{and} \quad \mathrm{Ext}_A^2(N,X) = 0.$$

Proof. We shall prove only the first statement, because the proof of the second one is dual. We have the inequality

$$\langle \mathbf{dim}\, X, \boldsymbol{d} \rangle_A = \langle \mathbf{dim}\, X, \mathbf{dim}\, M \rangle_A = -\dim_K \mathrm{Ext}_A^1(X,M) \leq 0,$$

since $\mathrm{id}_A M \leq 1$. We know that $N = X \oplus L$ for some A-module L. From Proposition 1.4 we get $\mathrm{Ext}_A^1(X, L) = 0$. Thus

$$\begin{aligned} 0 &\geq \langle \mathbf{dim}\, X, \boldsymbol{d} \rangle_A \\ &= q_A(\mathbf{dim}\, X) + \langle \mathbf{dim}\, X, \mathbf{dim}\, L \rangle_A \\ &= q_A(\mathbf{dim}\, X) + \dim_K \mathrm{Hom}_A(X, L) + \dim_K \mathrm{Ext}_A^2(X, L) \geq 0. \end{aligned}$$

It follows that $\langle \mathbf{dim}\, X, \boldsymbol{d} \rangle_A = 0$, $q_A(\mathbf{dim}\, X) = 0$ and $\mathrm{Ext}_A^2(X, L) = 0$. It finishes the proof since $\mathrm{Ext}_A^2(X, X) = 0$. □

We have the following consequence of the above lemma.

Proposition 2.2. *For each maximal orbit $\mathcal{O}(N)$ in $\mathrm{mod}_A(\boldsymbol{d})$, we have $\mathrm{Ext}_A^2(N, N) = 0$.*

Proof. Fix a maximal orbit $\mathcal{O}(N)$ in $\mathrm{mod}_A(\boldsymbol{d})$. We show $\mathrm{Ext}_A^2(X, N) = 0$ for each indecomposable direct summand X of N. If $\mathrm{Hom}_A(X, M) \neq 0$ then $\mathrm{pd}_A X \leq 1$ and the claim follows. If $\mathrm{Hom}_A(X, M) = 0$ then we get it by the previous lemma. □

The immediate consequence of the above proposition and Proposition 1.3 is the following corollary.

Corollary 2.3. *The variety $\mathrm{mod}_A(\boldsymbol{d})$ is a complete intersection of dimension $a_A(\boldsymbol{d})$. Moreover maximal orbits in $\mathrm{mod}_A(\boldsymbol{d})$ consist of nonsingular points, and $N \in \mathrm{mod}_A(\boldsymbol{d})$ is nonsingular if and only if $\mathrm{Ext}_A^2(N, N) = 0$.*

Since $\mathrm{Ext}_A^1(M, M) = 0$ and $\mathrm{Ext}_A^2(M, M) = 0$ we have by Proposition 1.3 $\dim \mathcal{O}(M) = a_A(\boldsymbol{d}) - \dim_K \mathrm{Ext}_A^1(M, M) + \dim_K \mathrm{Ext}_A^2(M, M) = a_A(\boldsymbol{d})$. In particular, $\overline{\mathcal{O}(M)}$ is the irreducible component of $\mathrm{mod}_A(\boldsymbol{d})$ and $\mathcal{O}(M)$ is the orbit of maximal dimension in $\mathrm{mod}_A(\boldsymbol{d})$. We shall show that it is the unique orbit with this property. We start with the following lemma.

Lemma 2.4. *Let $\mathcal{O}(N)$ be a maximal orbit in $\mathrm{mod}_A(\boldsymbol{d})$. If $N \not\simeq M$ then $\dim \mathcal{O}(N) \leq a_A(\boldsymbol{d}) - 2$.*

