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Remarks on Littlewood-Paley functions and singular integrals

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Abstract. We consider the generalized Littlewood-Paley square functions arising from rough kernels and prove the L^p-boundedness for a certain range of p depending on the kernel. We also study a class of singular integrals by similar methods.

1. Introduction.

Let $\boldsymbol{R}^n$ be the n-dimensional Euclidean space. We take $\psi \in L^1(\boldsymbol{R}^n)$ satisfying

$$\int_{\boldsymbol{R}^n} \psi(x)\, dx = 0, \tag{1.1}$$

and define the Littlewood-Paley function on $\boldsymbol{R}^n$ by

$$S(f)(x) = S_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

The theory of the Littlewood-Paley functions has been an important part of harmonic analysis. We are referred to [**17**], [**18**] and [**19**] for its history and significance. Readers also can see [**2**], [**3**], [**4**], [**5**], [**9**], [**15**] for recent developments by the authors.

Among many well-known results for the L^p boundedness of S_ψ, one is the following (see Benedek, Calderón and Panzone [**1**]):

THEOREM A. *Let ψ satisfy, in addition to* (1.1),

$$|\psi(x)| \le c(1+|x|)^{-n-\varepsilon} \quad \text{for some } \varepsilon > 0, \tag{1.2}$$

$$\int_{\boldsymbol{R}^n} |\psi(x-y) - \psi(x)|\, dx \le c|y|^\varepsilon \quad \text{for some } \varepsilon > 0. \tag{1.3}$$

Then the operator S_ψ is bounded on $L^p(\boldsymbol{R}^n)$ for all $p \in (1, \infty)$.

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Let $P_t(x) = c_n t/(|x|^2 + t^2)^{(n+1)/2}$ be the Poisson kernel for the upper half space $\boldsymbol{R}^n \times (0, \infty)$. If $Q(x) = ((\partial/\partial t)P_t(x))_{t=1}$, then $S_Q(f)$ is the Littlewood-Paley g function, while if $H(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x)$ is the Haar function on $\boldsymbol{R}$, where χ_E denotes the characteristic function of a set E, then $S_H(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $F(x) = \int_0^x f(y)\,dy$. It is easy to see that Q and H satisfy the conditions (1.2) and (1.3), and hence Theorem A can be applied to the proof of the L^p boundedness of g and μ.

Recently, Ding-Fan-Pan [**2**] considered the Littlewood-Paley functions with rough kernels and proved the following:

THEOREM B. *Let* $\psi \in L^1(\boldsymbol{R}^n)$ *and suppose* (1.1) *and the following*:

(1) $\|\sup_{t>0}||\psi_t| * f|\|_r \le C_r\|f\|_r$ *for some* $r \in (1, \infty)$;

(2) *there exist* $\alpha, \beta > 0$ *such that* $|\hat{\psi}(\xi)| \le C\min(|\xi|^\alpha, |\xi|^{-\beta})$ *for all* $\xi \in \boldsymbol{R}^n\backslash\{0\}$.

Then

$$\|S_\psi(f)\|_p \le C_p\|f\|_p \quad \textit{for all } p \in (2r/(r+1), 2r/(r-1)).$$

In [**2**], they also studied singular integral operators of the form:

$$T_\psi(f)(x) = \int_0^\infty \psi_t * f(x)\frac{dt}{t}$$

and obtained the following result:

THEOREM C. *Suppose that* $\psi \in L^1(\boldsymbol{R}^n)$ *satisfies* (1.1) *and the conditions* (1) *and* (2) *of Theorem B. Then* T_ψ *is bounded on* $L^p(\boldsymbol{R}^n)$ *for all* $p \in (2r/(r+1), 2r/(r-1))$.

We have stated Theorems B and C in slightly different forms from what are presented in [**2**]; however, one can easily see that their proofs imply the results as claimed.

In this note we improve Theorems B and C by essentially relaxing the conditions imposed on ψ. The methods we use will apply to the generalized Littlewood-Paley functions and singular integrals which we now define. Let $b(x)$ be a measurable function on $\boldsymbol{R}^n$ and let $\gamma(t)$ $(t \ge 0)$ be a continuous curve. For $(x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}$, we define the generalized Littlewood-Paley function on $\boldsymbol{R}^{n+1}$, initially for $f \in \mathscr{S}(\boldsymbol{R}^{n+1})$ (the Schwartz space), by

$$S(f)(x,z) = S_{\psi,\gamma}(f)(x,z) = \left(\int_0^\infty |\psi_t \sharp f(x,z)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$\psi_t \sharp f(x,z) = \int_{\boldsymbol{R}^n} b(y)\psi_t(y) f(x-y, z-\gamma(|y|))\,dy.$$

Then it is easy to check

$$\mathscr{F}(\psi_t \sharp f)(\xi,\omega) = \hat{f}(\xi,\omega) A_t(\xi,\omega),$$

where

$$A_t(\xi,\omega) = \int_{\boldsymbol{R}^n} b(y)\psi_t(y) e^{-2\pi i(\langle y,\xi\rangle + \gamma(|y|)\omega)}\,dy.$$

Here $\mathscr{F}(F), \hat{f}$ denote the Fourier transforms and $\langle \cdot\,, \cdot \rangle$ denotes the inner product: $\langle y, \xi \rangle = y_1\xi_1 + \cdots + y_n\xi_n$. We also define a singular integral of the form:

$$T_{\psi,\gamma}(f)(x,z) = \int_0^\infty \psi_t \sharp f(x,z) \frac{dt}{t}.$$

We present the theorems for $S_{\psi,\gamma}$ in Section 2 and give their proofs in Sections 3 and 4. In Section 2, we also give applications of the theorems, where we see examples of curves γ and functions ψ which satisfy the requirements assumed in the theorems. We give the results for S_ψ in Section 5 as immediate corollaries to the results for $S_{\psi,\gamma}$. In Section 5, we also give some remarks concerning the conditions imposed on the functions ψ. Finally, we study $T_{\psi,\gamma}$ and T_ψ in Section 6, where we see these operators are closely related to the classical Calderón-Zygmund singular integrals arising from homogeneous kernels and techniques similar to those used to prove the L^p boundedness of the Littlewood-Paley functions also apply to $T_{\psi,\gamma}$ and T_ψ.

2. Results for the generalized Littlewood-Paley functions.

We define two maximal functions. Let

$$N_{\psi,\gamma}(f)(x,z) = \sup_{k\in\boldsymbol{Z}} \left| \int_{\boldsymbol{R}^n} |b(y)|^2 p_{2^k}(y) f(x-y, z-\gamma(|y|))\,dy \right|,$$

where $p(x) = \int_1^2 |\psi_t(x)|\,dt/t$ and $\boldsymbol{Z}$ denotes the set of integers; and

$$M_{\psi,\gamma}(f)(x,z) = \sup_{t>0} \left| \int_{\boldsymbol{R}^n} |b(y)|^2 |\psi_t(y)| f(x-y, z-\gamma(|y|))\,dy \right|.$$

Note that $N_{\psi,\gamma}(f) \le M_{\psi,\gamma}(|f|)$.

Our results involve the maximal functions $N_{\psi,\gamma}$ and $M_{\psi,\gamma}$, and can be formulated as follows:

Theorem 1. *Suppose there exists* $\varepsilon > 0$ *such that*

$$\int_{2^k}^{2^{k+1}} |A_t(\xi,\omega)|^2 \, dt/t \le c \min(|2^k\xi|^{\varepsilon}, |2^k\xi|^{-\varepsilon}) \tag{2.1}$$

for all $\xi \in \boldsymbol{R}^n \backslash \{0\}$, $\omega \in \boldsymbol{R}$ *and* $k \in \boldsymbol{Z}$, *where* c *is independent of* ξ, ω *and* k. *Then* $S_{\psi,\gamma}$ *is bounded on* $L^2(\boldsymbol{R}^{n+1})$. *If we further assume that there exists* $r \in (1,\infty)$ *such that* $N_{\psi,\gamma}$ *is bounded on* $L^r(\boldsymbol{R}^{n+1})$, *then* $S_{\psi,\gamma}$ *is bounded on* $L^p(\boldsymbol{R}^{n+1})$ *for all* $p \in (2, 2r/(r-1))$.

Theorem 2. *Suppose that* (2.1) *holds and there exists* $r \in (1,\infty)$ *such that* $M_{\psi,\gamma}$ *is bounded on* $L^r(\boldsymbol{R}^{n+1})$. *Then* $S_{\psi,\gamma}$ *is bounded on* $L^p(\boldsymbol{R}^{n+1})$ *for all* $p \in (2r/(r+1), 2)$.

Let $b(y)$ be a radial function, $b(y) = b_0(|y|)$, and $\psi(y) = \chi_{(0,1]}(|y|)|y|^{-n+1}\Omega(y')$ $(y' = y/|y|)$, where $\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0$. Here σ denotes the Lebesgue surface measure on the unit sphere S^{n-1} in $\boldsymbol{R}^n$. Then we obtain the Marcinkiewicz integral along a curve (see also Remark 4 in Section 5):

$$S_{\psi,\gamma}(f)(x,z) = \mu_\gamma(f)(x,z) = \left(\int_0^\infty |F_{t,\gamma}(x,z)|^2 t^{-3}\, dt\right)^{1/2},$$

where

$$F_{t,\gamma}(x,z) = \int_{|y|\le t} \Omega(y')|y|^{-n+1} b_0(|y|) f(x-y, z-\gamma(|y|))\, dy.$$

To prove Theorems 1 and 2 we adapt the method of [**7**] for our case of square functions. We show two vector valued inequalities (see Lemmas 1 and 2 below), which are needed to apply the Littlewood-Paley method. Since the duality argument as in [**7**] does not work completely in our case, we need different assumptions involving $M_{\psi,\gamma}$ or $N_{\psi,\gamma}$ according as $1 < p < 2$ or $2 < p < \infty$.

By Theorem 1 we have the following:

Corollary 1. *For* $\psi \in L^1(\boldsymbol{R}^n)$ *suppose*

$$\int_{|y|<t} \psi(y)\, dy = 0 \tag{2.2}$$

for all $t > 0$. *We further assume that* ψ *is compactly supported and*

$$\int_{\boldsymbol{R}^n} |\psi(y)|^u \, dy < \infty \quad \textit{for some } u \in (1,\infty). \tag{2.3}$$

Let $\gamma(t) = c_1 t^{\alpha_1} + \cdots + c_m t^{\alpha_m}$, *with* $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$, $\alpha_i \neq 1$, $c_i \neq 0$ *for all* i, *and let* $b(y) \equiv 1$. *Then* $S_{\psi,\gamma}$ *is bounded on* L^p *for all* $p \in [2,\infty)$.

PROOF. By Theorem 1 it suffices to prove (2.1) and the L^r boundedness of $N_{\psi,\gamma}$ for all $r \in (1,\infty)$. First we prove (2.1). Note that

$$\int_{2^k}^{2^{k+1}} |A_t(\xi,\omega)|^2 \frac{dt}{t} = \iint_{\boldsymbol{R}^n \times \boldsymbol{R}^n} \psi(y)\bar{\psi}(v) I(y,v,\xi,\omega,k)\, dydv, \tag{2.4}$$

where

$$I(y,v,\xi,\omega,k) = \int_1^2 \exp(-2\pi i(2^k t\langle y-v,\xi\rangle + \omega(\gamma(2^k t|y|) - \gamma(2^k t|v|))))\frac{dt}{t}.$$

We have $|I(y,v,\xi,\omega,k)| \le \log 2$ and by the van der Corput estimate (see [**20**])

$$|I(y,v,\xi,\omega,k)| \le c\left(|2^k\langle y-v,\xi\rangle| + \sum_{i=1}^m |\omega 2^{k\alpha_i}(|y|^{\alpha_i} - |v|^{\alpha_i})|\right)^{-1/(m+1)}. \tag{2.5}$$

By (2.4) and (2.5)

$$\int_{2^k}^{2^{k+1}} |A_t(\xi,\omega)|^2 \frac{dt}{t} \le c|2^k\xi|^{-\varepsilon} J_\varepsilon(\psi)$$

for all sufficiently small $\varepsilon > 0$, where

$$J_\varepsilon(\psi) = \sup_{|\xi|=1} \iint_{\boldsymbol{R}^n \times \boldsymbol{R}^n} |\psi(y)\psi(v)|\,|\langle\xi, y-v\rangle|^{-\varepsilon}\, dydv.$$

In [**15**] it is proved that $J_\varepsilon(\psi) < \infty$ for a sufficiently small ε. The rest of the estimate (2.1) follows from (2.2) and the assumption that ψ is compactly supported.

Define a non-negative measure μ_k by

$$\hat{\mu}_k(\xi,\omega) = \int_{\boldsymbol{R}^n} p_{2^k}(y) e^{-2\pi i(\langle y,\xi\rangle + \gamma(|y|)\omega)}\, dy.$$

Then

$$N_{\psi,\gamma}(f)(x,z) = \sup_{k\in\boldsymbol{Z}} |\mu_k * f(x,z)|.$$

Let $N_\psi(f) = \sup_{k\in\boldsymbol{Z}} |p_{2^k} * f|$. Then N_ψ is bounded on $L^p(\boldsymbol{R}^n)$ for all $p\in(1,\infty)$ (see Corollary 3 in Section 5). Therefore by Theorem C of [**7**] the L^r boundedness, for all $r\in(1,\infty)$, of $N_{\psi,\gamma}$ follows from the estimates:

$$|\hat{\mu}_k(\xi,\omega)| \le c|a_k\omega|^{-\varepsilon} \quad \text{for some } \varepsilon > 0, \tag{2.6}$$

$$|\hat{\mu}_k(\xi,\omega) - \hat{\mu}_k(\xi,0)| \le c|a_k\omega|^{\varepsilon} \quad \text{for some } \varepsilon > 0, \tag{2.7}$$

where $a_k = 2^{\alpha_m k}$ for $k > 0$ and $a_k = 2^{\alpha_1 k}$ for $k \le 0$. Since ψ is compactly supported, we easily get (2.7).