Proof. Fix a directing tilting A-module T such that $M \in \mathrm{add}\, T$. We first show that $N \notin \mathcal{T}(T)$. Assume this in not the case. Then by the arguments presented in the proof of [**5**, Proposition 1.4(b)] there exists an exact sequence $0 \to L \to T_0 \to N \to 0$ such that $T_0 \in \mathrm{add}\, T$ and $L \in \mathcal{T}(T)$. We have $\mathrm{Hom}_A(X, T) \neq 0$ and $\mathrm{Hom}_A(T, X) \neq 0$ for each indecomposable direct summand X of L, hence $L \in \mathrm{add}\, T$, since T is a directing tilting A-module. Note $\mathbf{dim}\, T_0 = \mathbf{dim}\, L + \mathbf{dim}\, N =$

$\mathbf{dim}\, L + \mathbf{dim}\, M = \mathbf{dim}(L \oplus M)$. Since $T_0 \in \operatorname{add} T$ and $L \oplus M \in \operatorname{add} T$, it follows that $T_0 \simeq L \oplus M$, because modules in $\operatorname{add} T$ are uniquely determined by their dimension vectors. Indeed, from the proof of [**15**, Proposition (3.2)] we know that the dimension vectors of indecomposable direct summands of T form a basis of $K_0(A)$. Thus we get an exact sequence $0 \to L \to L \oplus M \to N \to 0$ and by the result of Riedtmann [**19**, Proposition 3.4] $\mathcal{O}(N) \subset \overline{\mathcal{O}(M)}$. Consequently $\mathcal{O}(N) = \mathcal{O}(M)$ and $N \simeq M$, since $\mathcal{O}(N)$ is a maximal orbit in $\operatorname{mod}_A(\boldsymbol{d})$, contradiction.

It follows from the above considerations that there exists an indecomposable direct summand X of N such that $X \in \mathcal{F}(T)$, since the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ is splitting. Then $\operatorname{Hom}_A(M, X) = 0$. Since T is a cotilting module we show dually that there exists an indecomposable direct summand $Y \in \mathcal{T}(T) \setminus \operatorname{add} T$ of N such that $\operatorname{Hom}_A(Y, M) = 0$. Note that $X \not\simeq Y$. By Lemma 2.1 we get $q_A(\mathbf{dim}\, X) = 0 = q_A(\mathbf{dim}\, Y)$. Particularly, $\operatorname{Ext}_A^1(X, X) \neq 0 \neq \operatorname{Ext}_A^1(Y, Y)$ and $\dim_K \operatorname{Ext}_A^1(N, N) \geq 2$. From Proposition 2.2 we have $\operatorname{Ext}_A^2(N, N) = 0$, hence we obtain $\dim \mathcal{O}(N) = a_A(\boldsymbol{d}) - \dim_K \operatorname{Ext}_A^1(N, N) \leq a_A(\boldsymbol{d}) - 2$ using Proposition 1.3. □

The immediate consequence of the above lemma is the following corollary.

Corollary 2.5. *The orbit $\mathcal{O}(M)$ is the unique orbit of maximal dimension in* $\operatorname{mod}_A(\boldsymbol{d})$.

We finish our considerations in the general case by remarks on codimension 1 orbits.

Proposition 2.6. *There is only a finite number of orbits of codimension* 1 *and they are contained in* $\overline{\mathcal{O}(M)}$. *Moreover, if $\mathcal{O}(N)$ is an orbit of codimension* 1 *in* $\operatorname{mod}_A(\boldsymbol{d})$ *then* $\operatorname{Ext}_A^2(N, N) = 0$.

Proof. By Lemma 2.4 it follows that all orbits of codimension 1 are contained in $\overline{\mathcal{O}(M)}$. Thus obviously there is only a finite number of them.

Let $\mathcal{O}(N)$ be an orbit of codimension 1 in $\operatorname{mod}_A(\boldsymbol{d})$. Since $\mathcal{O}(N)$ is not maximal in $\operatorname{mod}_A(\boldsymbol{d})$ we have by Proposition 1.4 $\operatorname{Ext}_A^1(N_1, N_2) \neq 0$ for some decomposition $N \simeq N_1 \oplus N_2$ of N. Without loss of generality we may assume that N_1 is indecomposable. We get a nonsplit exact sequence $0 \to N_2 \to L \to N_1 \to 0$ for some A-module L. Then $\mathcal{O}(N) \subset \overline{\mathcal{O}(L)} \setminus \mathcal{O}(L)$. Since $\mathcal{O}(N)$ is an orbit of codimension 1 and $\mathcal{O}(M)$ is the unique orbit of codimension 0, it follows that $L \simeq M$. Hence L is a sincere directing A-module. In particular, we get $\operatorname{pd}_A N_2 \leq 1$ and $\operatorname{id}_A N_1 \leq 1$, since $\operatorname{Hom}_A(X, L) \neq 0$ for each indecomposable direct summand X of N_2, and $\operatorname{Hom}_A(L, N_1) \neq 0$. Thus we only need to show that $\operatorname{Ext}_A^2(N_1, N_2) = 0$.

We have $\operatorname{Hom}_A(L, \tau_A N_2) = 0$, as L is directing and $\operatorname{Hom}_A(X, L) \neq 0$ for each indecomposable direct summand X of N_2. As a consequence we obtain, using the Auslander-Reiten formula ([**20**, (2.4)]), that $\operatorname{Ext}_A^1(N_2, L) = 0$. Moreover, if $\mathrm{Hom}_A(N_1, L) \neq 0$ then $\mathrm{pd}_A N_1 \leq 1$ and we are done. Thus we may assume $\mathrm{Hom}_A(N_1, L) = 0$ and proceed exactly in the same way as in the proof of [**4**, Lemma 7]. □

Now we assume that M is tilting and show that in this case the variety $\mathrm{mod}_A(\boldsymbol{d})$ is irreducible and normal. We have $\langle \boldsymbol{d}, \mathbf{dim}\, X\rangle_A < 0$ or $\langle \mathbf{dim}\, X, \boldsymbol{d}\rangle_A < 0$ for each indecomposable A-module X such that $X \notin \mathrm{add}\, M$. Indeed, if $X \in \mathscr{T}(M)\backslash \mathrm{add}\, M$ then $\mathrm{Hom}_A(X, M) = 0 = \mathrm{Ext}_A^2(X, M)$ and $\mathrm{Ext}_A^1(X, M) \neq 0$, hence $\langle \mathbf{dim}\, X, \boldsymbol{d}\rangle_A < 0$. Similarly, we show that $\langle \boldsymbol{d}, \mathbf{dim}\, X\rangle_A < 0$ if $X \in \mathscr{F}(M)$. Since the torsion pair $(\mathscr{T}(M), \mathscr{F}(M))$ is splitting, it finishes the proof. Using Lemma 2.1 we conclude that if $\mathcal{O}(N)$ is a maximal orbit in $\mathrm{mod}_A(\boldsymbol{d})$ then $\mathrm{Hom}_A(X, M) \neq 0$ and $\mathrm{Hom}_A(M, X) \neq 0$ for each indecomposable direct summand X of N. It means that $N \in \mathrm{add}\, M$. Hence $N \simeq M$, since $\mathbf{dim}\, N = \mathbf{dim}\, M$. Thus we get the following fact.

Proposition 2.7. *If M is tilting then $\mathcal{O}(M)$ is the unique maximal orbit in $\mathrm{mod}_A(\boldsymbol{d})$. In particular, $\mathrm{mod}_A(\boldsymbol{d})$ is the closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$, hence is irreducible.*

The only thing which remains to show is the normality of $\mathrm{mod}_A(\boldsymbol{d})$ for M tilting. Since $\mathrm{mod}_A(\boldsymbol{d})$ is a complete intersection, hence due to Serre's criterion [**14**, Theorem 11.5] (see also [**3**, Theorem 2.4]) we have to show that $\mathrm{mod}_A(\boldsymbol{d})$ is nonsingular in codimension one, that is, the set of singular points in $\mathrm{mod}_A(\boldsymbol{d})$ is of codimension at least 2. By Proposition 2.6 we already know that the orbits of codimension 1 consist of nonsingular points. To treat the general case we introduce some notation.