To prove (2.6), put

$$B(\xi, \omega, k) = \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(y)\psi(v)| I(y, v, \xi, \omega, k)\, dydv.$$

We note that

$$(2.8) \qquad |\hat{\mu}_k(\xi, \omega)| = \left| \int_1^2 \int_{\mathbf{R}^n} |\psi_t(y)| e^{-2\pi i (2^k \langle y, \xi \rangle + \gamma(2^k |y|)\omega)}\, dydt/t \right|$$
$$\le \left(\int_1^2 \left| \int_{\mathbf{R}^n} |\psi_t(y)| e^{-2\pi i (2^k \langle y, \xi \rangle + \gamma(2^k |y|)\omega)}\, dy \right|^2 dt \right)^{1/2}$$
$$\le c |B(\xi, \omega, k)|^{1/2}.$$

Now by (2.5) we have

$$(2.9) \qquad |B(\xi, \omega, k)| \le c \min_{1 \le i \le m} |2^{k\alpha_i}\omega|^{-\varepsilon} L_{\varepsilon, i}(\psi)$$

for sufficiently small ε, where

$$L_{\varepsilon, i}(\psi) = \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(y)\psi(v)|\, ||y|^{\alpha_i} - |v|^{\alpha_i}|^{-\varepsilon}\, dydv.$$

For simplicity we assume that ψ is supported in $\{|x| < 1\}$. Then by Hölder's inequality and (2.3) we have

$$L_{\varepsilon, i}(\psi) \le \|\psi\|_u^2 \left(\iint_{\{|y|<1\} \times \{|v|<1\}} ||y|^{\alpha_i} - |v|^{\alpha_i}|^{-\varepsilon u'}\, dydv \right)^{1/u'},$$

where u' is the conjugate exponent to u. It is easy to see that the last integral is finite if $u'\varepsilon < \min(1, 1/\alpha_i)$. Thus by (2.8) and (2.9) we get (2.6), which completes the proof of the corollary. □

REMARK 1. Let $L : \mathbf{R}^n \to \mathbf{R}^m$, $n \ge m$, be a linear transformation. Then the condition (2.1) in Theorems 1 and 2 can be replaced by

$$(2.10) \qquad \int_{2^k}^{2^{k+1}} |A_t(\xi, \omega)|^2\, dt/t \le c \min(|2^k L\xi|^{\varepsilon}, |2^k L\xi|^{-\varepsilon}) \quad \text{for some } \varepsilon > 0,$$

for all $\xi \in \mathbf{R}^n \backslash \{0\}$, $\omega \in \mathbf{R}$ and $k \in \mathbf{Z}$. Also the bounds in Theorems are independent of the linear transform L. This can be seen as follows. Let Π_m be the projection $\Pi_m x = (x_1, \dots, x_m)$ for $x \in \boldsymbol{R}^n$. Then by checking the proofs of Theorems 1 and 2, we can easily see that (2.1) can be replaced by

$$\int_{2^k}^{2^{k+1}} |A_t(\xi, \omega)|^2 \, dt/t \le c \min(|2^k \Pi_m \xi|^{\varepsilon}, |2^k \Pi_m \xi|^{-\varepsilon}) \quad \text{for some } \varepsilon > 0.$$

Thus the rest of the proof follows that in Corollary in [**2**].

As an application of Remark 1, we now state a result on the Marcinkiewicz integral along a curve. We define the two dimensional maximal function

$$g_\gamma^*(u, v) = \sup_{k \in \boldsymbol{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |g(u - t, v - \gamma(t))| \, dt.$$

Clearly $g_\gamma^*(u, v) \approx \sup_{s>0} s^{-1} \int_0^s |g(u - t, v - \gamma(t))| \, dt$.

Corollary 2. *Let $n \ge 2$, and let $\mu_\gamma(f)$ be the Marcinkiewicz integral along γ. If b is bounded and $\Omega \in H^1(S^{n-1})$ satisfies $\int_{S^{n-1}} \Omega \, d\sigma = 0$, then we have*

$$\|\mu_\gamma(f)\|_{L^p(\boldsymbol{R}^{n+1})} \le C_p \|f\|_{L^p(\boldsymbol{R}^{n+1})}$$

for all $p \in (2r/(r+1), 2r/(r-1))$ provided

$$\|g_\gamma^*\|_{L^r(\boldsymbol{R}^2)} \le C_r \|g\|_{L^r(\boldsymbol{R}^2)} \quad \text{for some } r \in (1, \infty). \tag{2.11}$$

This corollary shows that the requirement of γ being convex and increasing in Theorem 4 of [**5**] is superfluous.

Proof. By the atomic decomposition of $H^1(S^{n-1})$ (see [**3**]) one can assume $\Omega(y') = a(y')$ is an L^∞-atom centered at $\mathbf{1} = (1, 0, \dots, 0)$, and it suffices to show

$$\|\mu_\gamma(f)\|_p \le C \|f\|_p$$

with C independent of $a(y')$.

Recall an L^∞-atom $a(y')$ centered at $\mathbf{1}$ is an L^∞ function $a(y')$ on S^{n-1} satisfying

$$\operatorname{supp}(a) \subset B(\mathbf{1}, \rho) \cap S^{n-1}, \quad \rho \in (0, 2],$$

$$\int_{S^{n-1}} a(y') \, d\sigma(y') = 0,$$

$$\|a\|_\infty \le \rho^{-(n-1)}$$

(where $B(\mathbf{1}, \rho)$ denotes the ball in $\boldsymbol{R}^n$ with center $\mathbf{1}$ and radius ρ). For this ρ, let $L_\rho \xi = (\rho^2 \xi_1, \rho \xi_2, \dots, \rho \xi_n)$. Then by checking the proof in [**3**], one easily sees that

$$\int_{2^k}^{2^{k+1}} |A_t(\xi, \omega)|^2 \, dt/t \le c \min(|2^k L_\rho \xi|^{\varepsilon}, |2^k L_\rho \xi|^{-\varepsilon}) \tag{2.12}$$

for some $\varepsilon > 0$. Thus by Remark 1 it remains to show

$$\|M_{\psi,\gamma}(f)\|_r \le C\|f\|_r.$$

It is easy to see that

$$M_{\psi,\gamma}(f)(x,z) \le c \sup_{k\in\boldsymbol{Z}} 2^{-k} \int_{|y|\le 2^k} \frac{|\Omega(y')|}{|y|^{n-1}} |f(x-y, z-\gamma(|y|))|\, dy.$$

By using the polar coordinates, the right hand side of the above inequality is

$$c \sup_{k\in\boldsymbol{Z}} 2^{-k} \int_0^{2^k} \int_{S^{n-1}} |\Omega(y')|\, |f(x-ty', z-\gamma(t))|\, d\sigma(y')\, dt,$$

which is bounded, up to a constant, by

$$\int_{S^{n-1}} |\Omega(y')| \mathscr{M}_{y',\gamma}(f)(x,z)\, d\sigma(y'),$$

where

$$\mathscr{M}_{y',\gamma}(f)(x,z) = \sup_{k\in\boldsymbol{Z}} 2^{-k} \int_0^{2^k} |f(x-ty', z-\gamma(t))|\, dt.$$

Now we have

$$\|M_{\psi,\gamma}(f)\|_r \le c \int_{S^{n-1}} |\Omega(y')|\, \|\mathscr{M}_{y',\gamma}(f)\|_r\, d\sigma(y').$$

By the Calderón-Zygmund rotation method, one easily sees that (2.11) implies

$$\|\mathscr{M}_{y',\gamma}(f)\|_{L^r(\boldsymbol{R}^{n+1})} \le C\|f\|_{L^r(\boldsymbol{R}^{n+1})}$$

with C independent of y', and hence the corollary is proved. $\square$

3. Proof of Theorem 1.

Let $\mathscr{H}$ denote the Hilbert space $L^2((0,\infty), dt/t)$. For each $k \in \boldsymbol{Z}$ we consider an operator T_k mapping functions on $\boldsymbol{R}^{n+1}$ to $\mathscr{H}$-valued functions on $\boldsymbol{R}^{n+1}$, which is defined by

$$(T_k(f)(x,z))(t) = T_k(f)(x,z;t) = \psi_t \sharp f(x,z)\chi_{[1,2)}(2^{-k}t).$$

Note that

$$|T_k(f)(x,z)|_{\mathscr{H}} = \left(\int_{2^k}^{2^{k+1}} |\psi_t \sharp f(x,z)|^2 \frac{dt}{t}\right)^{1/2}.$$

Lemma 1. *Let* $2 < s < \infty$, $r = (s/2)' = s/(s-2)$. *If* $N_{\psi,\gamma}$ *is bounded on* $L^r(\boldsymbol{R}^{n+1})$, *then*

$$\left\|\left(\sum_k |T_k(f_k)|_{\mathscr{H}}^2\right)^{1/2}\right\|_s \le c\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_s.$$

Proof. Take a non-negative $g \in L^r$ satisfying $\|g\|_r \le 1$ and

$$I := \left\|\left(\sum_k |T_k(f_k)|_{\mathscr{H}}^2\right)^{1/2}\right\|_s^2 = \int\left(\sum_k |T_k(f_k)|_{\mathscr{H}}^2\right) g\, dxdz.$$

Then since

$$|T_k(f_k)|_{\mathscr{H}}^2 \le \|\psi\|_1 \int_{\boldsymbol{R}^n} |b(y)|^2 p_{2^k}(y) |f_k(x-y, z-\gamma(|y|))|^2\, dy,$$

we have

$$\begin{aligned} I &\le c \sum_k \int |f_k(x,z)|^2 \left(\int_{\boldsymbol{R}^n} |b(y)|^2 p_{2^k}(y) g(x+y, z+\gamma(|y|))\, dy\right) dxdz \\ &\le c \sum_k \int |f_k|^2 \tilde{N}_{\psi,\gamma}(g)\, dxdz, \end{aligned}$$

where

$$\tilde{N}_{\psi,\gamma}(f) = \sup_{k \in \boldsymbol{Z}} \left|\int_{\boldsymbol{R}^n} |b(y)|^2 p_{2^k}(y) f(x+y, z+\gamma(|y|))\, dy\right|.$$

By Hölder's inequality we have

$$\begin{aligned} \sum_k \int |f_k|^2 \tilde{N}_{\psi,\gamma}(g)\, dxdz &\le c\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_s^2 \|\tilde{N}_{\psi,\gamma}(g)\|_r \\ &\le c\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_s^2, \end{aligned}$$

where the L^r-boundedness of $\tilde{N}_{\psi,\gamma}$ follows from that of $N_{\psi,\gamma}$. This completes the proof. □

To prove Theorem 1 we use Lemma 1 and the ordinary Littlewood-Paley decomposition. Take $\Psi \in C^\infty(\boldsymbol{R}^n)$ supported in $\{1/2 \le |\xi| \le 2\}$ and satisfying

$$\sum_{j\in\boldsymbol{Z}}\Psi(2^j\xi)=1\quad\text{for all }\ \xi\neq 0.$$

Define

$$\widehat{\Delta_j(f)}(\xi,\omega)=\Psi(2^j\xi)\hat{f}(\xi,\omega)\quad\text{for all }\ j\in\boldsymbol{Z}.$$

Then we note that $\Delta_j(\psi_t\sharp f)=\psi_t\sharp\Delta_j(f)$. Decompose

$$\psi_t\sharp f(x,z)=\sum_{j\in\boldsymbol{Z}}F_j(x,z;t),$$

where

$$F_j(x,z;t)=\sum_{k\in\boldsymbol{Z}}\Delta_{j+k}(\psi_t\sharp f)(x,z)\chi_{[2^k,2^{k+1})}(t).$$

Set

$$U_j(f)(x,z)=\left(\int_0^\infty|F_j(x,z;t)|^2\frac{dt}{t}\right)^{1/2}=\left(\sum_k|T_k(\Delta_{j+k}(f))|_{\mathscr{H}}^2\right)^{1/2}.$$

Then

$$S_{\psi,\gamma}(f)(x,z)\leq\sum_{j\in\boldsymbol{Z}}U_j(f)(x,z).$$

We prove the estimate:

$$\|U_j(f)\|_2\leq c2^{-\varepsilon|j|/2}\|f\|_2, \tag{3.1}$$

where ε is the same as that in (2.1). Put $E_j=\{2^{-1-j}\leq|\xi|\leq 2^{1-j}\}$. Then by the Plancherel theorem and (2.1) we have

$$\begin{aligned}\|U_j(f)\|_2^2&=\sum_{k\in\boldsymbol{Z}}\int_{\boldsymbol{R}^{n+1}}\int_{2^k}^{2^{k+1}}|\Delta_{j+k}(\psi_t\sharp f)(x,z)|^2\frac{dt}{t}dxdz\\&\leq\sum_{k\in\boldsymbol{Z}}c\int_{E_{j+k}\times\boldsymbol{R}}\left(\int_{2^k}^{2^{k+1}}|A_t(\xi,\omega)|^2\frac{dt}{t}\right)|\hat{f}(\xi,\omega)|^2\,d\xi d\omega\\&\leq\sum_{k\in\boldsymbol{Z}}c\int_{E_{j+k}\times\boldsymbol{R}}\min(|2^k\xi|^\varepsilon,|2^k\xi|^{-\varepsilon})|\hat{f}(\xi,\omega)|^2\,d\xi d\omega\\&\leq c2^{-\varepsilon|j|}\sum_{k\in\boldsymbol{Z}}\int_{E_{j+k}\times\boldsymbol{R}}|\hat{f}(\xi,\omega)|^2\,d\xi d\omega\leq c2^{-\varepsilon|j|}\|f\|_2^2,\end{aligned}$$

where the last inequality holds since the sets E_j are finitely overlapping.