An indecomposable A-module X is called homogeneous if $\tau_A X \simeq X$. It follows from [**17**] that if X is an indecomposable homogeneous A-module then we have $q_A(\mathbf{dim}\, X) = 0$ (recall that A is assumed to be tame tilted). On the other hand, if $\boldsymbol{h} \in K_0(A)$ is a connected positive vector such that $q_A(\boldsymbol{h}) = 0$, then all but a finite number of indecomposable A-modules with dimension vector $\boldsymbol{h}$ are homogeneous. Moreover, there exists an open subset $\mathscr{U}$ of $\boldsymbol{P}^1(K)$ and a regular map $\rho : \mathscr{U} \to \mathrm{mod}_A(\boldsymbol{h})$ such that $\rho(\lambda)$ is an indecomposable homogeneous A-module for $\lambda \in \mathscr{U}$ and for each indecomposable homogeneous A-module X with dimension vector $\boldsymbol{h}$ there exists $\lambda \in \mathscr{U}$ with the property $\rho(\lambda) \simeq X$. Indeed, there exists a tame canonical (in the sense of Ringel [**20**]) algebra Λ and a tilting Λ-module T such that $\mathrm{End}_\Lambda(T)$ is (isomorphic to) the support algebra of $\boldsymbol{h}$ (see [**18**, Section 5]). In addition, each indecomposable homogeneous A-module of dimension vector $\boldsymbol{h}$ is of the form $\mathrm{Hom}_\Lambda(T, Y)$ for some indecomposable homogeneous Λ-module Y, where $\boldsymbol{c} := \mathbf{dim}\, Y$ has the property $\boldsymbol{c}_x = 1$ for each x. Since there exists an open subset $\mathscr{V}$ of $\boldsymbol{P}^1(K)$ and a regular map $\sigma : \mathscr{V} \to \mathrm{mod}_\Lambda(\boldsymbol{c})$ such that $\sigma(\lambda)$ is an indecomposable homogeneous Λ-module for each $\lambda \in \mathscr{V}$ and for each indecomposable homogeneous A-module Y there exists $\lambda \in \mathscr{V}$ with the property $\sigma(\lambda) \simeq Y$ (see for example [**2**, Lemma 4.9]), the claim follows from [**12**, Lemma 6.3].

Let L be a direct sum of indecomposable nonhomogeneous A-modules and $\boldsymbol{h}_1, \dots, \boldsymbol{h}_k$ be connected positive vectors from $K_0(A)$ such that $q_A(\boldsymbol{h}_i) = 0$ for each i. Assume that $\mathbf{dim}\, L + \boldsymbol{h}_1 + \cdots + \boldsymbol{h}_k = \boldsymbol{d}$ and let $\mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k)$ be the set of all $N \in \mathrm{mod}_A(\boldsymbol{d})$ such that $N \simeq L \oplus H_1 \oplus \cdots \oplus H_k$, where H_i is an indecomposable homogeneous A-module with dimension vector $\boldsymbol{h}_i$, $i = 1, \dots, k$. It follows from the above remarks that $\mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k)$ is an irreducible constructible set of dimension $\max_{N \in \mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k)} \dim \mathscr{O}(N) + k$.

We keep the above notation in the following lemma.

LEMMA 2.8. *If M is tilting and $k \geq 1$ then* $\dim \mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k) \leq a_A(\boldsymbol{d}) - 2$.

PROOF. For simplicity we write $\mathscr{W}$ instead of $\mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k)$. Let H be an indecomposable homogeneous A-module of dimension vector $\boldsymbol{h}_1$. Since $\mathrm{Ext}_A^1(H, H) \neq 0$ we have $H \notin \mathrm{add}\, M$. Moreover, $H \in \mathscr{T}(M)$ or $H \in \mathscr{F}(M)$, as the torsion pair $(\mathscr{T}(M), \mathscr{F}(M))$ is splitting. Without loss of generality we may assume that the first possibility holds. Then $\mathrm{Hom}_A(H, M) = 0 = \mathrm{Ext}_A^2(H, M)$, thus $\langle \boldsymbol{h}_1, \boldsymbol{d} \rangle_A = -\dim_K \mathrm{Ext}_A^1(H, M) \leq 0$. Then using Lemma 1.2, we get $\langle \boldsymbol{h}_1, \boldsymbol{d} \rangle_A \leq -2$.