The L^2 boundedness of $S_{\psi,\gamma}$ is an immediate consequence of (3.1). Suppose that $1 < r < \infty$, $p \in (2, 2r/(r-1))$ and $N_{\psi,\gamma}$ is bounded on L^r. Note that if s is related to r as in Lemma 1, then $p < s = 2r/(r-1)$. By the Littlewood-Paley inequality and Lemma 1 we see that

$$\begin{aligned}(3.2) \qquad \|U_j(f)\|_s &= \left\| \left(\sum_{k \in \boldsymbol{Z}} |T_k(\Delta_{j+k}(f))|_{\mathscr{H}}^2 \right)^{1/2} \right\|_s \\ &\le c \left\| \left(\sum_{k \in \boldsymbol{Z}} |\Delta_{j+k}(f)|^2 \right)^{1/2} \right\|_s \\ &\le c\|f\|_s.\end{aligned}$$

Interpolating between the two estimates (3.1) and (3.2), we get

$$\|U_j(f)\|_p \le c2^{-\theta\varepsilon|j|/2}\|f\|_p$$

for some $\theta \in (0, 1)$. Thus

$$\|S_{\psi,\gamma}(f)\|_p \le \sum_{j \in \boldsymbol{Z}} \|U_j(f)\|_p \le c\|f\|_p.$$

This completes the proof.

4. Proof of Theorem 2.

Lemma 2. *Let $1 < s < 2$, $r = (s'/2)' = s/(2-s)$. If $M_{\psi,\gamma}$ is bounded on L^r, then*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_s \le c \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s.$$

For a function h on $\boldsymbol{R}^{n+1} \times (0, \infty)$, define an $\mathscr{H}$-valued function $P_k(h)$ by

$$(P_k(h)(x,z))(t) = P_k(h)(x,z;t) = h(x,z;t)\chi_{[1,2)}(2^{-k}t).$$

We also let T_k act on such h by $(T_k(h)(x,z))(t) = T_k(h)(x,z;t) = T_k(h(\cdot\,,\cdot\,;t)) \cdot (x,z;t)$. To prove Lemma 2 we need the following.

Lemma 3. *Under the assumptions of Lemma 2, for a sequence $\{h_k(x,z;t)\}$ we have*

$$\left\| \left(\sum_k |T_k(h_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_{s'} \le c \left\| \left(\sum_k |P_k(h_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_{s'}.$$

Proof. As in the proof of Lemma 1, take a non-negative $g \in L^r$ with $\|g\|_r \le 1$ and

$$I := \left\| \left(\sum_k |T_k(h_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_{s'}^2 = \int \left(\sum_k |T_k(h_k)|_{\mathscr{H}}^2 \right) g\, dxdz.$$

Note that

$$\int |T_k(h_k)|_{\mathscr{H}}^2 g\, dxdz \le \|\psi\|_1 \int \tilde{M}_{\psi,\gamma}(g) |P_k(h_k)|_{\mathscr{H}}^2\, dxdz,$$

where

$$\tilde{M}_{\psi,\gamma}(f)(x,z) = \sup_{t>0} \left| \int_{\boldsymbol{R}^n} |b(y)|^2\, |\psi_t(y)| f(x+y, z+\gamma(|y|))\, dy \right|.$$

Therefore, as in the proof of Lemma 1 we have

$$\begin{aligned} I &\le c \left\| \left(\sum_k |P_k(h_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_{s'}^2 \|\tilde{M}_{\psi,\gamma}(g)\|_r \\ &\le c \left\| \left(\sum_k |P_k(h_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_{s'}^2. \end{aligned}$$

This completes the proof. □

Now we give the proof of Lemma 2. Let $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ denote the inner product in $\mathscr{H}$. Then

$$\int \langle T_k(f_k)(x,z;\cdot), h_k(x,z;\cdot) \rangle_{\mathscr{H}}\, dxdz = \int \langle P_k(f_k)(x,z;\cdot), \tilde{T}_k(h_k)(x,z;\cdot) \rangle_{\mathscr{H}}\, dxdz,$$

where

$$\tilde{T}_k(h)(x,z;t) = \chi_{[1,2)}(2^{-k}t) \int_{\boldsymbol{R}^n} \bar{b}(y) \bar{\psi}_t(y) h(x+y, z+\gamma(|y|); t)\, dy,$$

and $P_k(f_k)(x,z;t) = f_k(x,z)\chi_{[1,2)}(2^{-k}t)$. Therefore by Hölder's inequality and Lemma 3 we see that

$$\begin{aligned} &\left| \int \sum_k \langle T_k(f_k)(x,z;\cdot), h_k(x,z;\cdot) \rangle_{\mathscr{H}}\, dxdz \right| \\ &\qquad \le c \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s \left\| \left(\sum_k |P_k(h_k)|_{\mathscr{H}}^2 \right)^{1/2} \right\|_{s'}. \end{aligned}$$

Taking the supremum over $\{h_k(x,z;t)\}$ with $\|(\sum_k |P_k(h_k)|_{\mathscr{H}}^2)^{1/2}\|_{s'} \le 1$, we get the conclusion.

We turn to the proof of Theorem 2. Let p, r, with $p \in (2r/(r+1), 2)$, be as in Theorem 2, and $r = s/(2-s)$. Then note that $2r/(r+1) = s < p$. By Lemma 2 and the Littlewood-Paley inequality we have $\|U_j(f)\|_s \le c\|f\|_s$, where U_j is as in the proof of Theorem 1. Since we also have the L^2-estimate (3.1), arguing as in the proof of Theorem 1 we can reach the conclusion.

5. Results for the Littlewood-Paley functions.

Let

$$N_\psi(f)(x) = \sup_{k \in \boldsymbol{Z}} |p_{2^k} * f(x)| \quad (x \in \boldsymbol{R}^n),$$

$$M_\psi(f)(x) = \sup_{t>0} ||\psi_t| * f(x)|,$$

where we recall $p(x) = \int_1^2 |\psi_t(x)|\,dt/t$. Note that $N_\psi(f) \le M_\psi(|f|)$. $M_\psi(f)$ is the maximal function in the statement of Theorem B and $N_\psi(f)$ is in the proof of Corollary 1. If ψ satisfies the conditions:

$$\left|\int_1^2 \mathscr{F}(|\psi|)(t\xi)\,dt\right| \le c|\xi|^{-\varepsilon} \quad \text{for some } \varepsilon > 0, \tag{5.1}$$

$$\left|\int_1^2 \mathscr{F}(|\psi|)(t\xi)\,dt - \|\psi\|_1\right| \le c|\xi|^{\varepsilon} \quad \text{for some } \varepsilon > 0, \tag{5.2}$$

then N_ψ is bounded on L^p for all $p \in (1, \infty)$ by [**7**, Theorem A].

Choosing $b(y) \equiv 1$, $\gamma(t) \equiv 0$ and $f(x,z) = f_1(x)f_2(z)$, as immediate consequences of Theorems 1 and 2 we have the following.

Theorem 3. *Suppose there exists $\varepsilon > 0$ such that*

$$\int_1^2 |\hat{\psi}(t\xi)|^2\,dt \le c\min(|\xi|^{\varepsilon}, |\xi|^{-\varepsilon}) \quad \textit{for all } \xi \in \boldsymbol{R}^n \backslash \{0\}. \tag{5.3}$$

Then S_ψ is bounded on $L^2(\boldsymbol{R}^n)$. If we further assume that there exists $r \in (1, \infty)$ such that N_ψ is bounded on $L^r(\boldsymbol{R}^n)$, then S_ψ is bounded on $L^p(\boldsymbol{R}^n)$ for all $p \in (2, 2r/(r-1))$.

Theorem 4. *Suppose that* (5.3) *holds and there exists $r \in (1, \infty)$ such that M_ψ is bounded on $L^r(\boldsymbol{R}^n)$. Then S_ψ is bounded on $L^p(\boldsymbol{R}^n)$ for all $p \in (2r/(r+1), 2)$.*

There exists ψ which satisfies $\int_1^2 |\hat{\psi}(t\xi)|^2\,dt \le c|\xi|^{-\varepsilon}$ for some $\varepsilon > 0$ but does not satisfy the pointwise estimate $|\hat{\psi}(\xi)| \le c|\xi|^{-\delta}$ for any $\delta > 0$ (see Remark 3), and for any $r \in (1,2)$ there exists ψ such that $|\hat{\psi}(\xi)| \le c \min(|\xi|^{\delta}, |\xi|^{-\delta})$ for some $\delta > 0$, which implies (5.3), and N_ψ is bounded on L^r but M_ψ is not (see Remark 2). From these facts we see that Theorems 3 and 4 essentially improve Theorem B.

Let

$$B_\varepsilon(\psi) = \int_{|x| \ge 1} |\psi(x)|\,|x|^{\varepsilon}\,dx \quad \text{for } \varepsilon > 0,$$

$$D_u(\psi) = \left(\int_{|x| \le 1} |\psi(x)|^u\,dx \right)^{1/u} \quad \text{for } u > 1.$$

Then by Theorems 3, 4 and the results of [**15**] we have the following:

COROLLARY 3. *Let $\psi \in L^1(\boldsymbol{R}^n)$ satisfy* (1.1).

(i) *Suppose the following conditions hold for ψ:*
  (1) $B_\varepsilon(\psi) < \infty$ *for some* $\varepsilon > 0$;
  (2) $D_u(\psi) < \infty$ *for some* $u > 1$;
  (3) $|\psi(x)| \le h(r)\Omega(\theta)$ $(r = |x|, \theta = x/|x|)$ *for all* $x \in \boldsymbol{R}^n \setminus \{0\}$, *for some non-negative functions h and Ω such that*
    (a) $h(r)$ *is non-increasing for* $r \in (0,\infty)$,
    (b) $h(|x|) \in L^1(\boldsymbol{R}^n)$,
    (c) $\Omega \in L^q(S^{n-1})$ *for some* q, $1 < q \le \infty$.

  Then S_ψ is bounded on L^p for all $p \in (1,\infty)$.

(ii) *If we assume that ψ satisfies the conditions* (1), (2) *of* (i) *and*
  (4) $|\psi(x)| \le h(r)\Omega(\theta)$ *for* $|x| \ge 1$,

  where h and Ω are as in (3) *of* (i), *then S_ψ is bounded on L^p for all $p \in [2,\infty)$. In particular, if ψ is supported in $\{|x| \le 1\}$, the condition* (2) *of* (i) *only is sufficient for the L^p-boundedness, $2 \le p < \infty$, of S_ψ.*

PROOF. The conditions (1), (2) and (4) imply (5.3), (5.1) and (5.2) (see [**15**, Lemmas 1–3]). As we have noted above, the L^p boundedness of N_ψ, for all $p \in (1,\infty)$, follows from (5.1) and (5.2). Thus, we have the second assertion by Theorem 3.

If the condition (3) holds, $M_\psi(f)$ is bounded, up to a constant, by the maximal function

$$\sup_{t>0} t^{-n} \int_{|y|<t} |f(x-y)|\Omega(y/|y|)\,dy,$$

which is bounded on L^p for all $p \in (1,\infty)$ (see [**15**] for this argument). Thus we get the first assertion by Theorems 3 and 4. This completes the proof. $\square$

REMARK 2. If $1 < p < 2$, then there exists ψ on $\boldsymbol{R}^1$ such that although ψ

is supported in $\{|x| \le 1\}$ and satisfies the condition (2) of Corollary 3, S_ψ is not bounded on L^p. Let $\psi^{(\alpha)}(x) = \alpha|1-|x||^{\alpha-1}\chi_{(-1,1)}(x)\,\mathrm{sgn}(x)$ $(\alpha > 0)$. Then if $2/(2\alpha+1) > p$, $S_{\psi^{(\alpha)}}$ is not bounded on L^p. To see this we consider the square function g_λ^*. It is known that if $\lambda > 1$ and $1 < p < 2/\lambda$, then g_λ^* is not bounded on L^p (see [**10**]). By this and the pointwise relation between $S_{\psi^{(\alpha)}}$ and $g_{2\alpha+1}^*$ (see [**21**] and also [**16**]) we see the unboundedness of $S_{\psi^{(\alpha)}}$.

We also note that $\psi^{(\alpha)}$ satisfies (5.3) and $N_{\psi^{(\alpha)}}$ is bounded on L^p (see the proof of Corollary 3 (ii); in fact, it is easy to see that $|\widehat{\psi^{(\alpha)}}(\xi)| \le c\min(|\xi|^\delta, |\xi|^{-\delta})$ for some $\delta > 0$), but $M_{\psi^{(\alpha)}}$ $(2/(2\alpha+1) > p)$ is not bounded on L^p since if it is bounded on L^p, by Theorem 4 $S_{\psi^{(\alpha)}}$ is bounded on L^p (note that $2p/(p+1) < p$), which contradicts the observation made in the previous paragraph.

Remark 3. For simplicity we consider on $\boldsymbol{R}^1$. It is known that there exists a bounded ψ supported in $\{|x| \le 1\}$ such that $\int \psi = 0$ and ψ does not satisfy the pointwise estimate $|\hat{\psi}(\xi)| \le c|\xi|^{-\delta}$ for any $\delta > 0$ (see, e.g., Theorem 2.5.2 in [**12**, p. 57]). For this ψ, by Lemmas 2 and 3 in [**15**] we have $\int_1^2 |\hat{\psi}(t\xi)|^2\,dt \le c_\varepsilon|\xi|^{-\varepsilon}$ for all $\varepsilon \in (0,1)$.

Remark 4. Let

$$\psi(x) = |x|^{-n+\rho}\Omega(x')\chi_{(0,1]}(|x|) \quad (\rho > 0, x' = x/|x|),$$

where $\Omega \in L^1(S^{n-1})$ and $\int_{S^{n-1}} \Omega(\theta)\,d\sigma(\theta) = 0$. Put $\mu_\rho(f) = S_\psi(f)$. Then, when $\rho = 1$, $\mu_\rho(f)$ is the Marcinkiewicz integral defined by Stein [**17**]. Weak $(1,1)$ and L^p estimates $(1 < p < \infty)$ were studied by [**17**] and [**11**] by assuming certain smoothness conditions on Ω. For the recent developments, see the works of the authors cited in Section 1 and also [**6**], [**14**].

Remark 5. It is easy to see that the conditions $|x| \ge 1$ and $|x| \le 1$ in the definitions of B_ε, D_u and Corollary 3 (4) can be replaced by $|x| \ge a$ and $|x| \le a$, respectively, for any $a > 0$.

6. Singular integrals and Littlewood Paley theory.

Let $\psi \in L^1$ satisfy (1.1). Define a function Ω_ψ of homogeneous of degree 0 by

$$\Omega_\psi(x) = |x|^n \int_0^\infty \psi(rx) r^n \frac{dr}{r} \quad (x \ne 0).$$

Note that the integral exists for almost every x and $\Omega_\psi \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega_\psi(\theta)\,d\sigma(\theta) = 0$.