According to the formula on the dimension of $\mathscr{W}$ and the formula on the dimension of an orbit given in Proposition 1.3 it is enough to show that $\dim_K \mathrm{Ext}_A^2(N, N) \leq \dim_K \mathrm{Ext}_A^1(N, N) - k - 2$ for $N \in \mathscr{W}$. Note that $\mathrm{Ext}_A^2(X, X) = 0$ for each indecomposable A-module X. Moreover, we have $\mathrm{Ext}_A^1(X, X) \neq 0$ for each indecomposable homogeneous A-module X. Using these remarks we have to show that $\sum_X \mathrm{Ext}_A^2(X, N_X) \leq \sum_X \mathrm{Ext}_A^1(X, N_X) - 2$, where the sums run over all indecomposable direct summands X of N. Here, for each indecomposable direct summand X of N, we denote by N_X such a direct summand of N that $N = X \oplus N_X$.

Fix $N \in \mathscr{W}$. Let H be a homogeneous indecomposable direct summand of N of dimension vector $\boldsymbol{h}_1$. We have

$$\begin{aligned}
&\dim_K \mathrm{Ext}_A^2(H, N_H) \\
&\qquad = \langle \boldsymbol{h}_1, \mathbf{dim}\, N_H \rangle_A - \dim_K \mathrm{Hom}_A(H, N_H) + \dim_K \mathrm{Ext}_A^1(H, N_H) \\
&\qquad \leq \langle \boldsymbol{h}_1, \boldsymbol{d} \rangle_A + \dim_K \mathrm{Ext}_A^1(H, N_H),
\end{aligned}$$

since $q_A(\boldsymbol{h}_1) = 0$. Using the inequality $\langle \boldsymbol{h}_1, \boldsymbol{d} \rangle_A \leq -2$ shown above we get $\dim_K \mathrm{Ext}_A^2(H, N_H) \leq \dim_K \mathrm{Ext}_A^1(H, N_H) - 2$.

Let X be an indecomposable direct summand of N different from H. If $\mathrm{Hom}_A(X, M) \neq 0$ then $\mathrm{pd}_A X \leq 1$, hence $\mathrm{Ext}_A^2(X, N_X) = 0$. In particular, $\dim_K \mathrm{Ext}_A^2(X, N_X) \leq \dim_K \mathrm{Ext}_A^1(X, N_X)$. If $\mathrm{Hom}_A(X, M) = 0$ then $\langle \mathbf{dim}\, X, \boldsymbol{d} \rangle_A =$

$-\dim_K \operatorname{Ext}_A^1(X, M) \le 0$. Since $q_A(\mathbf{dim}\, X) \ge 0$ we get $\langle \mathbf{dim}\, X, \mathbf{dim}\, N_X \rangle_A \le 0$. Using this inequality we obtain similarly as above that $\dim_K \operatorname{Ext}_A^2(X, N_X) \le \dim_K \operatorname{Ext}_A^1(X, N_X)$, and this finishes the proof. $\square$

The following consequence of the above lemma finishes the proof of Main Theorem.

Proposition 2.9. *If M is tilting then* $\operatorname{mod}_A(\boldsymbol{d})$ *is normal.*

Proof. We need to show that $\operatorname{mod}_A(\boldsymbol{d})$ is nonsingular in codimension one. By [**17**] it follows that $\operatorname{mod}_A(\boldsymbol{d})$ is a finite union of sets $\mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k)$ defined above. Thus we only need to show that $\dim \mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k) \le a_A(\boldsymbol{d}) - 2$ if $\mathscr{W}(L, \boldsymbol{h}_1, \dots, \boldsymbol{h}_k)$ contains a nonsingular point in $\operatorname{mod}_A(\boldsymbol{d})$. It trivially follows from the previous lemma if $k \ge 1$. If $k = 0$ then $\mathscr{W}(L) = \mathscr{O}(L)$ and the claim follows from Proposition 2.6. $\square$

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