Lemma 4. *Suppose that the condition* (2.2) *holds and that there exists* $\varepsilon \in (0,1)$ *such that*

$$|\gamma(t)-\gamma(0)| \le ct^{\varepsilon} \quad \textit{for all} \quad t \in [0,1]. \tag{6.1}$$

We further assume that $b(y)$ *is a bounded radial function*: $b(y) = b_0(|y|)$. *Then for* $f \in \mathscr{S}(\boldsymbol{R}^{n+1})$ *we have*

$$\lim_{N\to\infty,\,\varepsilon\to 0}\int_{\varepsilon}^{N}\psi_t\sharp f(x,z)\frac{dt}{t} = \text{p.v.}\int f(x-y,z-\gamma(|y|))b(y)\frac{\Omega_{\psi}(y)}{|y|^{n}}\,dy.$$

Proof. To prove the lemma, without loss of generality, we may assume $\gamma(0)=0$. Let

$$I_{\varepsilon,N} = \int_{|y|<1}(f(x-y,z-\gamma(|y|)) - f(x,z))b(y)\int_{\varepsilon}^{N}\psi_t(y)\frac{dt}{t}\,dy,$$

$$II_{\varepsilon,N} = \int_{|y|>1} f(x-y,z-\gamma(|y|))b(y)\int_{\varepsilon}^{N}\psi_t(y)\frac{dt}{t}\,dy.$$

Then by (2.2) we have

$$\int_{\varepsilon}^{N}\psi_t\sharp f(x)\frac{dt}{t} = \int f(x-y,z-\gamma(|y|))b(y)\int_{\varepsilon}^{N}\psi_t(y)\frac{dt}{t}\,dy = I_{\varepsilon,N} + II_{\varepsilon,N}.$$

Using the polar coordinates, we have

$$I_{\varepsilon,N} = \int_0^1\int_{S^{n-1}}\left((f(x-r\theta,z-\gamma(r)) - f(x,z))b_0(r)\int_{\varepsilon}^{N}\psi_t(r\theta)\frac{dt}{t}r^{n-1}\right)d\sigma(\theta)\,dr.$$

By the mean value theorem we see that

$$\begin{aligned}
&\left|(f(x-r\theta,z-\gamma(r)) - f(x,z))b_0(r)\int_{\varepsilon}^{N}\psi_t(r\theta)\frac{dt}{t}r^{n-1}\right| \\
&\qquad \le cr^{n+\varepsilon-1}\int_{\varepsilon}^{N}t^{-n}|\psi(t^{-1}r\theta)|\frac{dt}{t} \\
&\qquad \le cr^{\varepsilon-1}\int_0^{\infty}t^{-n}|\psi(t^{-1}\theta)|\frac{dt}{t}.
\end{aligned}$$

Thus by the dominated convergence theorem

$$\begin{aligned}
&\lim_{N\to\infty,\,\varepsilon\to 0} I_{\varepsilon,N} \\
&\quad = \int_0^1\int_{S^{n-1}}((f(x-r\theta,z-\gamma(r)) - f(x,z))b_0(r)r^{-n}\Omega_{\psi}(\theta)r^{n-1})\,d\sigma(\theta)\,dr \\
&\quad = \int_{|y|<1}(f(x-y,z-\gamma(|y|)) - f(x,z))b(y)\frac{\Omega_{\psi}(y)}{|y|^{n}}\,dy.
\end{aligned}$$

Similarly we have

$$\lim_{N\to\infty,\,\varepsilon\to 0} II_{\varepsilon,N} = \int_{|y|>1} f(x-y, z-\gamma(|y|))b(y)\frac{\Omega_\psi(y)}{|y|^n}\,dy.$$

On the other hand, as usual, we see that

$$\begin{aligned}\lim_{\delta\to 0}\int_{|y|>\delta} & f(x-y, z-\gamma(|y|))b(y)\frac{\Omega_\psi(y)}{|y|^n}\,dy \\ &= \int_{|y|<1}(f(x-y, z-\gamma(|y|)) - f(x,z))b(y)\frac{\Omega_\psi(y)}{|y|^n}\,dy \\ &\quad + \int_{|y|>1} f(x-y, z-\gamma(|y|))b(y)\frac{\Omega_\psi(y)}{|y|^n}\,dy.\end{aligned}$$

Combining these observations, we get the conclusion. □

Under the assumption of Lemma 4 we can define

$$\begin{aligned}\text{(6.2)}\qquad T_{\psi,\gamma}(f)(x,z) &:= \lim_{N\to\infty,\,\varepsilon\to 0}\int_\varepsilon^N \psi_t \sharp f(x,z)\frac{dt}{t} \\ &= \text{p.v.}\int f(x-y, z-\gamma(|y|))b(y)\frac{\Omega_\psi(y)}{|y|^n}\,dy\end{aligned}$$

for $f \in \mathscr{S}(\boldsymbol{R}^{n+1})$. On the other hand, if the condition (2.1) holds, the limit in (6.2) exists in L^2 (see Remark 6 below). So, in this case, we define $T_{\psi,\gamma}(f)$ by the L^2 limit. This causes no ambiguity; if both the condition (2.1) and the assumptions of Lemma 4 hold, these definitions are the same.

Let $\Omega \in L^1(S^{n-1})$ satisfy $\int_{S^{n-1}} \Omega(\theta)\,d\sigma(\theta) = 0$. Put

$$\psi_\Omega(x) = (\log 2)^{-1}|x|^{-n}\Omega(x')\chi_{[1,2)}(|x|).$$

Then $\psi_\Omega \in L^1(\boldsymbol{R}^n)$ and (2.2) is satisfied. We also note that $\Omega_{\psi_\Omega} = \Omega$. So, if γ satisfies (6.1) and $b(y)$ is bounded and radial, by (6.2) we see that

$$\text{(6.3)}\qquad T_{\psi_\Omega,\gamma}(f)(x,z) = \text{p.v.}\int f(x-y, z-\gamma(|y|))b(y)\Omega(y')|y|^{-n}\,dy.$$

The Littlewood-Paley theory used to study $S_{\psi,\gamma}$ also can be applied to prove the L^p boundedness of $T_{\psi,\gamma}$.

Theorem 5. *Suppose that* $N_{\psi,\gamma}$ *is bounded on* $L^s(\boldsymbol{R}^{n+1})$, $1 < s < \infty$, *and the condition* (2.1) *holds. Then* $T_{\psi,\gamma}$ *is bounded on* $L^p(\boldsymbol{R}^{n+1})$ *for* $2s/(s+1) < p < 2s/(s-1)$.

PROOF. By the inequalities of Littlewood-Paley, Schwarz and Minkowski we have, for $1 < p < \infty$,

$$\begin{aligned}\left\|\int_\varepsilon^N \psi_t\sharp f\frac{dt}{t}\right\|_p &\le c\left\|\left(\sum_j\left|\int_\varepsilon^N \sum_k T_{k+j}(\Delta_j(f))(\cdot\,,\cdot\,;t)\frac{dt}{t}\right|^2\right)^{1/2}\right\|_p \\ &\le c\left\|\left(\sum_j\left(\sum_k |T_{k+j}(\Delta_j(f))|_{\mathscr{H}}\right)^2\right)^{1/2}\right\|_p \\ &\le c\left\|\sum_k\left(\sum_j |T_{k+j}(\Delta_j(f))|_{\mathscr{H}}^2\right)^{1/2}\right\|_p \\ &= c\left\|\sum_k U_k(f)\right\|_p.\end{aligned}$$

Let $2 \le p < 2s/(s-1)$. Then as in the proof of Theorem 1, we have

$$\|U_k(f)\|_p \le c2^{-\delta|k|}\|f\|_p$$

for some $\delta > 0$. Combining these estimates, we see that

$$\left\|\int_\varepsilon^N \psi_t\sharp f\frac{dt}{t}\right\|_p \le c\|f\|_p.$$

Letting $\varepsilon \to 0$, $N \to \infty$, we get the conclusion for $2 \le p < 2s/(s-1)$. The result for $2s/(s+1) < p < 2$ follows by duality. □

REMARK 6. Let $f \in \mathscr{S}(\boldsymbol{R}^{n+1})$. Suppose that the condition (2.1) holds. Then the limit

$$\lim_{N\to\infty,\,\varepsilon\to 0}\int_\varepsilon^N \psi_t\sharp f(x,z)\frac{dt}{t}$$

exists in L^2. To see this, put, for $0 < M < N$,

$$T_k^{M,N}(f)(x,z;t) = \psi_t\sharp f(x,z)\chi_{[1,2)}(2^{-k}t)\chi_{[M,N]}(t)$$

and define

$$U_j^{M,N}(f) = \left(\sum_k |T_k^{M,N}(\Delta_{j+k}(f))|_{\mathscr{H}}^2\right)^{1/2}.$$

Then as in the proof of Theorem 5 we have

$$\left\|\int_M^N \psi_t \sharp f \frac{dt}{t}\right\|_2 \le c \left\|\sum_j U_j^{M,N}(f)\right\|_2.$$

Note that $U_j^{M,N}(f) \le U_j(f)$, $\lim_{M,N\to\infty} U_j^{M,N}(f) = 0$ a.e. and $\|\sum_j U_j(f)\|_2 < \infty$. Therefore by the dominated convergence theorem we see that

$$\lim_{M,N\to\infty}\left\|\int_M^N \psi_t \sharp f \frac{dt}{t}\right\|_2 = 0.$$

Similarly we see that

$$\lim_{\varepsilon,\delta\to 0}\left\|\int_\varepsilon^\delta \psi_t \sharp f \frac{dt}{t}\right\|_2 = 0.$$

Thus the Cauchy criterion implies the conclusion.

Corollary 4. *Let ψ, b and γ be as in Corollary* 1. *Then $T_{\psi,\gamma}$ is bounded on $L^p(\boldsymbol{R}^{n+1})$ for all $p \in (1,\infty)$.*

Proof. In the proof of Corollary 1, it is shown that $N_{\psi,\gamma}$ is bounded on $L^p(\boldsymbol{R}^{n+1})$ for all $p \in (1,\infty)$ and the condition (2.1) holds. Thus the result follows from Theorem 5. □

If (5.3) or (2.2) is satisfied, we can also define

$$T_\psi(f)(x) = \lim_{N\to\infty,\varepsilon\to 0}\int_\varepsilon^N \psi_t * f(x)\frac{dt}{t} \quad (x \in \boldsymbol{R}^n).$$

Then as a corollary to Theorem 5, we have the following.

Theorem 6. *Suppose that N_ψ is bounded on $L^s(\boldsymbol{R}^n)$, $1 < s < \infty$, and ψ satisfies* (5.3). *Then T_ψ is bounded on $L^p(\boldsymbol{R}^n)$ for $2s/(s+1) < p < 2s/(s-1)$.*

Theorem 6 essentially improves Theorem C in the same way that Theorems 3 and 4 improve Theorem B.

Remark 7. Let $n \ge 2$. Let Ω be in the Hardy space $H^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega\, d\sigma = 0$, and let $\Gamma(r)$ be a function on $(0,\infty)$ such that $\int_0^\infty |\Gamma(r)| r^{n-1}\, dr < \infty$. Put $\psi(x) = \Gamma(|x|)\Omega(x')$. Then by Lemma 4

$$T_\psi(f)(x) = \text{p.v.}\, c\int f(x-y)\frac{\Omega(y)}{|y|^n}\, dy$$

for some constant c. Thus by [**8**] T_ψ is bounded on L^p for all $p \in (1,\infty)$. This improves Theorem 5 in [**2**]; in [**2**] the result is proved under a stronger assumption that $|\Gamma(r)| \le c\min(r^{-n+\rho}, r^{-n-\rho})$ for some $\rho > 0$.

REMARK 8. It is not difficult to see that we can replace the condition (2.1) by (2.10) in all the results of this section which require (2.1) (see Remark 1).

By Remark 8, Lemma 4 and Theorem 5 we have the following singular integral analog of Corollary 2.

COROLLARY 5. *Let $n \ge 2$, and let a curve γ and a function b be as in Lemma 4. Let $\Omega \in H^1(S^{n-1})$ satisfy $\int_{S^{n-1}} \Omega\, d\sigma = 0$. Define*

$$T(f)(x,z) = \text{p.v.}\int f(x-y, z-\gamma(|y|))b(y)\Omega(y')|y|^{-n}\,dy,$$

initially for $f \in \mathscr{S}(\boldsymbol{R}^{n+1})$. Suppose

$$\|g_\gamma^*\|_{L^r(\boldsymbol{R}^2)} \le C_r\|g\|_{L^r(\boldsymbol{R}^2)}$$

for some $r \in (1,\infty)$, where g_γ^* is as in Corollary 2. Then we have

$$\|T(f)\|_{L^p(\boldsymbol{R}^{n+1})} \le C_p\|f\|_{L^p(\boldsymbol{R}^{n+1})}$$

for all $p \in (2r/(r+1), 2r/(r-1))$.

PROOF. The proof is similar to that of Corollary 2. So, our proof here is brief. As in the proof of Corollary 2 we may assume that Ω is an L^∞-atom. Let Ω be the L^∞-atom considered in the proof of Corollary 2, and put

$$\psi(x) = (\log 2)^{-1}|x|^{-n}\Omega(x')\chi_{[1,2]}(|x|).$$

Then by (6.3) $T(f) = T_{\psi,\gamma}(f)$. Define $A_t(\xi,\omega)$ by this ψ. Then we also have the estimate (2.12). Thus by Remark 8 and Theorem 5, to prove the corollary it suffices to show the $L^r(\boldsymbol{R}^{n+1})$ boundedness of $N_{\psi,\gamma}$, but this also can be proved in the same way as the $L^r(\boldsymbol{R}^{n+1})$ boundedness of $M_{\psi,\gamma}$ in the proof of Corollary 2 by using the $L^r(\boldsymbol{R}^2)$ boundedness of g_γ^*. This completes the proof. $\square$

Corollary 5 is the main theorem in a recent paper by Lu, Pan and Yang [**13**], in which they obtained the theorem by a different proof based on estimates of certain oscillatory integrals.

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